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WiSe 2019/2010

7th November 2019

Stochastics II

4. Tutorial

Exercise 1 **(4 Points)** Let $\mathcal{B}$ be the Borel σ-field on $\mathbb{R}$. Show that

(i) for each $A \in \mathcal{B}^{[0,\infty)}$ there exists a sequence $(t_n)_{n\in\mathbb{N}}$ in $[0,\infty)$ and a set $B \in \mathcal{B}^{\mathbb{N}}$ such that

$$A = \{x \in \mathbb{R}^{[0,\infty)} : \ (x_{t_1}, x_{t_2}, \ldots) \in B\}.$$

(ii) the set

$$C := \{x \in \mathbb{R}^{[0,\infty)} : x \text{ is continuous}\}$$

is not contained in $\mathcal{B}^{[0,\infty)}$.

Exercise 2 **(4 Points)** Show that a stochastic process $X := (X_t)_{t\in[0,\infty)}$, which is non-decreasing and integrable, has a modification which is P-a.s. rightcontinuous with left limits, if and only if the function $t \mapsto E[X_t]$ is rightcontinuous.

Exercise 3 **(5 Points)** Let $\lambda : [0,\infty) \to [0,\infty)$ be a function with $\int_s^t \lambda(u)du < \infty$ for all $s,t \in \mathbb{R}$. Let

$$\{P_{t_1,\ldots,t_n} : n \in \mathbb{N}, 0 \le t_1 < \ldots < t_n\}$$

be a family of finite dimensional distributions, where $P_{t_1,\ldots,t_n}$ is a probability measure on $(\mathbb{N}_0^n, 2^{\mathbb{N}_0^n})$ for $n \in \mathbb{N}$ and $0 \le t_1 < \ldots < t_n$, which is given by

$$P_{t_1,\ldots,t_n}(\{k_1,\ldots,k_n\}) = \begin{cases} \prod_{j=1}^n \frac{e^{-\lambda_j}\lambda_j^{k_j-k_{j-1}}}{(k_j-k_{j-1})!}, & k_1 \le \ldots \le k_n \\ 0, & \textit{else} \end{cases}$$

with $k_0 := 0$, $t_0 := 0$ and $\lambda_j = \int_{t_{j-1}}^{t_j} \lambda(u)du$ for $j = 1,\ldots,n$. Show that this family of finite dimensional distributions is consistent.

Exercise 4 **(5 Points)** Show that a family

$$\mathcal{P} := \{P_{t_1,\ldots,t_n} : n \in \mathbb{N}, 0 \le t_1 < \ldots < t_n\}$$

of finite dimensional distributions is consistent, if and only if the following identity holds for all $n \in \mathbb{N}$, $0 \le t_1 < \ldots, t_n$ and $j = 1,\ldots,n$:

$$\varphi_{P_{t_1,\ldots,t_n}}(u_1,\ldots,u_{j-1},0,u_{j+1},\ldots,u_n) = \varphi_{P_{t_1,t_{j-1},t_{j+1},\ldots,t_n}}(u_1,\ldots,u_{j-1},u_{j+1},\ldots,u_n).$$

Hint: It could be helpful to define a measure $\tilde{P}$ on $(\mathbb{R}^{n-1}, \mathcal{B}(\mathbb{R}^{n-1}))$ such that

$$\tilde{P}(A_1, \ldots, A_{j-1}, A_{j+1}, \ldots, A_n) = P_{t_1,\ldots,t_n}(A_1, \ldots, A_{j-1}, \mathbb{R}, A_{j+1}, \ldots, A_n)$$

for $P_{t_1,\ldots,t_n} \in \mathcal{P}$ and all $A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R})$.