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Variants of the Stokes Problem: the Case of Anisotropic Potentials

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Dedicated to O. A. Ladyzhenskaya on the occasion of her 80^{th} birthday

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Abstract. We investigate the smoothness properties of local solutions of the nonlinear Stokes problem

$$-\mathrm{div}\,\{T(\varepsilon(v))\} + \nabla\pi \;=\; g \text{ on } \Omega,$$
$$\mathrm{div}\, v \;\equiv\; 0 \text{ on } \Omega,$$

where v: $\Omega \to \mathbb{R}^n$ is the velocity field, π: $\Omega \to \mathbb{R}$ denotes the pressure function, and g: $\Omega \to \mathbb{R}^n$ represents a system of volume forces, Ω denoting an open subset of $\mathbb{R}^n$. The tensor T is assumed to be the gradient of some potential f acting on symmetric matrices. Our main hypothesis imposed on f is the existence of exponents $1 < p \le q < \infty$ such that

$$\lambda(1+|\varepsilon|^2)^{\frac{p-2}{2}}|\sigma|^2 \le D^2 f(\varepsilon)(\sigma,\sigma) \le \Lambda(1+|\varepsilon|^2)^{\frac{q-2}{2}}|\sigma|^2$$

holds with suitable constants λ, $\Lambda > 0$, i.e. the potential f is of anisotropic power growth. Under natural assumptions on p and q we prove that velocity fields from the space $W^1_{p,\mathrm{loc}}(\Omega;\mathbb{R}^n)$ are of class $C^{1,\alpha}$ on an open subset of Ω with full measure. If $n = 2$, then the set of interior singularities is empty.

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1. Introduction

Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \ge 2$, and a system of volume forces g: $\Omega \to \mathbb{R}^n$ together with a boundary function v_0: $\partial\Omega \to \mathbb{R}^n$, the Stokes problem in the classical formulation reads as follows (see [La], p. 35): find a velocity field v: $\Omega \to \mathbb{R}^n$ and a pressure function π: $\Omega \to \mathbb{R}$ such that the following system of equations is satisfied

$$\begin{cases} \nu\Delta v = \nabla\pi - g & \text{on } \Omega, \\ \operatorname{div} v = 0 & \text{on } \Omega, \\ v = v_0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here ν denotes the kinematic viscosity which is assumed to be constant. For a detailed overview concerning the existence and regularity of solutions (even in the case of unbounded domains) we refer to the monograph [La] of Ladyzhenskaya. The velocity field v solving (1.1) is easily seen to be the solution of the minimization problem

$$J[u] := \int_\Omega \big\{ f(\varepsilon(u)) - g \cdot u \big\} \,\mathrm{d}x \to \min \quad \text{in } \mathcal{C}, \tag{1.2}$$

where $\mathcal{C}$ represents the class of all solenoidal vector fields with trace v_0 on $\partial\Omega$, and where f denotes the quadratic potential $f(\varepsilon) = |\varepsilon|^2$. (For simplicity we let $\nu = 1$, and $\varepsilon(u)$ is the symmetric gradient of u: $\Omega \to \mathbb{R}^n$.) A natural extension of problem (1.2) arises when we consider more general convex potentials f which immediately leads to corresponding nonlinear variants of equation (1.1). As a rule, the question of existence of minimizers can be easily settled by working in appropriate energy classes and by using Korn's inequality but the question of regularity becomes much more delicate: since we are in the nonlinear vector-valued setting, only partial regularity results can be expected, if $n \geq 3$. Before going into details, let us first look at some examples.

Power growth potentials f fall into the class of models proposed by Ladyzhenskaya in [La], p. 192. Roughly speaking, we require f to satisfy

$$\lambda(1+|\varepsilon|^2)^{\frac{p-2}{2}}|\sigma|^2 \leq D^2 f(\varepsilon)(\sigma,\sigma) \leq \Lambda(1+|\varepsilon|^2)^{\frac{p-2}{2}}|\sigma|^2 \tag{1.3}$$

with positive constants λ, Λ and with some exponent $p > 1$. Of course, (1.3) is some kind of uniform ellipticity estimate which is required to be valid for all symmetric $(n \times n)$-matrices. A typical example is given by $f(\varepsilon) = (1+|\varepsilon|^2)^{p/2}$, and for this potential the solution v of (1.2) represents the stationary flow of a "power-law fluid" with small velocity. Another example of this category arises if we let (see e.g. [AM], [BAH])

$$f(\varepsilon) = \mu_\infty|\varepsilon|^2 + \mu_0(1+|\varepsilon|^2)^{\frac{p}{2}}$$

with numbers μ_0, $\mu_\infty > 0$ and $p > 1$.

