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Partial regularity for a class of anisotropic variational integrals with convex hull property

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Abstract. We consider integrands $f:\mathbb{R}^{nN} \to \mathbb{R}$ which are of lower (upper) growth rate $s \geqslant 2$ ($q > s$) and which satisfy an additional structural condition implying the convex hull property, i.e., if the boundary data of a minimizer $u:\Omega \to \mathbb{R}^N$ of the energy $\int_\Omega f(\nabla u)\,\mathrm{d}x$ respect a closed convex set $K \subset \mathbb{R}^N$, then so does u on the whole domain. We show partial $C^{1,\alpha}$-regularity of bounded local minimizers if $q < \min\{s + 2/3, sn/(n-2)\}$ and discuss cases in which the latter condition on the exponents can be improved. Moreover, we give examples of integrands which fit into our category and to which the results of Acerbi and Fusco [2] do not apply, in particular, we give some extensions to the subquadratic case.

Keywords: regularity, minimizers, anisotropic growth

1. Introduction

As a model for our investigations we consider the anisotropic energy (in which $1 + |\nabla u|^2$ can also be replaced by $|\nabla u|^2$)

$$I[u] = \int_\Omega \left[1 + |\nabla u|^2 + \left(1 + |\partial_n u|^2\right)^{q/s}\right]^{s/2} \mathrm{d}x \tag{1.1}$$

defined on suitable classes of vectorial functions $u:\Omega \to \mathbb{R}^N$, where Ω denotes a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geqslant 2$, and $s, q \in \mathbb{R}$ are fixed exponents such that for the moment $2 \leqslant s < q$. We are interested in the partial regularity properties of local minimizers of (1.1) but unfortunately we cannot refer to the paper [2] of Acerbi and Fusco since our energy density does not decompose in the form "$(1 + |\nabla u|^2)^{s/2} + (1 + |\partial_n u|^2)^{q/2}$". More precisely, the subject of [2] are energy densities f which can be written as

$$f(\nabla u) = h(\nabla u) + \sum_{\alpha \in I} h_\alpha(\partial_\alpha u),$$

where I is a subset of $\{1, \dots, n\}$ and h is an elliptic integrand of growth order s. The functions h_α are strictly convex and of growth order q_α (to be defined in terms of $D^2 h_\alpha$), $2 \leqslant s < q_\alpha$, we refer the reader to Theorem 2.3 and Proposition 4.1 of [2] for a detailed discussion.

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By elementary calculations (see Appendix A) it is easy to see that in our example the integrand f satisfies

$$\lambda\bigl(1+|Z|^2\bigr)^{(s-2)/2}|U|^2 \leqslant D^2 f(Z)(U,U) \leqslant \Lambda\bigl(1+|Z|^2\bigr)^{(q-2)/2}|U|^2 \tag{1.2}$$

for all $Z, U \in \mathbb{R}^{nN}$ with constants $0 < \lambda < \Lambda < \infty$. Moreover, f can be represented as

$$f(Z) = g\bigl(|Z_1|, \dots, |Z_n|\bigr), \quad Z = (Z_1, \dots, Z_n) \in \mathbb{R}^{nN}, \tag{1.3}$$

where g is an increasing function of each argument.

Definition 1.1. Let $f : \mathbb{R}^{nN} \to [0, \infty)$ denote a function of class C^2 satisfying (1.2) with exponents $2 \leqslant s < q$. $u \in W^1_{s,\mathrm{loc}}(\Omega; \mathbb{R}^N)$ is termed a local minimizer of the energy $J[w] = \int_\Omega f(\nabla w)\,\mathrm{d}x$ if and only if

(a) $\int_{\Omega'} f(\nabla u)\,\mathrm{d}x < \infty, \forall \Omega' \Subset \Omega$ and
(b) $\int_{\mathrm{spt}(u-v)} f(\nabla u)\,\mathrm{d}x \leqslant \int_{\mathrm{spt}(u-v)} f(\nabla v)\,\mathrm{d}x, \forall v \in W^1_{s,\mathrm{loc}}(\Omega; \mathbb{R}^N)$, $\mathrm{spt}(u-v) \Subset \Omega$.

Since we are in the convex case (which follows from the first inequality in (1.2)), it is easy to show that for boundary values u_0 with the property $J[u_0] < \infty$ there exists a unique J-minimizer in the class $u_0 + \overset{\circ}{W}{}^1_s(\Omega; \mathbb{R}^N)$ (see Lemma 1.2 below) which under reasonable assumptions on the exponents s and q in fact is in the space $W^1_{q,\mathrm{loc}}(\Omega; \mathbb{R}^N)$.

Let us briefly recall the partial regularity results which are known to be valid for local J-minimizers u in case that f satisfies condition (1.2).

(i) In [27] it is shown that there is an open subset Ω_0 of Ω such that $|\Omega - \Omega_0| = 0$ and $u \in C^{1,\alpha}(\Omega_0; \mathbb{R}^N)$ provided we know

$$q < \min\left\{s+1, s\frac{n}{n-1}\right\}. \tag{1.4}$$

Note that in [27] the second inequality of (1.2) is not required, they work with the weaker upper bound $f(\xi) \leqslant C(1 + |\xi|^q)$ which is a consequence of (1.2). We remark that the restriction $q < sn/(n-1)$ enters their arguments through the use of a lemma on Sobolev functions due to Fonseca and Malý [15], $q < s+1$ is needed for establishing the Euler–Lagrange equation for local minimizers (see [27, Section 2, Remark 1]).

(ii) Also partial $C^{1,\alpha}$-regularity of local J-minimizers has been established in the paper [6] in case that (let $\mu = 2 - s$ in condition (1.8) of [6])

$$q < s\frac{n+2}{n}. \tag{1.5}$$

We remark that in [6] also subquadratic growth is considered and that the left-hand side of (1.2) can be replaced by a weaker estimate but then (1.5) becomes more complicated. Condition (1.5) is mainly used to prove that the gradient of a local minimizer actually belongs to the space $L^t_{\mathrm{loc}}(\Omega; \mathbb{R}^{nN})$ for some $t > q$, during the blow-up procedure we just need $q < sn/(n-2)$ (if $n \geqslant 3$), we refer to Section 4 of [6].

Since $\min\{s+1, sn/(n-1)\} \leqslant s(n+2)/n$, (1.5) is less restrictive than condition (1.4) of [27]. In particular, if $n = 2$, then (1.4) reads as $q < s + 1$ and from (1.5) we get $q < 2s$ (compare Remark 4.1).

The purpose of our note is to improve both results with the help of the additional requirement (1.3), precisely:

Theorem 1.1. *Let $f : \mathbb{R}^{nN} \to [0,\infty)$ denote a C^2-integrand with* (1.2) *and* (1.3), $2 \leqslant s < q < \infty$, *and let u denote a locally bounded local minimizer in the sense of Definition* 1.1. *Then u is an element of the space $W^1_{q,\mathrm{loc}}(\Omega;\mathbb{R}^N)$ and also a function of class $C^{1,\alpha}(\Omega_0;\mathbb{R}^N)$ for any $0 < \alpha < 1$ on an open subset Ω_0 of Ω with full Lebesgue measure provided we impose the bound*

$$q < \min\left\{s + \frac{2}{3}, s\frac{n}{n-2}\right\} \quad \text{if } n \geqslant 3, \tag{1.6}$$

and $q < \frac{2}{3} + s$, if $n = 2$.

Remark 1.1.

(a) Clearly, at least for large n, (1.6) is less restrictive than (1.5), so assumption (1.3) together with the local boundedness of the minimizer allows more flexibility concerning the choices for s and q. If n is small, than [27,6] lead to better results. Let us briefly comment on an other case for which (1.6) improves (1.5) by the way extending the theorems of [6,27]. If we choose s according to

$$\frac{n-2}{3} < s < \frac{n}{3},$$

then it is easy to show that

$$s + \frac{2}{3} < s\frac{n}{n-2} \quad \text{and} \quad s\frac{n+2}{n} < s + \frac{2}{3},$$

which means that (1.6) reduces to the requirement $q < \frac{2}{3} + s$, and the latter inequality is weaker than (1.5).

(b) Condition (1.6) in the form $q < sn/(n-2)$ also occurs in the paper [21], where Marcellini considers the scalar case $N = 1$ and shows everywhere regularity of local minimizers u assuming already that u is in the space $W^1_{q,\mathrm{loc}}(\Omega)$. In [22] Marcellini then uses (1.5) to get the existence of a minimizer in $W^1_{q,\mathrm{loc}}(\Omega)$.

(c) Let us look at the example

$$f(\xi_1,\dots,\xi_n) = |\xi|^2 + \left(1 + |\xi_n|^2\right)^{q/2}, \quad \xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^{nN},$$

with $q > 2$. This integrand is covered by [2], and according to this result (n large)

$$q < 2 + \frac{4}{n-3} \tag{1.7}$$

is sufficient for partial regularity of local minimizers $u \in W^1_{2,\mathrm{loc}}(\Omega;\mathbb{R}^N)$. In this case (1.6) reads (n large) $q < 2 + 4/(n-2)$, thus Acerbi and Fusco obtain better results even without the assumption

$u \in L^{\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ (which is not so restrictive since in our example f satisfies (1.3), compare the discussion after Lemma 1.1).

