Strong law of large numbers for fragmentation processes

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Introduction

In the spirit of a classical result for Crump–Mode–Jagers processes (general branching processes), cf. [Ner81], we present a strong law of large numbers for fragmentation processes.

For self–similar fragmentation processes we show the almost sure convergence of an empirical measure associated with the stopping line corresponding to the first fragments of size smaller than $\eta \in (0,1]$.

Some of the mathematical roots of fragmentation processes lay with older families of spatial branching processes such as branching random walks and C–M–J processes.

Such stochastic processes exemplify the phenomena of random splitting according to systematic rules and they may be seen as modelling the growth of special types of multi–particle systems.

For a comprehensive treatise on fragmentation processes see the monograph [Ber06] by J. Bertoin.

Self–similar mass fragmentation processes

Set

$$\mathcal{S} := \left\{ \mathbf{s} := (s_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_0^+ : \sum_{n\in\mathbb{N}} s_n \le 1,\ s_i \ge s_j\, \forall\, i \le j \right\}.$$

Definition *We call an $\mathcal{S}$–valued Markov process $(\lambda(t))_{t\in\mathbb{R}_0^+}$, continuous in probability, a self–similar mass fragmentation process with index $\alpha \in \mathbb{R}$ if*

- $\lambda(0) = (1,0,\ldots)$
- *For any $s,t \in \mathbb{R}_0^+$, given that $\lambda(t) = (s_n)_{n\in\mathbb{N}}$, we have*

$$\lambda(s+t) \overset{\mathrm{d}}{=} \left(s_n \lambda^{(n)}(s_n^\alpha s) \right)_{n\in\mathbb{N}}^{\downarrow},$$

where the $\lambda^{(n)}$ are i.i.d. copies of λ.
If $\alpha = 0$, then the process is called homogenous.

Characterisation [Ber02]: Self–similar mass fragmentation processes are characterised by

- the index of self–similarity $\alpha \in \mathbb{R}$,
- a nonnegative rate of erosion,
- a so–called *dislocation measure* ν on $\mathcal{S}$ which satisfies

$$\nu(\{1,0,\ldots\}) = 0 \quad \text{and} \quad \int_{\mathcal{S}} (1-s_1)\nu(\mathrm{d}\mathbf{s}) < \infty.$$

Interpretation

- The dislocation measure ν specifies the rate at which blocks split.
- A block of mass x dislocates into a mass partition $x\mathbf{s}$, where $\mathbf{s} \in \mathcal{S}$, at rate $\nu(d\mathbf{s})$.

Poissonian structure ($\alpha = 0$)

Let $\{(\mathbf{s}(t), k(t)) : t \ge 0\}$ be an $\mathcal{S}\times\mathbb{N}$–valued Poisson point process with characteristic measure $\nu \otimes \sharp$, where $\sharp$ is the counting measure on $\mathbb{N}$, and let $\lambda := (\lambda(t))_{t\in\mathbb{R}_0^+}$ be the homogenous mass fragmentation process associated with ν.

- λ changes state at all times $t \ge 0$ for which an atom $(\mathbf{s}(t), k(t))$ occurs in $\mathcal{S} \setminus \{(1,0,\cdots)\} \times \mathbb{N}$.
- At such a time t the sequence $\lambda(t)$ is obtained from $\lambda(t-)$ by replacing its $k(t)$–th term, $\lambda_{k(t)}(t-)$, with the sequence $\lambda_{k(t)}(t-)\mathbf{s}(t)$ and ranking the terms in decreasing order.

Finite vs. infinite dislocation measure

$\nu(\mathcal{S}) < \infty$:

- A block of size x remains unchanged for exponential periods of time with parameter $x^\alpha \nu(\mathcal{S})$.
- There are finitely many dislocations over any finite time horizon.
- By taking the negative logarithm of fragment sizes the fragmentation process is closely related to continuous–time Markov branching random walks and C–M–J processes.

$\nu(\mathcal{S}) = \infty$:

- Blocks dislocate immediately.
- There is a countably infinite number of dislocations over any finite time horizon.

The case $\nu(\mathcal{S}) = \infty$ is the more interesting one and, moreover, in [BM05] it is shown that our main result can be inferred from [Ner81] if $\nu(\mathcal{S}) < \infty$. Hence, we are mainly interested in the case of an infinite dislocation measure. However, our results hold true for finite and infinite dislocation measures.

Main result

Preliminaries

Stopping line Let $(\lambda_{\eta,i})_{i\in\mathbb{N}}$ be an arbitrarily ordered set of the elements in the stopping line of those fragments which are the first in their line of descent that are smaller than η.