In [Fu] and [FS1] the following results on the regularity of minimizing velocity fields were established: let (1.3) hold with $p \geq 2$ and consider the solution u of (1.2) to be sought in the Sobolev-space $W^1_p(\Omega;\mathbb{R}^n)$. Then, if $n \geq 3$, there exists an open subset Ω_0 of Ω with full measure such that $u \in C^{1,\alpha}(\Omega_0;\mathbb{R}^n)$ for any $0 < \alpha < 1$, i.e. we have partial C^1-regularity (compare [FS1], Chapter 3, Theorem 3.1.3 and 3.1.4). In the twodimensional case together with $p = 2$ (this includes the Powell–Eyring model, see [PE]) the singular set is empty, we again refer to [FS1], Chapter 3, Theorem 3.3.2. The case $1 < p < 2$ was treated in [Re] (see also [FR]): for $n = 2$ there are no singular points, in case $n \geq 3$ with $p \geq 2 - 4/n$

partial regularity holds. Recently, Kaplický, Málek and Stará ([KMS]) considered the Dirichlet problem

$$\begin{aligned} (\nabla v)v - \operatorname{div}\{T(\varepsilon(v))\} + \nabla\pi &= g \text{ on } \Omega, \\ \operatorname{div} v &= 0 \text{ on } \Omega, \\ v &= 0 \text{ at } \partial\Omega \end{aligned}$$

for twodimensional domains Ω, where T is the gradient of some potential f with (1.3) satisfying in addition the structural condition $f(\varepsilon) = \tilde{f}(|\varepsilon|^2)$. In case $p \in (3/2, \infty)$ they construct global $C^{1,\alpha}$-solutions up to the boundary, for $p > 6/5$ solutions with interior $C^{1,\alpha}$-regularity are obtained. It should be emphasized that their paper covers the case of a non-vanishing convective term $(\nabla v)v$ which makes it necessary to impose a lower bound on the exponent p. From the physical point of view their assumption that f is just a function of $|\varepsilon|$ seems to be quite natural.

Potentials of logarithmic type are related to the Prandtl–Eyring fluid model: we have (up to physical constants)

$$f(\varepsilon) = \int_0^{|\varepsilon|} \operatorname{ar\,sinh} t \, \mathrm{d}t$$

and the corresponding version of (1.3) reads as

$$\lambda(1 + |\varepsilon|)^{-1}|\sigma|^2 \le D^2 f(\varepsilon)(\sigma,\sigma) \le \Lambda|\varepsilon|^{-1} \ln(1 + |\varepsilon|)|\sigma|^2. \tag{1.4}$$

The potential f is now of nearly linear growth which means that obviously there exists no exponent $p > 1$ such that (1.3) is satisfied. In [FS2] we discussed the existence of minimizers in certain Orlicz-type classes and proved partial $C^{1,\alpha}$-regularity, if $n = 3$, as well as full regularity, if $n = 2$. Further extensions are given in the paper [FO], Section 7. In connection with the twodimensional case we should mention the related work of Frehse and Seregin ([FrS]) on plastic materials with logarithmic hardening where they prove differentiability in case $n = 2$.

The purpose of our note is to introduce a third class of potentials allowing different growth rates from above and below. More precisely, suppose that $f \ge 0$ is of class C^2 satisfying

$$\lambda(1 + |\varepsilon|^2)^{\frac{p-2}{2}}|\sigma|^2 \le D^2 f(\varepsilon)(\sigma,\sigma) \le \Lambda(1 + |\varepsilon|^2)^{\frac{q-2}{2}}|\sigma|^2 \tag{1.5}$$

for exponents $1 < p \le q < \infty$. (Note that the logarithmic potential is not of this type since according to (1.4) we would have $p = 1$ and $q = 1 + \rho$ (with any $\rho > 0$) but $p = 1$ is excluded for obvious reasons.) A typical example for this anisotropic behavior is given by

$$f(\varepsilon) = |\varepsilon|^2 + (1 + |\varepsilon_{11}|^2)^{\frac{q}{2}}$$

with $q > 2$. Another example arises if we let $f(\varepsilon) = |\varepsilon|^2 + h(\varepsilon)$ where h satisfies the estimate $0 \le D^2 h(\varepsilon)(\sigma,\sigma) \le \Lambda(1 + |\varepsilon|^2)^{(q-2)/2}|\sigma|^2$, i.e. we allow that $D^2 h$ is degenerated together with the upper growth rate $q - 2$ for some $q > 2$. In standard variational calculus the anisotropic situation modelled by (1.5) has been intensively studied in recent years, we mention the papers [Ma], [PS], [AF], [BF1], [BF2] and

[BF4], but also in the mathematical theory of non-Newtonian fluids anisotropic growth seems to be of some interest, we refer to [MNRR], p. 193. Again we look at the minimization problem (1.2), and it is an easy exercise to show the existence of a unique J-minimizing map u provided we work in the class

$$\mathcal{C} = \Big\{ w \in v_0 + \overset{\circ}{W}{}^1_p(\Omega;\mathbb{R}^n) : \operatorname{div} w = 0,\ J[w] < \infty \Big\}$$

assuming in addition that $v_0 \in \mathcal{C}$. Our main result concerns the regularity of this solution.

Main Theorem. *Let (1.5) hold and suppose that the volume forces g are sufficiently regular.*

a) *If $q \geq 2$ and $q < p(1 + 2/n)$, then partial C^1-regularity holds.*

b) *Let $q = 2$ together with $n = 2$. Then the singular set is empty.*

Let us add some

Remarks. i) In [BF1] we established this theorem in the framework of classical variational calculus, and of course we here benefit from the general line of [BF1] but the arguments given there have to be adjusted in a non-trivial way.

ii) In part a) the restriction $q \geq 2$ is a technical one but not of principal nature. In fact, the case $q < 2$ would at least require a separate proof of the blow-up lemma given in Section 5, and even for the standard setting described in [BF1] this step causes a lot of technical difficulties if $q < 2$.