(d) Let in extension of (1.1)

$$f(\xi) = \Big(1 + |\xi|^2 + \sum_{\alpha\in I}(1+|\xi_\alpha|^2)^{q_\alpha/s}\Big)^{s/2},$$

where I is a subset of $\{1,\dots,n\}$ and $2 \leqslant s < q_\alpha$, $\alpha \in I$. Then (1.3) holds and (1.2) is satisfied with $q = \max\{q_\alpha\colon\ \alpha \in I\}$. Clearly the theorems of [2] cannot be applied, but we get partial regularity under condition (1.6). The same is true for the energy

$$f(\xi) = (1+|\xi|^2)^{s/2} + \sum_{\alpha\in I} h_\alpha\big(|\xi_\alpha|\big)$$

if we choose h_α of the form $(1+t^2)^{q_\alpha/2}$ but with infinitely many linear pieces so that ellipticity is destroyed (see [8]) which means that for $H_\alpha(\xi_\alpha) = h_\alpha(|\xi_\alpha|)$ we just have the estimate

$$0 \leqslant D^2H_\alpha(\xi_\alpha)(U_\alpha,U_\alpha) \leqslant c_\alpha\big(1+|\xi_\alpha|^2\big)^{(q-2)/2}|U_\alpha|^2.$$

Let us finally look at the integrand

$$f(\xi_1,\dots,\xi_n) = \Phi_1(\xi_1) + \cdots + \Phi_n(\xi_n), \quad \Phi_k(\xi_k) = \big(1+|\xi_k|^2\big)^{q_k/2},\ \xi_k \in \mathbb{R}^N,$$

with exponents $q_k \geqslant 2$. In contrast to [2] an elliptic part involving “the whole gradient” is missing. We have

$$D^2f(\xi)(U,U) \geqslant c\sum_{k=1}^{n}\big(1+|\xi_k|^2\big)^{(q_k-2)/2}|U_k|^2 \geqslant c|U|^2,$$
$$D^2f(\xi)(U,U) \leqslant c\big(1+|\xi|^2\big)^{(q-2)/2}|U|^2,$$

where $q := \max\{q_k\colon\ k = 1,\dots,n\}$. Thus (1.2) holds for the choice $s = 2$, and for n large Theorem 1.1 implies that bounded local minima are partially regular if $q_k < 2 + 4/(n-2)$, $k = 1,\dots,n$.

(e) We do not touch the question of everywhere regularity in the vectorial case which besides an appropriate ratio of q and s also requires a structural condition of the form $f(\xi) = G(|\xi|)$, we refer to [23,5], where the interested reader will find further references.

In order to finish our discussion on the various choices for the exponents s and q and to include the results of [6] we reformulate Theorem 1.1 in the following way.

Theorem 1.2. *Suppose that all the hypotheses of Theorem* 1.1 *are valid but with* (1.6) *being replaced by*

$$q < 2s \quad \text{if } n = 2, \tag{1.6*}$$
$$q < s\frac{n}{n-2} \quad \text{and} \quad q < q_0 := \max\Big\{s + \frac{2}{3}, s\frac{n+2}{n}\Big\} \quad \text{if } n \geqslant 3. \tag{1.6**}$$

Then partial $C^{1,\alpha}$-regularity holds for locally bounded local minimizers u.

In case $n = 2$ or $n \geqslant 3$ together with $q_0 = s(n+2)/n$ we can drop the assumption $u \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ and also (1.3) *is superfluous. Moreover, under the above assumptions, u is in the space $W^1_{q,\mathrm{loc}}(\Omega; \mathbb{R}^N)$.*

Note that the last statement of Theorem 1.2 is a direct consequence of [6, Theorem 1.1] with the choice $\mu = 2 - s$. Note also that the requirement $q < q_0$ is sufficient to show $\nabla u \in L^{q+\varepsilon}_{\mathrm{loc}}(\Omega; \mathbb{R}^{nN})$ for some $\varepsilon > 0$ (see Theorem 3.1 below). With this information we get partial $C^{1,\alpha}$-regularity just assuming $q < sn/(n-2)$. The local boundedness of u and condition (1.3) do not enter the blow-up procedure.

Up to now we limited our discussion to the case $2 \leqslant s < q$ in order to compare our results to the ones of [2,27]. To our knowledge anisotropic power growth with leading exponent $q < 2$ only occurs in [6]: with the choice $\mu = 2 - s$ in [6, Theorem 1.1] we see that condition (1.5) is sufficient for partial regularity. But in fact we have a stronger result:

Theorem 1.3. *The statements of Theorems* 1.1 *and* 1.2 *remain true if we consider arbitrary exponents* $1 < s < q$ (*provided the other hypotheses are valid*).

Thus we can also include models like

$$f(\xi) = \left(1 + |\xi|^2\right)^{(1+\varepsilon)/2} + |\xi_n|^2, \quad \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^{nN},$$

with appropriate choice of $\varepsilon \in (0,1)$ which are not covered by [2,27].

Next we discuss our assumption (1.3) by showing that it is a kind of natural hypothesis giving boundedness of minimizers. First of all we collect some consequences of (1.2).

Lemma 1.1. *Let $f : \mathbb{R}^{nN} \to [0, \infty)$ satisfy* (1.2) *with $2 \leqslant s < q$. Then we have*:

(i) *f is strictly convex*;
(ii) $f(Z) \leqslant c_1(|Z|^q + 1)$;
(iii) $|\nabla f(Z)| \leqslant c_2(|Z|^{q-1} + 1)$;
(iv) $\nabla f(Z) : Z \geqslant c_3|Z|^s - c_4$;
(v) $f(Z) \geqslant c_5|Z|^s - c_6$,

the estimates being valid for all $Z \in \mathbb{R}^{nN}$. Here c_i denote positive constants.

Proof. (i) Is a consequence of the first inequality in (1.2). (ii) Follows from the second part of (1.2). (iii) Compare [10, Lemma 2.2, p. 156].

(iv) We have

$$\nabla f(Z) : Z = \int_0^1 D^2 f(tZ)(Z, Z)\, \mathrm{d}t + \nabla f(0) : Z,$$

and the lower bound in (1.2) gives the result.

(v) Convexity of f implies

$$f(Z) \geqslant f\left(\frac{Z}{2}\right) + \nabla f\left(\frac{Z}{2}\right) : \frac{Z}{2},$$

and the claim follows from $f \geqslant 0$ together with (iv). □

Lemma 1.2. *Let* $u_0 \in W^1_s(\Omega;\mathbb{R}^N)$ *satisfy* $J[u_0] = \int_\Omega f(\nabla u_0)\,\mathrm{d}x < \infty$, *where* $f \geqslant 0$ *is of class* C^2 *with* (1.2). *Then the variational problem*

$$J[u] \to \min \quad \textit{in } u_0 + \overset{\circ}{W}{}^1_s(\Omega;\mathbb{R}^N) \tag{1.8}$$

has a unique solution.

Proof. Obviously $\inf\{J[w]\colon w \in u_0 + \overset{\circ}{W}{}^1_s(\Omega;\mathbb{R}^N)\} < \infty$, thus on account of Lemma 1.1(v), any minimizing sequence $\{u_m\} \subset u_0 + \overset{\circ}{W}{}^1_s(\Omega;\mathbb{R}^N)$ is uniformly bounded in $W^1_s(\Omega;\mathbb{R}^N)$, hence $u_m \rightharpoondown: u$ in $W^1_s(\Omega;\mathbb{R}^N)$. Convexity of f together with De Giorgi's theorem on lower semicontinuity proves that u solves (1.8). Uniqueness follows from strict convexity of f. $\square$

Lemma 1.3 (Convex hull property). *In addition to the hypotheses of Lemma* 1.2 *assume* $\mathrm{Im}(u_0) \subset K$ *for a compact convex set* $K \subset \mathbb{R}^N$. *Then the solution* u *of* (1.8) *also satisfies* $\mathrm{Im}(u) \subset K$ *provided that* (1.3) *holds.*

Remark 1.2.

(a) From the paper [11] we deduce

$$\sup_\Omega |u^i| \leqslant \sup_\Omega |u_0^i|, \quad i = 1,\dots,N,$$

thus $u \in L^\infty(\Omega;\mathbb{R}^N)$, if u_0 is bounded. In case of (1.3) this is a trivial observation since we may use comparison functions like $v = (\Phi(u^1), u^2, \dots, u^N)$, where $m := \sup_\Omega |u_0^1|$ and

$$\Phi(t) = \begin{cases} m, & t \geqslant m, \\ t, & -m \leqslant t \leqslant m, \\ -m, & t \leqslant -m. \end{cases}$$

Then $|\partial_\alpha v| \leqslant |\partial_\alpha u|$ and (1.3) implies $v = u$.

(b) Lemma 1.3 motivates the study of locally bounded local minimizers in Theorem 1.1. It should however be noted that our proof of Theorem 1.1 does not work if we just consider solutions of class $L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ and impose the growth condition (1.2). In the next sections Lemma 1.3 will be an important tool.

Proof of Lemma 1.3. Let $\pi : \mathbb{R}^N \to K$ denote the projection onto K which satisfies $\mathrm{Lip}(\pi) = 1$. From Lemma B.1 below we deduce $|\partial_\alpha(\pi \circ u)| \leqslant |\partial_\alpha u|$, $\alpha = 1,\dots,n$, thus ($v := \pi \circ u$)

$$g(|\partial_1 v|,\dots,|\partial_n v|) \leqslant g(|\partial_1 u|,\dots,|\partial_n u|).$$

So v is minimizing which implies $v = u$. $\square$

The rest of our paper is organized as follows: in Section 2 we discuss some (appropriate) local approximation, Section 3 contains the proof of uniform higher integrability of these approximations under the condition $q < s + \frac{2}{3}$, and in Section 4 we use this result for obtaining Theorem 1.1 via a standard blow-up procedure which works in case $q < sn/(n-2)$, if $n > 2$. The subquadratic case is briefly considered in Section 5. In Appendix A we discuss our example (1.1), in Appendix B we give a short proof of the chain rule inequality needed for Lemma 1.3.