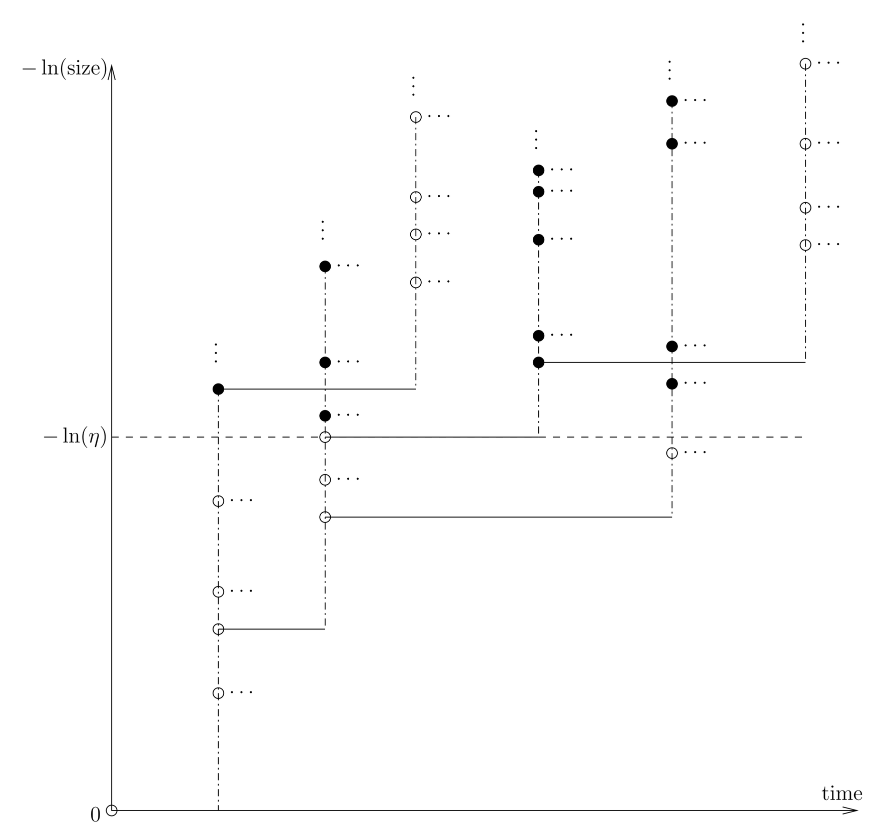

Figure: Stopping line at $\eta \in (0,1]$. The black dots indicate the particles belonging to $(\lambda_{\eta,i})_{i\in\mathbb{N}}$, since their size is smaller than η and they result from the dislocation of a particle of size greater than or equal to η.

Set

$$\underline{p} := \inf\left\{ p \in \mathbb{R} : \int_{\mathcal{S}} \left| 1 - \sum_{n\in\mathbb{N}} s_n^{1+p} \right| \nu(\mathrm{d}\mathbf{s}) < \infty \right\} \in (-1, 0].$$

It is well known that

$$\Phi : (\underline{p}, \infty) \to \mathbb{R},\ p \mapsto \Phi(p) = \int_{\mathcal{S}} \left(1 - \sum_{n\in\mathbb{N}} s_n^{1+p} \right) \nu(\mathrm{d}\mathbf{s})$$

is strictly monotonically increasing and concave.

Assumption (A)

(i) Erosion rate $= 0$.
(ii) If $\underline{p} = 0$, then

$$\Phi'(0+) = \int_{\mathcal{S}} \left(\sum_{n\in\mathbb{N}} s_n \ln\left(s_n^{-1}\right) \right) \nu(\mathrm{d}\mathbf{s}) < \infty.$$

(iii) If $\nu\left(\sum_{n\in\mathbb{N}} s_n < 1\right) > 0$, then there exists a $p^* \in (\underline{p}, 0]$ such that $\Phi(p^*) = 0$.

If $\nu\left(\sum_{n\in\mathbb{N}} s_n < 1\right) = 0$, then $\Phi(0) = 0$, and thus we set $p^* := 0$ in this case.

Martingale $\left(\sum_{i\in\mathbb{N}} \lambda_{\eta,i}^{1+p^*}\right)_{1\ge\eta>0}$ is a nonnegative, uniformly integrable martingale with unit mean.

We denote the $\mathbb{P}$–a.s. limit of this martingale by $\Lambda(p^*)$.

Preparations

Define

$$\mathcal{B}^+ := \{ f : [0,1] \to \mathbb{R}_0^+ : f \text{ is bounded and measurable} \}.$$

For any $\eta \in (0,1]$ and $f \in \mathcal{B}^+$ set

$$\langle \rho_\eta, f \rangle := \sum_{i\in\mathbb{N}} \lambda_{\eta,i}^{1+p^*} f\left(\frac{\lambda_{\eta,i}}{\eta} \right).$$

Objective We aim at showing that $\langle \rho_\eta, f \rangle$ behaves asymptotically, as $\eta \downarrow 0$, $\mathbb{P}$–a.s. like $\Lambda(p^*)$ up to a multiplicative constant which depends on f.

For every $f \in \mathcal{B}^+$ define

$$\langle \rho, f \rangle := \frac{1}{\Phi'(p^*)} \int_0^1 f(u) \left(\int_{\mathcal{S}} \sum_{n\in\mathbb{N}} \mathbb{1}_{\{u > s_n\}} s_n^{1+p^*} \nu(\mathrm{d}\mathbf{s}) \right) \frac{du}{u}.$$

Main result

Remark

$$\forall\, f \in \mathcal{B}^+ : \langle \rho, f \rangle = \lim_{\eta\downarrow 0} \mathbb{E}(\langle \rho_\eta, f \rangle).$$

Theorem [HKK09] *For any self–similar fragmentation process satisfying* **(A)** *we have*

$$\lim_{\eta\downarrow 0} \langle \rho_\eta, f \rangle = \langle \rho, f \rangle \Lambda(p^*)$$

$\mathbb{P}$–a.s. for all $f \in \mathcal{B}^+$.

References

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