Nevertheless it is possible to include the case $q < 2$ up to a certain extend: if we have (1.5) with exponents $1 < p < q < 2$, then (1.5) clearly is satisfied with q replaced by $\bar{q} = 2$ which means that in case $n = 3$ partial regularity holds under the additional assumption $2 < p(1 + 2/3)$. i.e. $p > 6/5$, whereas for $n = 2$ we have full regularity without further restriction on p. Let us remark again that in the isotropic case $1 < p = q < 2$ together with $n = 3$ we always have partial regularity since then $p > 2 - 4/n$ (see [Re], [FR]).

iii) Let us again look at the case $1 < p < q < 2$ and define $f(\varepsilon) = (1 + |\varepsilon|^2)^{p/2} + (1 + |\varepsilon_{11}|^2)^{q/2}$. In case $n = 3$ we see from Remark ii) that partial regularity holds for $p > 6/5$, thus the whole scale $p \in (1, 6/5)$ is excluded, which motivates the study of the regularity theory in the subquadratic case $1 < p < q < 2$ together with the appropriate condition $q < p(1 + 2/3)$. But even with this extension it is not possible to get better results for the above example since f satisfies (1.5) with new exponent $\bar{q} = 2$, and this choice is optimal, which again leads us to the condition $p > 6/5$ excluding values of q and p close to 1. In [Bi] and [BF3] an appropriate regularity theory in the variational setting was developed which requires as main ingredient the bound $q < 2 + p$ (for any dimension n) which is clearly satisfied if $1 < p < q < 2$, and it remains a challenging task to transfer this result to anisotropic Stokes problems.

iv) It would also be desirable to give a version of part b) for the superquadratic case $q > 2$ together with the restriction $q < 2p$ (compare [BF4]). As it stands, b) implies full regularity for energy densities like $f(\varepsilon) = (1 + |\varepsilon|^2)^{p/2} + |\varepsilon_{11}|^2$ with $p \in (1, 2]$. More generally, we can consider f of the form $f(\varepsilon) = (1+|\varepsilon|^2)^{p/2}+h(\varepsilon)$ with h satisfying $0 \le D^2h(\varepsilon)(\sigma, \sigma) \le \Lambda|\sigma|^2$ and exponent p in $(1, 2]$.

v) Without further comment it is not clear that the minimizer $u \in \mathcal{C}$ is also a solution of the corresponding Euler–Lagrange equation ("the anisotropic Stokes system"), see Remark 2.2.

Our paper is organized as follows: in Section 2 we fix our notation and give a precise formulation of our main theorem. Section 3 is a collection of results on higher weak differentiability. In Section 4 we regularize our original variational problem and prove Caccioppoli-type inequalities as well as uniform higher integrability of the regularizing sequence. Using these preparations partial regularity is established via blow-up in Section 5. The final Section 6 contains some comments on the case $n = 2$.

2. Notation and results

Let $\mathbb{S}$ denote the space of all symmetric matrices of order n. We will use the following notation

$$u \cdot v = u_i v_i, \quad |u| = \sqrt{u \cdot u},$$
$$u \otimes v = (u_i v_j), \quad u \odot v = \frac{1}{2}(u \otimes v + v \otimes u)$$

for $u = (u_i)$, $v = (v_i) \in \mathbb{R}^n$,

$$\varepsilon : \sigma = \varepsilon_{ij}\sigma_{ij}, \quad |\varepsilon| = \sqrt{\varepsilon : \varepsilon}, \quad \sigma\nu = (\sigma_{ij}\nu_j) \in \mathbb{R}^n$$

for ε, $\sigma \in \mathbb{S}$, $\nu \in \mathbb{R}^n$. Here the convention of summation over repeated indices running from 1 to n is adopted. Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, denote an open set. For functions u: $\Omega \to \mathbb{R}^n$ we let

$$\varepsilon(u) = \frac{1}{2}\big(\partial_i u^j + \partial_j u^i\big),$$

whenever this expression makes sense. For a definition of the standard Lebesgue and Sobolev spaces like $L^p_{(\mathrm{loc})}(...)$, $W^k_{p(,\mathrm{loc})}(...)$, $\overset{\circ}{W}{}^k_p(...)$, etc., we refer to e.g. [Ad].

Assume that we are given a function f: $\mathbb{S} \to [0, \infty)$ of class C^2 satisfying for all ε, $\sigma \in \mathbb{S}$

$$\lambda(1 + |\varepsilon|^2)^{\frac{p-2}{2}}|\sigma|^2 \le D^2f(\varepsilon)(\sigma, \sigma) \le \Lambda(1 + |\varepsilon|^2)^{\frac{q-2}{2}}|\sigma|^2 \tag{2.1}$$

with positive constants λ, Λ and with exponents $1 < p \le q < +\infty$. Let us define the energy

$$J[u] = \int_\Omega f(\varepsilon(u))\,\mathrm{d}x. \tag{2.2}$$

Note that (2.1) immediately implies the growth estimate

$$a|\varepsilon|^p - b \le f(\varepsilon) \le A\big(|\varepsilon|^q + 1\big) \quad \text{for all } \varepsilon \in \mathbb{S} \tag{2.3}$$

with constants a, $A > 0$, $b \in \mathbb{R}$. Moreover, f is a convex function. In extension of the Main Theorem from Section 1 we do not restrict ourselves to global J-minimizers, we just look at the local situation which is quite natural since the question of boundary regularity is beyond the scope of our investigations.