2. Local approximation

We use a standard approximation procedure which in different situations also occurs in [18,12,13,25, 4,6,8]. From now on assume that all the assumptions of Theorem 1.1 are valid. Let $r > q$ and define for $\delta > 0$

$$f_\delta(\xi) = \delta\big(1 + |\xi|^2\big)^{r/2} + f(\xi), \quad \xi \in \mathbb{R}^{nN}.$$

Without loss of generality let $B_{2R}(0) \Subset \Omega$ and consider the mollification u_ε of our local J-minimizer u. Let $v_{\varepsilon,\delta} \in u_\varepsilon + \overset{\circ}{W}{}^1_r(B_{2R};\mathbb{R}^N)$ denote the solution of

$$J_\delta[w] := \int_{B_{2R}} f_\delta(\nabla w)\,\mathrm{d}x \to \min \quad \text{in } u_\varepsilon + \overset{\circ}{W}{}^1_r\big(B_{2R};\mathbb{R}^N\big). \tag{2.1}$$

According to [20, Theorem 5.1] (note that (5.3) of [20] holds on account of $r > q$), we have $\nabla v_{\varepsilon,\delta} \in L^\infty_{\mathrm{loc}}(B_{2R};\mathbb{R}^{nN})$, moreover (see, e.g., [9]) $\nabla v_{\varepsilon,\delta} \in W^1_{2,\mathrm{loc}}(B_{2R};\mathbb{R}^{nN})$. The stated initial regularity of $v_{\varepsilon,\delta}$ is crucial for our calculations in Section 3, therefore we cannot use the local regularization with exponent q as we did in [6] (compare Remark 3.1). Anyhow, Lemma 2.1 below is also true for $r = q$. With the choice

$$\delta = \delta(\varepsilon) = \frac{1}{1 + \varepsilon^{-1} + \|\nabla u_\varepsilon\|^{2r}_{L^r(B_{2R},\mathbb{R}^{nN})}}$$

we define $v_\varepsilon = v_{\varepsilon,\delta(\varepsilon)}$ and $\widetilde{f}_\varepsilon = f_{\delta(\varepsilon)}$. The minimality of v_ε implies

$$\int_{B_{2R}} f(\nabla v_\varepsilon)\,\mathrm{d}x \leqslant \int_{B_{2R}} \widetilde{f}_\varepsilon(\nabla v_\varepsilon)\,\mathrm{d}x \leqslant \int_{B_{2R}} \widetilde{f}_\varepsilon(\nabla u_\varepsilon)\,\mathrm{d}x$$

and from Jensen's inequality we deduce

$$\int_{B_{2R}} f(\nabla u_\varepsilon)\,\mathrm{d}x \leqslant \int_{B_{2R+\varepsilon}} f(\nabla u)\,\mathrm{d}x.$$

Next we claim

$$\delta(\varepsilon)\int_{B_{2R}} \big(1 + |\nabla u_\varepsilon|^2\big)^{r/2}\,\mathrm{d}x \leqslant c(R)\sqrt{\varepsilon} \tag{2.2}$$

with $c(R)$ independent of ε. For the proof we observe that by definition of $\delta(\varepsilon)$ the left-hand side of (2.2) is dominated by

$$c(R)\frac{1 + x}{1 + x^2 + \varepsilon^{-1}}, \quad x := \int_{B_{2R}} |\nabla u_\varepsilon|^r\,\mathrm{d}y.$$

Case 1: If $x \leqslant 1/\sqrt{\varepsilon}$, then

$$\frac{1 + x}{1 + x^2 + \varepsilon^{-1}} \leqslant \frac{1 + (\sqrt{\varepsilon})^{-1}}{1 + x^2 + \varepsilon^{-1}} \leqslant \frac{1 + (\sqrt{\varepsilon})^{-1}}{1 + \varepsilon^{-1}} = \frac{\varepsilon + \sqrt{\varepsilon}}{\varepsilon + 1} \leqslant \varepsilon + \sqrt{\varepsilon} \leqslant 2\sqrt{\varepsilon}.$$

Case 2: Consider the case $x \geqslant 1/\sqrt{\varepsilon}$. Using $1 + x^2 \geqslant \frac{1}{2}(1+x)^2$ we obtain

$$\frac{1+x}{1+x^2+\varepsilon^{-1}} \leqslant \frac{1+x}{1+x^2} \leqslant \frac{2}{1+x} \leqslant \frac{2}{x} \leqslant 2\sqrt{\varepsilon},$$

and (2.2) is established.

Putting together the various estimates we get

$$\int_{B_{2R}} f(\nabla v_\varepsilon)\,\mathrm{d}x \leqslant \int_{B_{2R}} \widetilde{f}_\varepsilon(\nabla v_\varepsilon)\,\mathrm{d}x \leqslant \int_{B_{2R}} \widetilde{f}_\varepsilon(\nabla u_\varepsilon)\,\mathrm{d}x \leqslant \int_{B_{2R}} f(\nabla u)\,\mathrm{d}x + \mathrm{O}(\varepsilon). \tag{2.3}$$

From (2.3) and the growth of f (see Lemma 1.1(v)) we deduce

$$\int_{B_{2R}} |\nabla v_\varepsilon|^s\,\mathrm{d}x \leqslant c\left(1 + \int_{B_{2R}} f(\nabla u)\,\mathrm{d}x\right) < \infty, \tag{2.4}$$

and using $v_\varepsilon - u_\varepsilon \in \overset{\circ}{W}{}^1_s(B_{2R};\mathbb{R}^N)$ together with uniform bounds on $\|u_\varepsilon\|_{W^1_s(B_{2R})}$, (2.4) implies

$$\|v_\varepsilon\|_{W^1_s(B_{2R})} \leqslant \text{const} < \infty \tag{2.5}$$

independent of ε. Let $\widetilde{u} \in W^1_s(B_{2R})$ denote a weak limit of some subsequence of $\{v_\varepsilon\}$ which exists by (2.5). By De Giorgi's theorem

$$\int_{B_{2R}} f(\nabla\widetilde{u})\,\mathrm{d}x \leqslant \liminf_{\varepsilon\to 0} \int_{B_{2R}} f(\nabla v_\varepsilon)\,\mathrm{d}x, \tag{2.6}$$

thus $\int_{B_{2R}} f(\nabla\widetilde{u})\,\mathrm{d}x \leqslant \int_{B_{2R}} f(\nabla u)\,\mathrm{d}x$ on account of (2.3). Therefore $\widetilde{u}$ is J-minimizing on $u + \overset{\circ}{W}{}^1_s(B_{2R};\mathbb{R}^N)$, strict convexity implies $\widetilde{u} = u$. Altogether we have:

Lemma 2.1. *With the notation introduced above the following statements are valid*:

(i) $\|v_\varepsilon\|_{W^1_s(B_{2R};\mathbb{R}^N)} \leqslant \text{const} < \infty$;
(ii) $v_\varepsilon \underset{\varepsilon\to 0}{\rightharpoondown} u$ *in* $W^1_s(B_{2R};\mathbb{R}^N)$;
(iii) $\sup_{B_{2R}}|v_\varepsilon| \leqslant \sup_{B_{2R+\varepsilon}}|u| < \infty$;
(iv) $\delta(\varepsilon)\int_{B_{2R}}(1+|\nabla v_\varepsilon|^2)^{r/2}\,\mathrm{d}x \underset{\varepsilon\to 0}{\longrightarrow} 0$;
(v) $\int_{B_{2R}} f(\nabla v_\varepsilon)\,\mathrm{d}x \underset{\varepsilon\to 0}{\longrightarrow} \int_{B_{2R}} f(\nabla u)\,\mathrm{d}x$;
vi) $\int_{B_{2R}} \widetilde{f}_\varepsilon(\nabla v_\varepsilon)\,\mathrm{d}x \underset{\varepsilon\to 0}{\longrightarrow} \int_{B_{2R}} f(\nabla u)\,\mathrm{d}x$.

Proof. (i) and (ii) are obvious, (iii) follows from the maximum-principle Lemma 1.3. Ad (v) and (vi): we recall $\widetilde{u} = u$ and get from (2.6)

$$\int_{B_{2R}} f(\nabla u)\,\mathrm{d}x \leqslant \liminf_{\varepsilon\to 0} \int_{B_{2R}} f(\nabla v_\varepsilon)\,\mathrm{d}x,$$

whereas (2.3) implies

$$\limsup_{\varepsilon\to 0}\int_{B_{2R}} f(\nabla v_\varepsilon)\,\mathrm{d}x \leqslant \limsup_{\varepsilon\to 0}\int_{B_{2R}} \widetilde{f}_\varepsilon(\nabla v_\varepsilon)\,\mathrm{d}x \leqslant \int_{B_{2R}} f(\nabla u)\,\mathrm{d}x,$$

thus we have (v) and (vi). Subtracting (v) from (vi) we finally obtain (iv). □

3. Uniform local higher integrability of the solutions of the approximative problems

Keeping our notation from Section 2 we want to show that the solutions v_ε of problem (2.1) (with the choice $\delta = \delta(\varepsilon)$) satisfy

$$\sup_{0<\varepsilon<1}\int_{B_\rho} |\nabla v_\varepsilon|^t\,\mathrm{d}x \leqslant c(\rho) < \infty, \tag{3.1}$$

where t is some exponent bigger than q and ρ denotes a radius less than $2R$. In former papers (see, e.g., [16,6,8] and the references quoted therein) we used the differentiated form of the Euler–Lagrange equation associated to (2.1) together with a Caccioppoli-type inequality to show that some power of $|\nabla v_\varepsilon|$ belongs uniformly in ε to the space $W^1_2(B_{2\rho})$, hence by Sobolev's embedding theorem the uniform local integrability of $|\nabla v_\varepsilon|$ can be increased to a power $t > q$ provided $q < s(n+2)/n$ is true.

Here we show (3.1) by assuming that $q < s + 2/3$ which for $s < n/3$ is less restrictive than $q < s(n+2)/n$. The main idea (originating in [4]) is that due to the uniform local boundedness of $|v_\varepsilon|$ (see Lemma 2.1(iii)) we may use test vectors of the form $\eta^2 v_\varepsilon(1+|\nabla v_\varepsilon|^2)^{\alpha/2}$ in the Euler–Lagrange equation for (2.1), where α is some number $\geqslant 0$, η denoting a cut-off function with $\operatorname{spt}\eta \subset B_{2R}$, and estimate $|v_\varepsilon|$ just by a constant. The result will be a uniform bound of the form (3.1). More precisely, we have:

Theorem 3.1. *Let the assumptions of Theorem* 1.1 *hold with* (1.6) *replaced by*

$$q < \max\left\{s\frac{n+2}{n}, s+\frac{2}{3}\right\}. \tag{3.2}$$

Then (3.1) *is true for some* $t > q$, *in particular we have* $\nabla u \in L^t_{\mathrm{loc}}(\Omega;\mathbb{R}^{nN})$.

Proof. Without loss of generality we may assume that $s(n+2)/n \leqslant s+2/3$, otherwise the claim follows from [6, Lemma 3.4] (note that the arguments from [6] also work for the approximation considered here or replace our sequence $\{v_\varepsilon\}$ by the sequence $\{v_\varepsilon\}$ from [6] by the way leading to the same result that ∇u is in $L^t_{\mathrm{loc}}(\Omega;\mathbb{R}^{nN})$). We also like to remark that in the case $s(2+n)/n \geqslant s + 2/3$ the statement of Theorem 3.1 remains true if we drop our assumption $u \in L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, also the structure condition (1.3) can be removed (see again [6]).

Let $\alpha \geqslant 0$, $\eta \in C^\infty_0(B_{2R})$, $0 \leqslant \eta \leqslant 1$, $\Gamma_\varepsilon = 1 + |\nabla v_\varepsilon|^2$ and recall $\nabla v_\varepsilon \in L^\infty_{\mathrm{loc}} \cap W^1_{2,\mathrm{loc}}(\Omega;\mathbb{R}^{nN})$ as well as

$$\|v_\varepsilon\|_{L^\infty(B_{2R})} \leqslant c(R) < \infty. \tag{3.3}$$

In what follows c always denotes a positive constant independent of ε whose value may change from line to line. Moreover, we write $\delta = \delta(\varepsilon)$ for the parameter defined in Section 2. From

$$\int_{B_{2R}} \nabla \widetilde{f}_\varepsilon(\nabla v_\varepsilon) : \nabla\big(\eta^2 \Gamma_\varepsilon^{\alpha/2} v_\varepsilon\big)\,\mathrm{d}x = 0$$

we deduce

$$\begin{aligned}
&\int_{B_{2R}} \nabla \widetilde{f}_\varepsilon(\nabla v_\varepsilon) : \nabla v_\varepsilon \Gamma_\varepsilon^{\alpha/2}\eta^2\,\mathrm{d}x \\
&\quad = -\int_{B_{2R}} \nabla \widetilde{f}_\varepsilon(\nabla v_\varepsilon) : 2\eta\nabla\eta \otimes v_\varepsilon \Gamma_\varepsilon^{\alpha/2}\,\mathrm{d}x - \int_{B_{2R}} \nabla \widetilde{f}_\varepsilon(\nabla v_\varepsilon) : \eta^2\, v_\varepsilon \otimes \nabla\big(\Gamma_\varepsilon^{\alpha/2}\big)\,\mathrm{d}x \\
&\quad =: -A - B.
\end{aligned} \tag{3.4}$$

By Lemma 1.1(iv), we see

$$\begin{aligned}
\text{left-hand side of (3.4)} \geqslant{}& c\int_{B_{2R}} \Gamma_\varepsilon^{(s+\alpha)/2}\eta^2\,\mathrm{d}x + c\delta\int_{B_{2R}} \eta^2\Gamma_\varepsilon^{(r+\alpha)/2}\,\mathrm{d}x - c\int_{B_{2R}} \eta^2\Gamma_\varepsilon^{\alpha/2}\,\mathrm{d}x \\
&- c\delta\int_{B_{2R}} \eta^2\Gamma_\varepsilon^{(r-2+\alpha)/2}\,\mathrm{d}x.
\end{aligned} \tag{3.5}$$

We have by Lemma 1.1(iii)

$$\begin{aligned}
|-A| &\overset{(3.3)}{\leqslant} c\int_{B_{2R}} \big|\nabla \widetilde{f}_\varepsilon(\nabla v_\varepsilon)\big|\eta|\nabla\eta|\Gamma_\varepsilon^{\alpha/2}\,\mathrm{d}x \\
&\leqslant c\int_{B_{2R}} |\nabla v_\varepsilon|^{q-1}\Gamma_\varepsilon^{\alpha/2}\eta|\nabla\eta|\,\mathrm{d}x + c\int_{B_{2R}} \eta|\nabla\eta|\Gamma_\varepsilon^{\alpha/2}\,\mathrm{d}x \\
&\quad + c\delta\int_{B_{2R}} \Gamma_\varepsilon^{\alpha/2}\big(1+|\nabla v_\varepsilon|^2\big)^{(r-2)/2}|\nabla v_\varepsilon|\eta|\nabla\eta|\,\mathrm{d}x \\
&\leqslant c\int_{B_{2R}} \Gamma^{(q-1+\alpha)/2}\eta|\nabla\eta|\,\mathrm{d}x + c\delta\int_{B_{2R}} \Gamma_\varepsilon^{(r-1+\alpha)/2}\eta|\nabla\eta|\,\mathrm{d}x \\
&= c\int_{B_{2R}} \Gamma_\varepsilon^{(s+\alpha)/4}\eta|\nabla\eta|\Gamma_\varepsilon^{(q-1+\alpha)/2-(s+\alpha)/4}\,\mathrm{d}x \\
&\quad + c\delta\int_{B_{2R}} \Gamma_\varepsilon^{(r+\alpha)/4}\eta|\nabla\eta|\Gamma_\varepsilon^{(r-1+\alpha)/2-(r+\alpha)/4}\,\mathrm{d}x,
\end{aligned}$$

and by applying Young's inequality to the last two integrals and by absorbing terms on the right-hand side of (3.5) we arrive at

$$\begin{aligned}
&\int_{B_{2R}} \eta^2\Gamma_\varepsilon^{(s+\alpha)/2}\,\mathrm{d}x + \delta\int_{B_{2R}} \eta^2\Gamma_\varepsilon^{(r+\alpha)/2}\,\mathrm{d}x \\
&\quad \leqslant c\int_{B_{2R}} \eta^2\Gamma_\varepsilon^{\alpha/2}\,\mathrm{d}x + c\delta\int_{B_{2R}} \eta^2\Gamma_\varepsilon^{(r-2+\alpha)/2}\,\mathrm{d}x + c\int_{B_{2R}} |\nabla\eta|^2\Gamma_\varepsilon^{q-1+\alpha-(s+\alpha)/2}\,\mathrm{d}x \\
&\qquad + c\delta\int_{B_{2R}} |\nabla\eta|^2\Gamma_\varepsilon^{r-1+\alpha-(r+\alpha)/2}\,\mathrm{d}x + c|-B|.
\end{aligned} \tag{3.6}$$

Let us discuss $|-B|$: using $|\nabla(\Gamma_\varepsilon^{\alpha/2})| \leqslant c\Gamma_\varepsilon^{\alpha/2-1/2}|\nabla^2 v_\varepsilon|$, Lemma 1.1(iii), and (3.3) we find (summation with respect to $\mu = 1,\dots,n$)

$$\begin{aligned}
|-B| &\leqslant c\int_{B_{2R}} \Gamma_\varepsilon^{(q+\alpha-2)/2}\eta^2|\nabla^2 v_\varepsilon|\,\mathrm{d}x + c\delta\int_{B_{2R}} \Gamma_\varepsilon^{(r+\alpha-2)/2}\eta^2|\nabla^2 v_\varepsilon|\,\mathrm{d}x =: T_1 + T_2,\\
T_1 &= c\int_{B_{2R}} \Gamma_\varepsilon^{(s-2)/4}|\nabla^2 v_\varepsilon|\Gamma_\varepsilon^{(q+\alpha-2)/2-(s-2)/4}\eta^2\,\mathrm{d}x\\
&\overset{(1.2)}{\leqslant} c\int_{B_{2R}} D^2 f(\nabla v_\varepsilon)(\partial_\mu\nabla v_\varepsilon, \partial_\mu\nabla v_\varepsilon)\eta^2\,\mathrm{d}x + c\int_{B_{2R}} \eta^2\Gamma_\varepsilon^{q+\alpha-1-s/2}\,\mathrm{d}x,\\
T_2 &= c\delta\int_{B_{2R}} \Gamma_\varepsilon^{(r-2)/4}|\nabla^2 v_\varepsilon|\Gamma_\varepsilon^{(r+\alpha-2)/2-(r-2)/4}\eta^2\,\mathrm{d}x\\
&\leqslant c\delta\int_{B_{2R}} \Gamma_\varepsilon^{(r-2)/2}|\nabla^2 v_\varepsilon|^2\eta^2\,\mathrm{d}x + c\delta\int_{B_{2R}} \Gamma_\varepsilon^{r/2+\alpha-1}\eta^2\,\mathrm{d}x,
\end{aligned}$$

hence

$$\begin{aligned}
|-B| \leqslant{}& c\int_{B_{2R}} D^2\widetilde{f}_\varepsilon(\nabla v_\varepsilon)(\partial_\mu\nabla v_\varepsilon, \partial_\mu\nabla v_\varepsilon)\eta^2\,\mathrm{d}x\\
&+ c\int_{B_{2R}} \eta^2\Gamma_\varepsilon^{q+\alpha-1-s/2}\,\mathrm{d}x + c\delta\int_{B_{2R}} \Gamma_\varepsilon^{r/2+\alpha-1}\eta^2\,\mathrm{d}x.
\end{aligned}\tag{3.7}$$

From Lemma 3.1 below we get

$$\begin{aligned}
\int_{B_{2R}} \eta^2 D^2\widetilde{f}_\varepsilon(\nabla v_\varepsilon)(\partial_\mu\nabla v_\varepsilon, \partial_\mu\nabla v_\varepsilon)\,\mathrm{d}x &\leqslant c\|\nabla\eta\|^2_{L^\infty(B_{2R})}\int_{\operatorname{spt}\nabla\eta} |D^2\widetilde{f}_\varepsilon(\nabla v_\varepsilon)||\nabla v_\varepsilon|^2\,\mathrm{d}x\\
&\leqslant c\|\nabla\eta\|^2_{L^\infty(B_{2R})}\int_{\operatorname{spt}\nabla\eta}\left(\Gamma_\varepsilon^{q/2} + \delta\Gamma_\varepsilon^{r/2}\right)\mathrm{d}x.
\end{aligned}$$

Now we use (3.7) and the latter estimate to rewrite (3.6) in the following form:

$$\begin{aligned}
I_1 + I_2 :={}& \int_{B_{2R}} \eta^2\Gamma_\varepsilon^{(s+\alpha)/2}\,\mathrm{d}x + \delta\int_{B_{2R}} \eta^2\Gamma_\varepsilon^{(r+\alpha)/2}\,\mathrm{d}x\\
\leqslant{}& c\int_{B_{2R}} \eta^2\Gamma_\varepsilon^{\alpha/2}\,\mathrm{d}x + c\|\nabla\eta\|^2_\infty\int_{\operatorname{spt}\nabla\eta} \Gamma_\varepsilon^{q-1+\alpha-(s+\alpha)/2}\,\mathrm{d}x + c\int_{B_{2R}} \eta^2\Gamma_\varepsilon^{q+\alpha-1-s/2}\,\mathrm{d}x\\
&+ c\|\nabla\eta\|^2_\infty\int_{\operatorname{spt}\nabla\eta} \Gamma_\varepsilon^{q/2}\,\mathrm{d}x + c\delta\int_{B_{2R}} \eta^2\Gamma_\varepsilon^{(r-2+\alpha)/2}\,\mathrm{d}x\\
&+ c\delta\|\nabla\eta\|^2_\infty\int_{\operatorname{spt}\nabla\eta} \Gamma_\varepsilon^{r/2-1+\alpha/2}\,\mathrm{d}x + c\delta\int_{B_{2R}} \Gamma_\varepsilon^{r/2+\alpha-1}\eta^2\,\mathrm{d}x + c\delta\|\nabla\eta\|^2_\infty\int_{\operatorname{spt}\nabla\eta} \Gamma_\varepsilon^{r/2}\,\mathrm{d}x\\
=:{}& \sum_{i=1}^{8} K_i.
\end{aligned}\tag{3.8}$$