Definition 2.1. Let

$$\mathbb{K} = \Big\{ v \in W^1_{p,\mathrm{loc}}(\Omega;\mathbb{R}^n) : \operatorname{div} v = 0 \Big\}.$$

A mapping u: $\Omega \to \mathbb{R}^n$ is termed a local J-minimizer subject to the constraint $\operatorname{div} u = 0$ if and only if u belongs to the class $\mathbb{K}$ and satisfies:

a) $\displaystyle\int_{\Omega'} f(\varepsilon(u))\,\mathrm{d}x < +\infty \quad \text{for all } \Omega' \Subset \Omega;$

b) $\displaystyle\int_{\Omega'} f(\varepsilon(u))\,\mathrm{d}x \le \int_{\Omega'} f(\varepsilon(v))\,\mathrm{d}x \quad \text{for all } \Omega' \Subset \Omega$
and for all $v \in \mathbb{K}$ s.t. $\operatorname{spt}(u - v) \subset \Omega'$.

We have

Theorem 2.1. *Consider a local J-minimizer u where f is supposed to satisfy* (2.1).

a) *If $q \ge 2$ and $q < p(1 + 2/n)$, then there is an open set $\Omega_0 \subset \Omega$ with $|\Omega - \Omega_0| = 0$ such that $u \in C^{1,\alpha}(\Omega_0;\mathbb{R}^n)$ for any $0 < \alpha < 1$.*

b) *Let $n = 2$ together with $q = 2$. Then $\Omega_0 = \Omega$, i.e. full regularity holds.*

Remark 2.1. For technical simplicity we included no volume forces g in the energy density defined in (2.2). But Theorem 2.1 easily extends to forces g located in some appropriate Morrey space.

Remark 2.2. Since by definition a local minimizer is just of class $W^1_{p,\mathrm{loc}}(\Omega;\mathbb{R}^n)$ and since by (2.1) $Df(\varepsilon)$ can grow like $|\varepsilon|^{q-1}$ it is again unclear if the Euler equation holds. But we have the following intermediate regularity result (see Lemma 4.4 and Corollary 4.2): let (2.1) hold with exponents $1 < p \le q < +\infty$, $q \ge 2$. Then, if $q < p(1+2/n)$ (for any dimension $n \ge 2$), local J-minimizers belong to the space $W^1_{q,\mathrm{loc}}(\Omega;\mathbb{R}^n)$, and therefore solve the Euler-equation associated to (2.2) in the weak sense.

3. Auxiliary results

Here we collect some material which might be well-known to experts but which is hard to trace in the literature. Let $s \ge 2$ and consider a function F: $\mathbb{S} \to [0,\infty)$

of class C^2 satisfying the uniform estimate

$$a(1+|\varepsilon|^2)^{\frac{s-2}{2}}|\sigma|^2 \le D^2F(\varepsilon)(\sigma,\sigma) \le A(1+|\varepsilon|^2)^{\frac{s-2}{2}}|\sigma|^2 \tag{3.1}$$

for all ε, $\sigma \in \mathbb{S}$ with a, $A > 0$.

Lemma 3.1. *Suppose that* $v \in W^1_{s,\mathrm{loc}}(\tilde{\Omega};\mathbb{R}^n)$ *is a local minimizer of the energy* $w \mapsto \int_{\tilde{\Omega}} F(\varepsilon(w))\,\mathrm{d}x$ *subject to the constraint* $\operatorname{div} w = 0$*, where* $\tilde{\Omega}$ *denotes some arbitrary open set in* $\mathbb{R}^n$*. Then we have:*

a) $v \in W^2_{2,\mathrm{loc}}(\tilde{\Omega};\mathbb{R}^n)$*;*

b) $(1+|\varepsilon(v)|^2)^{s/4} \in W^1_{2,\mathrm{loc}}(\tilde{\Omega})$ *together with*

$$\nabla\big\{(1+|\varepsilon(v)|^2)^{s/4}\big\} = \frac{s}{2}(1+|\varepsilon(v)|^2)^{\frac{s}{4}-1}|\varepsilon(v)|\nabla|\varepsilon(v)|\,;$$

c) $DF(\varepsilon(v)) \in W^1_{s/(s-1),\mathrm{loc}}(\tilde{\Omega};\mathbb{S})$ *and*

$$\partial_k\big\{DF(\varepsilon(v))\big\} = D^2F(\varepsilon(v))(\partial_k\varepsilon(v),\cdot), \quad k=1,\dots,n.$$

Remark 3.1.

i) In the standard variational setting Lemma 3.1 is classical and can be found in the works of Ladyzhenskaya and Ural'tseva, Campanato or Morrey.

ii) In [Re] there is a variant of Lemma 3.1 covering the case $s < 2$.

iii) In Lemma 3.1 the notation of a local minimizer is the same as introduced in Section 2.