It is immediate that K_5 is bounded by K_7. Obviously $K_1 \leqslant \tau I_1 + c(\tau, R)$ for any $\tau > 0$, and the first term on the right-hand side can be absorbed into I_1. Let us assume $\alpha < 2$. Then $K_6 \leqslant K_8$ and

$$K_7 \leqslant c\tau I_2 + c(\tau, R)$$

on account of $\frac{1}{2}(r+2\alpha-2) < \frac{1}{2}(r+\alpha)$. Choosing τ small enough, we may absorb $c\tau I_2$ into I_2. Next we choose a ball $B_{t+\rho}(x_0) \Subset B_{2R}$ and take $\eta \equiv 1$ on $B_t(x_0)$ such that $\eta \equiv 0$ on $B_{2R} - B_{t+\rho}(x_0)$ together with $|\nabla\eta| \leqslant c/\rho$. Finally, we like to control K_2 in terms of K_4 which is possible if

$$q - 1 + \alpha - \frac{s+\alpha}{2} \leqslant \frac{q}{2},$$

i.e., we have to require at this stage

$$\alpha \leqslant 2 + s - q \tag{3.9}$$

(implying $\alpha < 2$ on account of $s < q$). Returning to (3.8) and exploiting the latter considerations we get

$$\begin{aligned}
&\int_{B_t(x_0)} \Gamma_\varepsilon^{(s+\alpha)/2}\,\mathrm{d}x + \delta \int_{B_t(x_0)} \Gamma_\varepsilon^{(r+\alpha)/2}\,\mathrm{d}x \\
&\quad \leqslant c(R)\left[1 + \int_{B_{t+\rho}(x_0)} \Gamma_\varepsilon^{(2q+2\alpha-2-s)/2}\,\mathrm{d}x + \rho^{-2}\int_{T_{t,\rho}(x_0)} \Gamma_\varepsilon^{q/2}\,\mathrm{d}x + \delta\rho^{-2}\int_{T_{t,\rho}(x_0)} \Gamma_\varepsilon^{r/2}\,\mathrm{d}x\right], \\
&T_{t,\rho}(x_0) = B_{t+\rho}(x_0) - B_t(x_0).
\end{aligned} \tag{3.10}$$

Let $\alpha_0 = 0$, $\alpha_{k+1} = -q+s+1+\frac{1}{2}\alpha_k$. Since our assumption on the ratio of q and s implies $q < s+1$, we see $\alpha_k > 0$ for $k \geqslant 1$. It is easy to check that $\alpha_k < \alpha_{k+1}$, moreover (3.9) is satisfied for the sequence α_k. We have $\alpha_\infty := \lim_{k\to\infty}\alpha_k = 2+2(-q+s)$, and if we want to have $s+\alpha_\infty > q$ we need our hypothesis $q < s + \frac{2}{3}$.

Next we fix $q^* \in (q, s+\alpha_\infty)$ and calculate $k \in \mathbb{N}$ such that $\alpha_k + s \geqslant q^*$. Given radii t, ρ and a center x_0 we apply (3.10) with $\alpha = \alpha_k$ and ρ replaced by ρ/k and get

$$\begin{aligned}
&\int_{B_t(x_0)} \Gamma_\varepsilon^{(s+\alpha_k)/2}\,\mathrm{d}x + \delta \int_{B_t(x_0)} \Gamma_\varepsilon^{(r+\alpha_k)/2}\,\mathrm{d}x \\
&\quad \leqslant c(k,R)\left[1 + \int_{B_{t+\rho/k}(x_0)} \Gamma_\varepsilon^{(s+\alpha_{k-1})/2}\,\mathrm{d}x + \rho^{-2}\int_{T_{t,\rho/k}(x_0)} \Gamma^{q/2}\,\mathrm{d}x + \delta\rho^{-2}\int_{T_{t,\rho/k}(x_0)} \Gamma_\varepsilon^{r/2}\,\mathrm{d}x\right].
\end{aligned}$$

In the next step we use (3.10) with t replaced by $t+\rho/k$ and ρ replaced by ρ/k and for the choice $\alpha = \alpha_{k-1}$. The final result is after iteration

$$\begin{aligned}
&\int_{B_t(x_0)} \Gamma_\varepsilon^{(s+\alpha_k)/2}\,\mathrm{d}x + \delta \int_{B_t(x_0)} \Gamma_\varepsilon^{(r+\alpha_k)/2}\,\mathrm{d}x \\
&\quad \leqslant c(k,R)\left[1 + \int_{B_{t+\rho}(x_0)} \Gamma_\varepsilon^{s/2}\,\mathrm{d}x + \rho^{-2}\int_{T_{t,\rho}(x_0)} \Gamma^{q/2}\,\mathrm{d}x + \delta\rho^{-2}\int_{T_{t,\rho}(x_0)} \Gamma_\varepsilon^{r/2}\,\mathrm{d}x\right].
\end{aligned} \tag{3.11}$$

In the last two integrals we may use Hölder's inequality to get

$$\rho^{-2}\int_{T_{t,\rho}(x_0)} \Gamma_\varepsilon^{q/2}\,\mathrm{d}x \leqslant c(R)\Big[\rho^{-\gamma}+\int_{T_{t,\rho}(x_0)} \Gamma^{(s+\alpha_k)/2}\,\mathrm{d}x\Big],$$
$$\rho^{-2}\int_{T_{t,\rho}(x_0)} \Gamma_\varepsilon^{r/2}\,\mathrm{d}x \leqslant c(R)\Big[\rho^{-\widetilde{\gamma}}+\int_{T_{t,\rho}(x_0)} \Gamma^{(r+\alpha_k)/2}\,\mathrm{d}x\Big]$$

with suitable positive exponents $\gamma, \widetilde{\gamma}$. By Lemma 2.1 the quantity $\int_{B_{t+\rho}(x_0)} \Gamma^{s/2}\,\mathrm{d}x$ is bounded by a local constant $c(R)$, therefore, filling the hole in (3.11) (add

$$c(k,R)\Big(\int_{B_t(x_0)} \Gamma_\varepsilon^{(s+\alpha_k)/2}\,\mathrm{d}x+\delta\int_{B_t(x_0)} \Gamma_\varepsilon^{(r+\alpha_k)/2}\,\mathrm{d}x\Big)$$

on both sides) implies (for some $\beta>0$)

$$\int_{B_t(x_0)} \Gamma_\varepsilon^{(s+\alpha_k)/2}\,\mathrm{d}x+\delta\int_{B_t(x_0)} \Gamma_\varepsilon^{(r+\alpha_k)/2}\,\mathrm{d}x$$
$$\leqslant c(k,R)\big[1+\rho^{-\beta}\big]+\Theta\Big[\int_{B_{t+\rho}(x_0)} \Gamma_\varepsilon^{(s+\alpha_k)/2}\,\mathrm{d}x+\delta\int_{B_{t+\rho}(x_0)} \Gamma_\varepsilon^{(r+\alpha_k)/2}\,\mathrm{d}x\Big]$$

with $\Theta<1$ not depending on ε. From [19, Lemma 3.1, p. 161] we get ($B_R=B_R(0)$)

$$\int_{B_R} \Gamma_\varepsilon^{(s+\alpha_k)/2}\,\mathrm{d}x+\delta\int_{B_R} \Gamma_\varepsilon^{(r+\alpha_k)/2}\,\mathrm{d}x \leqslant c(k,R)\big[1+R^{-\beta}\big], \tag{3.12}$$

where the local constant $c(k,R)$ involves positive powers of R and the bounds for $\sup_{B_{2R+\varepsilon}}|u|$, $\int_{B_{2R}} f(\nabla u)\,\mathrm{d}x$. Recalling the choice of k we have shown (3.1) for the exponent $t=q^*$.

Note that our calculations just required $r>q$, no further restriction on r is needed. Having established (3.1) we get $\nabla u\in L^{q^*}_{\mathrm{loc}}(B_{2R};\mathbb{R}^{nN})$ since $\nabla v_\varepsilon \rightharpoondown: W$ in $L^{q^*}_{\mathrm{loc}}(B_{2R};\mathbb{R}^{nN})$ but on account of Lemma 2.1(ii) we must have $W=\nabla u$. □

During the proof of Theorem 3.1 we made use of

Lemma 3.1. *There is a real number $c>0$ such that for all $\eta\in C_0^1(B_{2R})$, $0\leqslant\eta\leqslant 1$, and for all $Q\in\mathbb{R}^{nN}$ we have*

$$\int_{B_{2R}} \eta^2 D^2\widetilde{f}_\varepsilon(\nabla v_\varepsilon)(\partial_\mu\nabla v_\varepsilon,\partial_\mu\nabla v_\varepsilon)\,\mathrm{d}x \leqslant c\|\nabla\eta\|_\infty^2\int_{\mathrm{spt}\,\nabla\eta} \big|D^2\widetilde{f}_\varepsilon(\nabla v_\varepsilon)\big||\nabla v_\varepsilon-Q|^2\,\mathrm{d}x. \tag{3.13}$$

Proof. Here we just need to know that $f\geqslant 0$ is C^2 with $0\leqslant D^2f(Q)\leqslant \Lambda(1+|Q|^2)^{(q-2)/2}$ for some $q\geqslant 2$. Then, if $\widetilde{f}_\varepsilon$ is our regularization with exponent $r>q$, v_ε is still of class $W^1_{\infty,\mathrm{loc}}\cap W^2_{2,\mathrm{loc}}(B_{2R};\mathbb{R}^N)$ (see [20]) and we may differentiate the Euler equation for v_ε with the result

$$\int_{B_{2R}} D^2\widetilde{f}_\varepsilon(\nabla v_\varepsilon)(\partial_\mu\nabla v_\varepsilon,\nabla\varphi)\,\mathrm{d}x=0$$

for any φ with compact support. If we let $\varphi = \eta^2 \partial_\mu (v_\varepsilon - Qx)$, then (3.13) follows with elementary calculations. $\square$

Remark 3.1. In contrast to our arguments from above the proof of Lemma 3.1 in [6] requires much more work which is caused by the regularization with exponent $r = q$. On the other hand, higher integrability of ∇u is established in [6] under stronger hypotheses and with a completely different argument working in case $r = q$ once having proved Lemma 3.1 of [6]. Of course we could also choose some exponent $r > q$ in [6] trivializing Lemma 3.1. But then we have to take care of additional δ-terms occurring in the proof of [6, Lemma 3.4] which can be handled under the assumption $r < s(n+2)/n$. Recalling that in [6] we require for the anisotropic case $\mu = 2 - s$, $q < 2 - \mu + s2/n$, it is immediate that we can choose an admissible exponent $r \in (q, s(n+2)/n)$.

If we require f to satisfy (1.2), then the left-hand side of (3.13) gives an upper bound for

$$\int_{B_{2R}} \eta^2 |\nabla h_\varepsilon|^2 \, \mathrm{d}x, \quad h_\varepsilon = \left(1 + |\nabla v_\varepsilon|^2\right)^{s/4},$$

and if we take $Q = 0$, then (3.13) implies

$$\int_{B_{2R}} \eta^2 |\nabla h_\varepsilon|^2 \, \mathrm{d}x \leqslant c \|\nabla \eta\|_\infty^2 \int_{\operatorname{spt} \nabla \eta} \left(\Gamma_\varepsilon^{q/2} + \delta \Gamma_\varepsilon^{r/2}\right) \mathrm{d}x.$$

If we are in the situation of Theorem 3.1 we see from (3.12) that the right-hand side of the latter estimate is bounded by a local constant $c(R)$, thus

$$h_\varepsilon \rightharpoondown: h \quad \text{in } W^1_{2,\mathrm{loc}}(B_{2R}) \text{ as } \varepsilon \to 0$$

for some function h from this space.

The next lemma can be found for instance in [17,5,6].

Lemma 3.2. *We have* $h = (1 + |\nabla u|^2)^{s/4}$ *as well as*

$$\nabla v_\varepsilon \to \nabla u \quad \text{a.e. on } B_R \text{ as } \varepsilon \to 0.$$

Proof. As in [17, Lemma 4.1] or [5, Proposition III.4.3] we show that

$$\int_{B_R} \int_0^1 D^2 f\left(\nabla u + t(\nabla v_\varepsilon - \nabla u)\right)\left(\nabla v_\varepsilon - \nabla u, \nabla v_\varepsilon - \nabla u\right)(1-t) \, \mathrm{d}t \, \mathrm{d}x \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.14}$$

With (3.14), ellipticity implies $\nabla v_\varepsilon \to \nabla u$ a.e., in particular, $h_\varepsilon \to (1 + |\nabla u|^2)^{s/4}$ a.e., so that we have the formula for the limit function h. $\square$

Finally, we state a limit version of Lemma 3.1.

Lemma 3.3. *With the notation introduced above we have*

$$\int_{B_{2R}} \eta^2 |\nabla h|^2 \, dx \leqslant c \|\nabla \eta\|_\infty^2 \int_{\operatorname{spt} \nabla\eta} \left|D^2 f(\nabla u)\right| |\nabla u - Q|^2 \, dx \tag{3.15}$$

for any $\eta \in C_0^1(B_{2R})$, $0 \leqslant \eta \leqslant 1$, *and all* $Q \in \mathbb{R}^{nN}$.

Proof. We can follow [6, Lemma 3.6], we just have to check (see [6, formula (3.8)]) that in our case

$$\liminf_{\varepsilon \to 0} \int_{\operatorname{spt} \nabla\eta} \left|D^2 \widetilde{f}_\varepsilon(\nabla v_\varepsilon)\right| |\nabla v_\varepsilon - Q|^2 \, dx \leqslant \liminf_{\varepsilon \to 0} \int_{\operatorname{spt} \nabla\eta} \left|D^2 f(\nabla v_\varepsilon)\right| |\nabla v_\varepsilon - Q|^2 \, dx.$$

But since

$$\left|D^2 \widetilde{f}_\varepsilon(\nabla v_\varepsilon)\right| |\nabla v_\varepsilon - Q|^2 \leqslant \left|D^2 f(\nabla v_\varepsilon)\right| |\nabla v_\varepsilon - Q|^2 + c\delta(\varepsilon)\left(1 + |\nabla v_\varepsilon|^2\right)^{(r-2)/2} |\nabla v_\varepsilon - Q|^2,$$

the claim follows from our former observation that $\delta(\varepsilon) \int_{B_{R'}} \Gamma_\varepsilon^{r/2} \, dx \to 0$ for any radius $R' < 2R$. The rest of the proof is exactly the same as in Lemma 3.6 for [6], in particular we also make use of the fact that by (3.12) $|\nabla v_\varepsilon|^q$ is uniformly bounded in $L^{1+\tau}_{\mathrm{loc}}(B_{2R})$ for some $\tau > 0$ (compare (3.12) where obviously the radius R can be replaced by any number $R' < 2R$). $\square$

Remark 3.2. Note that u is a continuous function provided that $n = 2$ or $n = 3$. In the two-dimensional case this is a consequence of Theorem 3.1. In general, we observe that $h \in W^1_{2,\mathrm{loc}}(\Omega)$ implies $\nabla u \in L^{sn/(n-2)}_{\mathrm{loc}}(\Omega; \mathbb{R}^{nN})$ by Sobolev's embedding theorem, and continuity of u follows if $s > n - 2$.

4. Blow-up and partial regularity

Let the assumptions of Theorem 1.1 hold. If $B_r(x)$ is a ball in Ω, we introduce the excess of our local minimizer u with respect to this ball

$$E(x, r) = \fint_{B_r(x)} \left|\nabla u - (\nabla u)_{x,r}\right|^2 dy + \fint_{B_r(x)} \left|\nabla u - (\nabla u)_{x,r}\right|^q dy$$

which on account of Theorem 3.1 is well-defined. Here $\fint_{B_r(x)}$ and $(\cdot)_{x,r}$ denote the mean values of the corresponding quantities. As usual partial $C^{1,\alpha}$-regularity follows from

Lemma 4.1. *Fix a number* $L > 0$. *Then there is a constant* $C_*(L)$ *such that for every* $0 < \tau < \frac{1}{4}$ *there is an* $\varepsilon = \varepsilon(L, \tau)$ *satisfying*: *if* $B_r(x) \subset B_R(x_0)$ *for some fixed ball* $B_R(x_0) \Subset \Omega$ *and if we have*

$$\left|(\nabla u)_{x,r}\right| \leqslant L, \qquad E(x, r) \leqslant \varepsilon(L, \tau),$$

then

$$E(x, \tau r) \leqslant C_*(L) \tau^2 E(x, r).$$

Here the ball $B_R(x_0)$ *can be replaced by any subdomain* $\Omega' \Subset \Omega$, *and the restriction* $B_r(x) \subset \Omega'$ *is needed in order to apply our local estimate from Theorem* 3.1.

We argue by contradiction: assume that $L > 0$ is fixed and that for some $\tau \in (0, \frac{1}{4})$ there are balls $B_{r_m}(x_m) \subset B_R(x_0)$ such that

$$\left|(\nabla u)_{x_m,r_m}\right| \leqslant L, \qquad E(x_m, r_m) =: \lambda_m^2 \to 0$$

but

$$E(x_m, \tau r_m) > C_* \tau^2 \lambda_m^2$$

with C_* to be determined later. With $a_m = (u)_{x_m,r_m}$, $A_m = (\nabla u)_{x_m,r_m}$ we let

$$u_m(z) = \frac{1}{\lambda_m r_m}\left[u(x_m + r_m z) - a_m - r_m A_m z\right], \quad |z| < 1.$$

From our assumptions we get

$$\begin{aligned} &\fint_{B_1} |\nabla u_m|^2 \,\mathrm{d}z + \lambda_m^{q-2} \fint_{B_1} |\nabla u_m|^q \,\mathrm{d}z = 1, \\ &\fint_{B_\tau} \left|\nabla u_m - (\nabla u_m)_{0,\tau}\right|^2 \mathrm{d}z + \lambda_m^{q-2} \fint_{B_\tau} \left|\nabla u_m - (\nabla u_m)_{0,\tau}\right|^q \mathrm{d}z > C_* \tau^2, \end{aligned} \tag{4.1}$$

and after passing to subsequences we find

$$\begin{cases} A_m \to: A & \text{in } \mathbb{R}^{nN}, \\ u_m \rightharpoondown: \hat{u} & \text{in } W_2^1(B_1; \mathbb{R}^N), \\ \lambda_m \nabla u_m \to 0 & \text{in } L^2(B_1; \mathbb{R}^{nN}) \text{ and a.e.}, \\ \lambda_m^{1-2/q} \nabla u_m \rightharpoondown 0 & \text{in } L^q(B_1; \mathbb{R}^{nN}). \end{cases} \tag{4.2}$$

Following [14] or [6, Proposition 4.2] (and [5, Proposition III.4.7]) we see that $\hat{u}$ satisfies

$$\int_{B_1} D^2 f(A)(\nabla \hat{u}, \nabla \varphi) \,\mathrm{d}x = 0 \quad \forall \varphi \in C_0^1(B_1; \mathbb{R}^N)$$

and since this linear system is elliptic, we have the Campanato inequality

$$\fint_{B_\tau} \left|\nabla \hat{u} - (\nabla \hat{u})_{0,\tau}\right|^2 \mathrm{d}x \leqslant C^* \tau^2 \tag{4.3}$$

for some absolute constant C^*. Let $C_* = 2C^*$. Clearly (4.3) is in contradiction to (4.1) if we can improve the convergences stated in (4.2) to

$$\begin{cases} \nabla u_m \to \nabla \hat{u} & \text{in } L^2_{\mathrm{loc}}(B_1; \mathbb{R}^{nN}), \\ \lambda_m^{1-2/q} \nabla u_m \to 0 & \text{in } L^q_{\mathrm{loc}}(B_1; \mathbb{R}^{nN}). \end{cases} \tag{4.4}$$

The first statement in (4.4) follows from Proposition 4.3 in [6], “case $q \geqslant 2$, $\mu = 2 - s$”, for the second one we follow Proposition 4.5(ii) from [6], where the case $\mu = 2 - s \leqslant 0$ is the relevant one. With

$$\Psi_m = \lambda_m^{-1}\left[\left(1 + |A_m + \lambda_m \nabla u_m|^2\right)^{s/4} - \left(1 + |A_m|^2\right)^{s/4}\right]$$

(compare formula (4.18) in [6]) Lemma 3.3 implies

$$\sup_m \|\Psi_m\|_{W_2^1(B_\rho)} \leqslant c(\rho) < \infty \tag{4.5}$$

for any $\rho < 1$, thus we can follow the arguments presented in [6] after (4.20) to see that in case $n \geqslant 3$ (4.5) implies (4.4) provided $2q/s < 2n/(n-2)$, i.e., $q < sn/(n-2)$. If $n = 2$, no further restriction is needed. $\square$

We finish this section by adding some comments on the two-dimensional case.