Proof of Lemma 3.1. We have

$$\int_{\tilde{\Omega}} DF(\varepsilon(v)) : \varepsilon(\varphi)\,\mathrm{d}x = 0 \tag{3.2}$$

being valid for any $\varphi \in W^1_s(\tilde{\Omega};\mathbb{R}^n)$ with $\operatorname{div}\varphi = 0$ and compact support in $\tilde{\Omega}$. Let us introduce the difference quotient of a function g in the k^{th} direction through

$$\Delta_h g(x) := \frac{1}{h}\big\{g(x+he_k) - g(x)\big\}, \quad h \ne 0.$$

Next fix a ball $B_R \Subset \tilde{\Omega}$ and consider $\eta \in C^\infty_0(B_R)$ such that $\eta \equiv 1$ on B_r, $\eta \equiv 0$ outside of $B_{r'}$, $\eta \ge 0$ and $|\nabla\eta| \le c/(r'-r)$, where $0 < r < r' < R$. If we assume $|h|$ to be sufficiently small (depending on $\operatorname{spt}\varphi$), then (3.2) implies

$$\int_{\tilde{\Omega}} \big\{DF(\varepsilon(v)(x+he_k)) - DF(\varepsilon(v)(x))\big\} : \varepsilon(\varphi)\,\mathrm{d}x = 0. \tag{3.3}$$

Clearly $g := h^{-1}\operatorname{div}(\eta^2\Delta_h v)$ is in the space $L^s(B_{r'})$ together with $\int_{B_{r'}} g\,\mathrm{d}x = 0$, thus we can use the results of [LS] or [Pi] (see also [Ga], III, Theorem 3.2) on the existence of a function $\psi \in \overset{\circ}{W}{}^1_s(B_{r'};\mathbb{R}^n)$ such that $\operatorname{div}\psi = g$ on $B_{r'}$, i.e.

$$\operatorname{div}\psi = \frac{1}{h}\nabla\eta^2 \cdot \Delta_h v,$$

together with

$$\|\nabla\psi\|_{L^s(B_{r'})} \leq c\|g\|_{L^s(B_{r'})}. \tag{3.4}$$

Let us choose $\varphi := h^{-1}\eta^2\Delta_h v - \psi \in \overset{\circ}{W}{}^1_s(B_{r'};\mathbb{R}^n)$ in equation (3.3). We get

$$\begin{aligned}&\int_{B_{r'}} \Delta_h\big\{DF(\varepsilon(v))\big\} : \varepsilon(\Delta_h v)\eta^2 \,\mathrm{d}x \\ &= \int_{B_{r'}} \Delta_h\big\{DF(\varepsilon(v))\big\} : \big(h\varepsilon(\psi) - \nabla\eta^2 \odot \Delta_h v\big) \,\mathrm{d}x.\end{aligned} \tag{3.5}$$

Let us write

$$\begin{aligned}\Delta_h\big\{DF(\varepsilon(v))\big\}(x) &= \frac{1}{h}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} DF\big(\varepsilon(v)(x) + t[\varepsilon(v)(x+he_k) - \varepsilon(v)(x)]\big)\,\mathrm{d}t \\ &= \int_0^1 D^2F\big(\varepsilon(v) + th\varepsilon(\Delta_h v)\big)(\varepsilon(\Delta_h v), \cdot)\,\mathrm{d}t\end{aligned}$$

and introduce the parameter-dependent bilinear form

$$B_x := \int_0^1 D^2F\big(\varepsilon(v)(x) + th\varepsilon(\Delta_h v)(x)\big)\,\mathrm{d}t$$

acting on pairs of symmetric matrices. With this notation (3.5) takes the form

$$\int_{B_{r'}} B_x(\varepsilon(\Delta_h v), \varepsilon(\Delta_h v))\eta^2 \,\mathrm{d}x = \int_{B_{r'}} B_x(\varepsilon(\Delta_h v), h\varepsilon(\psi) - \nabla\eta^2 \odot \Delta_h v)\,\mathrm{d}x. \tag{3.6}$$

On the right-hand side we use the Cauchy–Schwarz inequality to get for any $0 < \delta < 1$

$$\begin{aligned}\left|\int_{B_{r'}} B_x(\varepsilon(\Delta_h v), \nabla\eta^2 \odot \Delta_h v)\,\mathrm{d}x\right| &\leq \delta\int_{B_{r'}} B_x(\varepsilon(\Delta_h v), \varepsilon(\Delta_h v))\eta^2\,\mathrm{d}x \\ &\quad + c\delta^{-1}\int_{B_{r'}} B_x(\nabla\eta \odot \Delta_h v, \nabla\eta \odot \Delta_h v)\,\mathrm{d}x \\ &=: \delta I_1 + c\delta^{-1} I_2,\end{aligned}$$

$$I_2 \leq c(r'-r)^{-2}\int_{B_{r'}}\int_0^1 \big(1 + |\varepsilon(v) + th\varepsilon(\Delta_h v)|^2\big)^{\frac{s-2}{2}} |\Delta_h v|^2 \,\mathrm{d}x\,\mathrm{d}t =: M,$$

where we made use of the right-hand side of (3.1). We further have $(0 < \delta' < 1)$

$$\begin{aligned}&\left|\int_{B_{r'}} B_x(\varepsilon(\Delta_h v), h\varepsilon(\psi))\,\mathrm{d}x\right| \\ &\leq \delta'\int_{B_{r'}} B_x(\varepsilon(\Delta_h v), \varepsilon(\Delta_h v))\,\mathrm{d}x + c\delta'^{-1}h^2\int_{B_{r'}} |B_x||\varepsilon(\psi)|^2\,\mathrm{d}x.\end{aligned}$$

Let us choose $\delta = 1/2$, thus δI_1 can be absorbed on the left-hand side of (3.6), and if we take $\delta' = 1/4$ we end up with

$$\int_{B_r} B_x(\varepsilon(\Delta_h v), \varepsilon(\Delta_h v))\,\mathrm{d}x$$
$$\leq \frac{1}{2}\int_{B_{r'}} B_x(\varepsilon(\Delta_h v), \varepsilon(\Delta_h v))\,\mathrm{d}x + c\left\{M + h^2 \int_{B_{r'}} |B_x||\varepsilon(\psi)|^2\,\mathrm{d}x\right\}. \quad (3.7)$$

Observe that (recall (3.4) and $s \geq 2$)

$$h^2 \int_{B_{r'}} |B_x||\varepsilon(\psi)|^2\,\mathrm{d}x \leq h^2 \left(\int_{B_{r'}} |\varepsilon(\psi)|^s\,\mathrm{d}x\right)^{\frac{2}{s}} \left(\int_{B_{r'}} |B_x|^{\frac{s}{s-2}}\,\mathrm{d}x\right)^{1-\frac{2}{s}}$$
$$\leq c(r'-r)^{-2} \left(\int_{B_{r'}} |\Delta_h v|^s\,\mathrm{d}x\right)^{\frac{2}{s}} \left(\int_{B_{r'}} |B_x|^{\frac{s}{s-2}}\,\mathrm{d}x\right)^{1-\frac{2}{s}},$$

and by applying Hölder's inequality also to the integral in M, we deduce from (3.7) by letting

$$\omega(r) := \int_{B_r} B_x(\varepsilon(\Delta_h v), \varepsilon(\Delta_h v))\,\mathrm{d}x$$

and recalling the definition of B_x:

$$\omega(r) \leq \frac{c}{(r'-r)^2} \left(\int_{B_{r'}} |\Delta_h v|^s\,\mathrm{d}x\right)^{\frac{2}{s}} \left(\int_{B_{r'}} (1 + |\varepsilon(v)|^2 + |h\varepsilon(\Delta_h v)|^2)^{\frac{s}{2}}\,\mathrm{d}x\right)^{1-\frac{2}{s}}$$
$$+\frac{1}{2}\omega(r'). \quad (3.8)$$

Since v is in the space $W^1_{s,\mathrm{loc}}(\tilde{\Omega}; \mathbb{R}^n)$ we have

$$\|\Delta_h v\|_{L^s(B_{r'})} \leq \|\nabla v\|_{L^s(B_{R+h})}$$

and by using similar bounds for the second integral in (3.8), we find

$$\omega(r) \leq \frac{1}{2}\omega(r') + c(r'-r)^{-2}\left(1 + \int_{B_{R+h}} |\nabla v|^s\,\mathrm{d}x\right).$$

Thus we may apply Lemma 3.1, p. 161, of [Gi] with the result

$$\omega(r) \leq c(r'-r)^{-2}\left(1 + \int_{B_{R+h}} |\nabla v|^s\,\mathrm{d}x\right), \quad 0 < r < r' \leq R. \quad (3.9)$$

Now, on account of $s \geq 2$, we have

$$\omega(r) \geq c|\varepsilon(\Delta_h v)|^2,$$

therefore (3.9) immediately implies (by quoting Korn's inequality) part a) of Lemma 3.1.

By [Mo], Theorem 3.6.8 (b), we then have

$$\varepsilon(\Delta_h v) \overset{h\to 0}{\to} \varepsilon(\partial_k v) \quad \text{in } L^2_{\mathrm{loc}},$$

in particular we may assume also convergence a.e. at least for a subsequence. This implies a.e.

$$\begin{aligned} 0 &\leq B_x(\varepsilon(\Delta_h v), \varepsilon(\Delta_h v)) \\ &= \int_0^1 D^2 F\big(\varepsilon(v)(x) + t[\varepsilon(v)(x + he_k) - \varepsilon(v)(x)]\big)(\varepsilon(\Delta_h v)(x), \varepsilon(\Delta_h v)(x))\,\mathrm{d}t \\ &\overset{h\to 0}{\to} D^2 F(\varepsilon(v)(x))(\varepsilon(\partial_k v)(x), \varepsilon(\partial_k v)(x)), \end{aligned}$$

where we now take the sum w.r.t. $k = 1, \dots, n$. If we apply Fatou's lemma and use (3.9) with the choice $r' = R$ we find

$$\int_{B_r} D^2 F(\varepsilon(v))(\varepsilon(\partial_k v), \varepsilon(\partial_k v))\,\mathrm{d}x \leq \frac{c}{(R-r)^2} \int_{B_R} (1 + |\nabla v|^s)\,\mathrm{d}x \tag{3.10}$$

for any $r < R$, in particular, the left-hand side of (3.10) is finite. We now claim

$$(1 + |\varepsilon(v)|^2)^{\frac{s}{4}} \in W^1_{2,\mathrm{loc}}(\tilde{\Omega}). \tag{3.11}$$