Proposition 4.1. *Let $n = 2$ and consider an integrand $f : \mathbb{R}^{2N} \to [0, \infty)$ of class C^2 just satisfying* (1.2) *with exponents $2 \leqslant s < q$ such that $q < 2s$. Consider a local minimizer $u \in W^1_{s,\mathrm{loc}}(\Omega; \mathbb{R}^N)$. Then there is an open subset Ω_0 of Ω, whose complement is of Hausdorff-dimension zero, such that $u \in C^{1,\alpha}(\Omega_0; \mathbb{R}^N)$ for any $0 < \alpha < 1$.*

Remark 4.1. For the definition of the Hausdorff-dimension we refer to [3]. In fact we have $\Omega_0 = \Omega$ but the proof requires different arguments which are presented in the paper [7].

Proof. According to Theorem 1.2 partial regularity holds in our situation, and, by Lemma 4.1, a point x_0 belongs to Ω_0 if and only if

$$\limsup_{r \to 0} \left|(\nabla u)_{x_0,r}\right| < \infty \quad \text{and} \quad E(x_0, r) \underset{r \to 0}{\longrightarrow} 0.$$

We recall inequality (3.13) from Lemma 3.1 in which we choose $Q = 0$. Using (1.2) we deduce

$$\begin{aligned}&\int_{B_{2R}} \eta^2 \left(1 + |\nabla v_\varepsilon|^2\right)^{(s-2)/2} \left|\nabla^2 v_\varepsilon\right|^2 \mathrm{d}x \\ &\quad \leqslant c\|\nabla \eta\|_\infty^2 \left\{\int_{\mathrm{spt}\,\nabla\eta} \delta(\varepsilon)\left(1 + |\nabla v_\varepsilon|^2\right)^{r/2} \mathrm{d}x + \int_{\mathrm{spt}\,\nabla\eta} \left|D^2 f(\nabla v_\varepsilon)\right| |\nabla v_\varepsilon|^2 \mathrm{d}x\right\},\end{aligned}$$

and by Lemma 2.1 together with our L^q-estimates from Theorem 3.1 we see

$$\int_{B_{2R}} \eta^2 \left(1 + |\nabla v_\varepsilon|^2\right)^{(s-2)/2} \left|\nabla^2 v_\varepsilon\right|^2 \mathrm{d}x \leqslant c(\eta) < \infty,$$

and $s \geqslant 2$ implies a uniform local bound for $\nabla^2 v_\varepsilon$ in L^2. Lemma 2.1(ii), immediately gives $u \in W^2_{2,\mathrm{loc}}(B_{2R}; \mathbb{R}^N)$, and since B_{2R} was arbitrary, we have $u \in W^2_{2,\mathrm{loc}}(\Omega; \mathbb{R}^N)$. Finally, we recall $n = 2$ and apply the Sobolev–Poincaré inequality to see that $E(x, r) \to 0$ as $r \to 0$ for any point $x \in \Omega$. Thus $x \in \Omega$ is a singular point if and only if $\limsup_{r \to 0} |(\nabla u)_{x,r}| = \infty$. But according to [19, IV, Theorem 2.1] these points form a subset of Hausdorff-dimension zero. $\square$

5. The subquadratic case

Up to now our considerations covered the case $2 \leqslant s < q$, next we analyze the situation for arbitrary exponents $1 < s < q$ and sketch the necessary adjustments which actually reduce to some formal remarks. First of all we observe that Lemma 1.1(iv), is also true for exponents $s \in (1,2)$. By Lemma 2.1 of [1] we obtain the upper bound Lemma 1.1(ii), for exponents $q < 2$. Neither the definition of local minimizers nor the results of Lemmas 1.2 and 1.3 are affected by the choices for s and q.

Ad Section 2. If $q \geqslant 2$ and $1 < s < q$, then f_δ is defined as before with any exponent $r > q$. Again we have

$$\nabla v_{\varepsilon,\delta} \in W^1_{2,\mathrm{loc}}\big(\Omega;\mathbb{R}^{nN}\big) \cap L^\infty_{\mathrm{loc}}. \tag{5.1}$$

If $q < 2$, then we may choose $r = 2$ still getting (5.1) on account of [20, Theorem 5.1] (with the choice $m = 2$). Clearly, Lemma 2.1 does not depend on the choices for s and q.

Ad Section 3. Due to (5.1) we still have the identity

$$\int_{B_{2R}} \nabla \widetilde{f}_\varepsilon(\nabla v_\varepsilon) : \nabla\big(\eta^2 \Gamma_\varepsilon^{\alpha/2} v_\varepsilon\big)\,\mathrm{d}x = 0, \quad \forall \alpha \geqslant 0,$$

and with the same calculations as before we obtain Theorem 3.1. Lemma 3.1 requires no changes, and for Lemmas 3.2 and 3.3 we can either quote the proofs of Proposition 3.5 and Lemma 3.6 of [6] with $\mu := 2 - s$ or – for a more detailed exposition – the proofs of [5, Proposition III.4.3 and Lemma III.4.4] (letting $\mu = 2 - s$). Again the information $\delta(\varepsilon)\int_{B_R} \Gamma_\varepsilon^{r/2}\,\mathrm{d}x \to 0$ as $\varepsilon \to 0$ is needed which is contained in Lemma 2.1.

Ad Section 4. The corresponding version of Lemma 3.1 can be found in [6, Lemma 4.1] where in case $q < 2$ the excess function $E(x,r)$ takes a form different to the one considered here. Anyhow, the general situation $1 < s < q$ is completely discussed in [6, Section 4] leading to the condition $q < sn/(n-2)$, if $n \geqslant 3$, which is sufficient for the blow-up procedure.

Appendix A

Let $f(\xi_1,\ldots,\xi_n) = (1 + |\xi|^2 + h(\xi_1))^{s/2}$ with $h(\xi_1) = (1 + |\xi_1|^2)^{q/s}$ for exponents $2 \leqslant s < q$. We have

$$\begin{aligned} D^2 f(\xi)(U,U) = {} & \frac{s}{2}\big[1 + |\xi|^2 + h(\xi_1)\big]^{(s-2)/2}\big\{2|U|^2 + D^2 h(\xi_1)(U_1,U_1)\big\} \\ & + \frac{s}{2}\frac{s-2}{2}[\ldots]^{s/2-2}\big\{2\xi : U + \nabla h(\xi_1)\cdot U_1\big\}^2, \end{aligned} \tag{A.1}$$

and since the second term on the right-hand side of (A.1) is $\geqslant 0$, we deduce on account of $D^2 h(\xi_1)(U_1,U_1) \geqslant 0$ the estimate

$$s|U|^2\big(1 + |\xi|^2\big)^{(s-2)/2} \leqslant D^2 f(\xi)(U,U). \tag{A.2}$$

In (A.2) we cannot replace $(s-2)/2$ by a larger exponent which is seen by choosing $\xi_1 = U_1 = 0$ in (A.1) and by considering large values of $|\xi|$. In this special case $|D^2 f(\xi)|$ can be controlled from above and from below by $|\xi|^{s-2}$. In order to obtain an upper bound for $D^2 f(\xi)(U,U)$ we observe

$$|\nabla h(\xi_1)| \leqslant c\left(1+|\xi_1|^2\right)^{q/s-1/2}, \qquad |D^2 h(\xi_1)| \leqslant c\left(1+|\xi_1|^2\right)^{q/s-1},$$

where c denotes various positive constants depending on the different parameters. From (A.1) we deduce:

$$|D^2 f(\xi)| \leqslant c\left[1+|\xi|^2+\left(1+|\xi_1|^2\right)^{q/s}\right]^{(s-2)/2}\left\{1+\left(1+|\xi_1|^2\right)^{q/s-1}\right\} + c[\ldots]^{s/2-2}\left\{|\xi|+\left(1+|\xi_1|^2\right)^{q/s-1/2}\right\}^2. \tag{A.3}$$

Since $s \geqslant 2$, the first term on the right-hand side of (A.3) is bounded from above by

$$c\left(1+|\xi|^2\right)^{(q/s)(s-2)/2}\left(1+|\xi|^2\right)^{q/s-1} = c\left(1+|\xi|^2\right)^{(q-2)/2}$$

which has the desired growth.

Next we discuss the second term by first observing

$$[\ldots]^{s/2-2}\left\{|\xi|+\left(1+|\xi_1|^2\right)^{q/s-1/2}\right\}^2 \leqslant c[\ldots]^{s/2-2}\left\{1+|\xi|^2+\left(1+|\xi_1|^2\right)^{2q/s-1}\right\} =: c\alpha\beta.$$

Case 1: In the case $s/2 \geqslant 2$ we have

$$\alpha\beta \leqslant c\left(1+|\xi|^2\right)^{(q/s)(s/2-2)}\left(1+|\xi|^2\right)^{2q/s-1} = c\left(1+|\xi|^2\right)^{(q-2)/2}.$$

Case 2: If $s/2 < 2$ then let us first assume in addition that

(a) $1+|\xi|^2 \leqslant (1+|\xi_1|^2)^{2q/s-1}$.