Let

$$\theta_L(t) := \left\{ \begin{array}{ll} \theta(t), & t \leq L \\ (1 + L^2)^{s/4}, & t \geq L \end{array} \right\}, \quad \theta(t) := (1 + t^2)^{\frac{s}{4}}.$$

$v \in W^2_{2,\mathrm{loc}}(\tilde{\Omega}; \mathbb{R}^n)$ implies $\theta_L(|\varepsilon(v)|) \in W^1_{1,\mathrm{loc}}(\tilde{\Omega})$ together with

$$|\nabla \theta_L(|\varepsilon(v)|)| \leq \theta'_L(|\varepsilon(v)|)|\nabla|\varepsilon(v)|| \leq \theta'(|\varepsilon(v)|)|\nabla|\varepsilon(v)||,$$

thus

$$\int_\omega |\nabla \theta_L(|\varepsilon(v)|)|^2\,\mathrm{d}x \leq c \int_\omega (1 + |\varepsilon(v)|^2)^{\frac{s-2}{2}} |\nabla \varepsilon(v)|^2\,\mathrm{d}x$$

for any subdomain $\omega \Subset \tilde{\Omega}$, and (3.10) in combination with a covering argument shows

$$\sup_{L>0} \|\theta_L(|\varepsilon(v)|)\|_{W^1_2(\omega)} < \infty,$$

hence

$$\theta_L(|\varepsilon(v)|) \rightharpoonup: \vartheta$$

weakly in $W^1_{2,\mathrm{loc}}(\tilde{\Omega})$ as $L \to \infty$. On the other hand

$$\theta_L(|\varepsilon(v)|) \to \theta(|\varepsilon(v)|) \text{ a.e.}, \quad L \to \infty,$$

thus $\vartheta = \theta(|\varepsilon(v)|)$ which proves (3.11). Finally, it is immediate that

$$\nabla \theta_L(|\varepsilon(v)|) = \theta'_L(|\varepsilon(v)|)\nabla|\varepsilon(v)| \to \theta'(|\varepsilon(v)|)\nabla|\varepsilon(v)|$$

a.e. as $L \to \infty$ which gives the required identity

$$\nabla \theta(|\varepsilon(v)|) = \theta'(|\varepsilon(v)|)\nabla|\varepsilon(v)|.$$

Let us recall the formula

$$\begin{aligned}&\Delta_h\big\{DF(\varepsilon(v))\big\}(x)\\&=\int_0^1 D^2F(\varepsilon(v)(x)+t[\varepsilon(v)(x+he_k)-\varepsilon(v)(x)]\big)(\varepsilon(\Delta_h v)(x),\cdot)\,\mathrm{d}t.\end{aligned}$$

The arguments outlined after (3.9) show that for almost all x the above expression converges to $D^2F(\varepsilon(v)(x))(\partial_k\varepsilon(v)(x),\cdot)$ as $h\to 0$, i.e.

$$\Delta_h\big\{DF(\varepsilon(v))\big\}\overset{h\to 0}{\to} D^2F(\varepsilon(v))(\partial_k\varepsilon(v),\cdot)\text{ a.e.}\tag{3.12}$$

In order to continue we observe (according to the above formula)

$$\begin{aligned}&\big|\Delta_h\big\{DF(\varepsilon(v))\big\}\big|^2\\&\leq B_x\big(\varepsilon(\Delta_h v),\Delta_h\big\{DF(\varepsilon(v))\big\}\big)\\&\leq \big(B_x(\varepsilon(\Delta_h v),\varepsilon(\Delta_h v))\big)^{\frac{1}{2}}\big(B_x\big(\Delta_h\big\{DF(\varepsilon(v))\big\},\Delta_h\big\{DF(\varepsilon(v))\big\}\big)\big)^{\frac{1}{2}}\\&\leq \big(B_x(\varepsilon(\Delta_h v),\varepsilon(\Delta_h v))\big)^{\frac{1}{2}}\sqrt{|B_x|}\,\big|\Delta_h\big\{DF(\varepsilon(v))\big\}\big|,\end{aligned}$$

so that

$$\big|\Delta_h\big\{DF(\varepsilon(v))\big\}\big|\leq\sqrt{|B_x|}\,\big(B_x(\varepsilon(\Delta_h v),\varepsilon(\Delta_h v))\big)^{\frac{1}{2}},$$

hence

$$\big|\Delta_h\big\{DF(\varepsilon(v))\big\}\big|^{\frac{s}{s-1}}\leq|B_x|^{\frac{1}{2}\frac{s}{s-1}}\big(B_x(\varepsilon(\Delta_h v),\varepsilon(\Delta_h v))\big)^{\frac{1}{2}\frac{s}{s-1}}.$$