Dropping $1+|\xi|^2$ in α, we get

$$\alpha \leqslant \left(1+|\xi_1|^2\right)^{(q/s)(s/2-2)}, \qquad \beta \leqslant c\left(1+|\xi_1|^2\right)^{2q/s-1},$$

hence

$$\alpha\beta \leqslant c\left(1+|\xi_1|^2\right)^{q/2-1} \leqslant c\left(1+|\xi|^2\right)^{(q-2)/2}.$$

Next let

(b) $1+|\xi|^2 \geqslant (1+|\xi|^2)^{2q/s-1}$.

Dropping $(1+|\xi_1|^2)^{q/s}$ in α, we get

$$\alpha \leqslant \left(1+|\xi|^2\right)^{s/2-2}.$$

For β we use our assumption (b) and get

$$\beta \leqslant c\left(1+|\xi|^2\right),$$

thus

$$\alpha\beta \leqslant c\big(1+|\xi|^2\big)^{s/2-1} \leqslant c\big(1+|\xi|^2\big)^{(q-2)/2}.$$

Altogether we obtain inequality (1.2) for the above example. It should be noted that the exponent $(q-2)/2$ occurring on the right-hand side is optimal.

Remark A.1. With similar arguments we can show the validity of (1.2) for the first example from Remark 1.1(d). Moreover, it is an easy exercise to check (1.2) if we modify the first example in Remark 1.1(d) by letting

$$f(\xi) = \left(|\xi|^2 + \sum_{\alpha\in I}\big(1+|\xi_\alpha|^2\big)^{q_\alpha/s}\right)^{s/2}$$

for some index set $\emptyset \neq I \subset \{1,\dots,n\}$ and exponents $2 \leqslant s < q_\alpha$, $q := \max\{q_\alpha\colon\ \alpha \in I\}$.

Appendix B

Here we give a proof of

Lemma B.1. *Let $v \in W^1_t(\Omega;\mathbb{R}^N)$ for some $1 \leqslant t < \infty$ and consider a Lipschitz function $\Phi:\mathbb{R}^N \to \mathbb{R}^k$. Then $\Phi\circ v$ is in the space $W^1_t(\Omega;\mathbb{R}^k)$ together with*

$$\big|\partial_\alpha(\Phi\circ v)\big| \leqslant \mathrm{Lip}(\Phi)\big|\partial_\alpha v\big|, \quad \alpha = 1,\dots,n. \tag{B.1}$$

Remark B.1. In the paper [24] the following weaker version of (B.1) is established (assuming $t > 1$)

$$\big|\nabla(\Phi\circ v)\big| \leqslant \sqrt{k}\,\mathrm{Lip}(\Phi)|\nabla v|, \tag{B.2}$$

but (B.2) does not imply the statement of Lemma 1.3.

Proof of Lemma B.1. From [26, Theorem 3.1.9] we get $\Phi\circ v \in W^1_t(\Omega;\mathbb{R}^k)$ and

$$\partial_\alpha(\Phi\circ v) = D\Phi(v)(\partial_\alpha v) \tag{B.3}$$

for any $v \in W^1_t(\Omega;\mathbb{R}^N)$ provided the Lipschitz function is in addition of class C^1. From (B.3) we deduce the estimate

$$\big|\nabla(\Phi\circ v)\big| \leqslant \|D\Phi\|_\infty|\nabla v|. \tag{B.4}$$

Clearly (B.4) is weaker than (B.1) since $\|D\Phi\|_\infty$ is of order $\sqrt{k}\,\mathrm{Lip}(\Phi)$.

Let us first consider the case $t > 1$. Given $v \in W^1_t(\Omega;\mathbb{R}^N)$ we also suppose that Φ is a C^1 Lipschitz function and choose a sequence $v_m \in C^\infty(\overline{\Omega};\mathbb{R}^N)$ such that

$$v_m \to v \quad \text{in } W^1_t(\Omega;\mathbb{R}^N) \text{ and a.e. on } \Omega.$$

We have ($1 \leqslant \alpha \leqslant n$)

$$\partial_\alpha(\Phi \circ v_m)(x) = \lim_{t\to 0} \frac{1}{t}\big(\Phi\big(v_m(x+te_\alpha)\big) - \Phi\big(v_m(x)\big)\big),$$

thus

$$\big|\partial_\alpha(\Phi \circ v_m)\big| \leqslant \mathrm{Lip}(\Phi)|\partial_\alpha v_m|. \tag{B.5}$$

From (B.4) or (B.5) we deduce

$$\sup_m \big\|\nabla(\Phi \circ v_m)\big\|_{L^t(\Omega)} < \infty,$$

and

$$\big|\Phi(v_m)\big| \leqslant \mathrm{Lip}(\Phi)\,|v_m| + \big|\Phi(0)\big|$$

gives

$$\sup_m \|\Phi \circ v_m\|_{L^t(\Omega)} < \infty,$$

thus ($t > 1$) $\Phi \circ v_m \rightharpoondown: w$ in $W^1_t(\Omega;\mathbb{R}^k)$ at least for a subsequence. Passing to a further subsequence we see $\Phi \circ v_m \to w$ a.e. on Ω, thus $w = \Phi \circ v$. Consider $B_r(z) \subset \Omega$ and observe that $\nabla(\Phi \circ v_m) \rightharpoondown \nabla(\Phi \circ v)$ in $L^t(\Omega;\mathbb{R}^{nk})$ implies

$$\begin{aligned}\int_{B_r(z)} \big|\partial_\alpha(\Phi \circ v)\big|\,\mathrm{d}x &\leqslant \liminf_{m\to\infty} \int_{B_r(z)} \big|\partial_\alpha(\Phi \circ v_m)\big|\,\mathrm{d}x \\ &\overset{(B.5)}{\leqslant} \mathrm{Lip}(\Phi) \liminf_{m\to\infty} \int_{B_r(z)} \big|\partial_\alpha v_m\big|\,\mathrm{d}x = \mathrm{Lip}(\Phi) \int_{B_r(z)} \big|\partial_\alpha v\big|\,\mathrm{d}x,\end{aligned}$$

therefore

$$\fint_{B_r(z)} \big|\partial_\alpha(\Phi \circ v)\big|\,\mathrm{d}x \leqslant \mathrm{Lip}(\Phi) \fint_{B_r(z)} |\partial_\alpha v|\,\mathrm{d}x,$$

and we have (B.1) in case $t > 1$ and $\Phi \in C^1 \cap \mathrm{Lip}\,(\mathbb{R}^N;\mathbb{R}^k)$.

Next we assume that Φ is merely Lipschitz. If Φ_ε is a mollification of Φ, we have

$$\mathrm{Lip}(\Phi_\varepsilon) \leqslant \mathrm{Lip}(\Phi), \qquad \Phi_\varepsilon \to \Phi \text{ uniformly}$$

as $\varepsilon \to 0$, moreover $\Phi_\varepsilon \circ v \to \Phi \circ v$ in $L^t(\Omega;\mathbb{R}^k)$ and a.e. The case considered before implies

$$\big|\partial_\alpha(\Phi_\varepsilon \circ v)\big| \leqslant \mathrm{Lip}(\Phi_\varepsilon)|\partial_\alpha v|, \tag{B.6}$$

thus ($t > 1$) $\Phi_\varepsilon \circ v \rightharpoondown \Phi \circ v$ in $W^1_t(\Omega;\mathbb{R}^k)$. Semicontinuity of the norm gives as before

$$\fint_{B_r(z)} \bigl|\partial_\alpha(\Phi \circ v)\bigr| \,\mathrm{d}x \leqslant \liminf_{\varepsilon\to 0} \fint_{B_r(z)} \bigl|\partial_\alpha(\Phi_\varepsilon \circ v)\bigr| \,\mathrm{d}x \overset{(B.6)}{\leqslant} \liminf_{\varepsilon\to 0} \mathrm{Lip}(\Phi_\varepsilon) \fint_{B_r(z)} |\partial_\alpha v| \,\mathrm{d}x.$$

Summing up Lemma B.1 is established in case $t > 1$. For completeness let $v \in W^1_1(\Omega;\mathbb{R}^N)$ and consider a Lipschitz function $\Phi : \mathbb{R}^N \to \mathbb{R}^k$. Let $v_m \in C^\infty(\overline{\Omega};\mathbb{R}^N)$ such that $v_m \to v$ in $W^1_1(\Omega;\mathbb{R}^N)$ and a.e. Clearly $\Phi(v_m) \to \Phi(v)$ in $L^1(\Omega;\mathbb{R}^k)$ and a.e., thus $\Phi(v) \in \mathrm{BV}(\Omega;\mathbb{R}^k)$ and

$$\bigl|\nabla(\Phi \circ v)\bigr|(\Omega) \leqslant \liminf_{m\to\infty} \int_\Omega \bigl|\nabla(\Phi \circ v_m)\bigr| \,\mathrm{d}x \overset{*}{\leqslant} \mathrm{Lip}(\Phi) \lim_{m\to\infty} \int_\Omega |\nabla v_m| \,\mathrm{d}x = \mathrm{Lip}(\Phi) \int_\Omega |\nabla v| \,\mathrm{d}x.$$

Observe that in $*$ we used (B.1) for the sequence $\{v_m\}$ which is in any space $W^1_t(\Omega;\mathbb{R}^N)$, $t > 1$. By the same reasoning we get for any ball $B_r(z) \subset \Omega$

$$\begin{aligned} \bigl|\nabla(\Phi \circ v)\bigr|\bigl(B_r(z)\bigr) &\leqslant \liminf_{m\to\infty} \int_{B_r(z)} \bigl|\nabla(\Phi \circ v_m)\bigr| \,\mathrm{d}x \leqslant \mathrm{Lip}(\Phi) \lim_{m\to\infty} \int_{B_r(z)} |\nabla v_m| \,\mathrm{d}x \\ &= \mathrm{Lip}(\Phi) \int_{B_r(z)} |\nabla v| \,\mathrm{d}x. \end{aligned}$$

Applying the Besicovitch derivation theorem (see [3, Theorem 2.22, p. 54]; choose $\nu = \nabla\Phi(v)$, $\mu = |\nabla v| \cdot \mathcal{L}^n$) we deduce $\nabla(\Phi \circ v) \in L^1(\Omega;\mathbb{R}^{nk})$ and $|\nabla(\Phi \circ v)| \leqslant \mathrm{Lip}(\Phi)|\nabla v|$. This is Lemma B.1 in the limit case $t = 1$ provided we replace ∇ by ∂_α in the above calculations. $\square$

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