On account of $s\geq 2$ we have $\frac{1}{2}\frac{s}{s-1}\leq 1$. Let us assume that $s>2$, the case $s=2$ follows by obvious simplifications. Then the right-hand side of the latter inequality is bounded from above by

$$c\Big\{|B_x|^{\frac{s}{s-2}}+B_x(\varepsilon(\Delta_h v),\varepsilon(\Delta_h v))\Big\}.$$

By definition of B_x and the growth of D^2F it is clear that $|B_x|^{s/(s-2)}$ is locally of class L^1 uniformly w.r.t. h, by (3.9) the same is true for $B_x(\varepsilon(\Delta_h v),\varepsilon(\Delta_h v))$, therefore $|\Delta_h\{DF(\varepsilon(v))\}|^{s/(s-1)}$ is uniformly bounded in L^1_{loc} which proves that $DF(\varepsilon(v))$ is of class $W^1_{s/(s-1),\text{loc}}(\tilde{\Omega};\mathbb{S})$ (recall $DF(\varepsilon(v))\in L^{s/(s-1)}(\tilde{\Omega};\mathbb{S})$ on account of the growth properties of DF). The formula for $\partial_k\{DF(\varepsilon(v))\}$ follows from (3.12) since also $\Delta_h\{DF(\varepsilon(v))\}\to\partial_k\{DF(\varepsilon(v))\}$ as $h\to 0$ in $L^{s/(s-1)}_{\text{loc}}(\tilde{\Omega};\mathbb{S})$ (see [Mo], Theorem 3.6.8 (b)). □

4. Regularization, higher integrability of the gradient and a Caccioppoli-type inequality

From now on we assume that we are in the situation of Theorem 2.1 a), i.e. $u\in\mathbb{K}$ is a local J-minimizer with f satisfying (2.1), moreover, we have the bounds

$$q\geq 2,\quad q<p(1+2/n).\tag{4.1}$$

Let $B_{2R} = B_{2R}(x_0)$ denote a ball compactly contained in Ω and consider a sequence $\{u_m\}$ of mollifications of u. We define

$$\delta_m = \big(1 + m + \|\varepsilon(u_m)\|^{2q}_{L^q(B_{2R})}\big)^{-1}$$

together with

$$f_m(\varepsilon) = f(\varepsilon) + \delta_m(1 + |\varepsilon|^2)^{\frac{q}{2}}, \quad \varepsilon \in \mathbb{S}.$$

Note that $\delta_m \to 0$ as $m \to \infty$. Next, let v_m denote the unique solution of the minimization problem

$$\int_{B_{2R}} f_m(\varepsilon(w))\,\mathrm{d}x \to \min$$

in the class $u_m + \overset{\circ}{W}{}^1_q(B_{2R};\mathbb{R}^n)$ subject to $\operatorname{div} w = 0$. We observe that the integrands f_m satisfy the hypotheses of Lemma 3.1 with s replaced by q. Further properties of the sequence $\{v_m\}$ are collected in

Lemma 4.1. *With the notation from above we have*

i) $\sup_m \|v_m\|_{W^1_p(B_{2R})} < \infty$;

ii) $v_m \rightharpoonup u$ *in* $W^1_p(B_{2R};\mathbb{R}^n)$ *as* $m \to \infty$;

iii) $\delta_m \int_{B_{2R}} (1 + |\varepsilon(v_m)|^2)^{\frac{q}{2}}\,\mathrm{d}x \to 0$ *as* $m \to \infty$;

iv) $\int_{B_{2R}} f_{(m)}(\varepsilon(v_m))\,\mathrm{d}x \to \int_{B_{2R}} f(\varepsilon(u))\,\mathrm{d}x$ *as* $m \to \infty$.

Proof. It is immediate that $\sup_m \|\varepsilon(v_m)\|_{L^p(B_{2R})} < \infty$, and from Korn's inequality (see e.g. [MM] or [Ko1], [Ko2], [Fi], [Fri], [St], [Ze]) we get

$$\|v_m - u_m\|_{L^p(B_{2R})} \leq c\|\varepsilon(u_m) - \varepsilon(v_m)\|_{L^p(B_{2R})},$$

hence $\sup_m \|v_m\|_{L^p(B_{2R})} < \infty$. Applying Korn's inequality in the form

$$\|\nabla v_m\|_{L^p(B_{2R})} \leq \big\{\|v_m\|_{L^p(B_{2R})} + \|\varepsilon(v_m)\|_{L^p(B_{2R})}\big\}$$

(again compare the above references or [FS1], Lemma 3.0.1) we deduce i).

Minimality of v_m and Jensen's inequality give

$$\begin{aligned}\int_{B_{2R}} f(\varepsilon(v_m))\,\mathrm{d}x &\leq \int_{B_{2R}} f_m(\varepsilon(v_m))\,\mathrm{d}x \leq \int_{B_{2R}} f_m(\varepsilon(u_m))\,\mathrm{d}x \\ &\leq \delta_m \int_{B_{2R}} (1 + |\varepsilon(u_m)|^2)^{\frac{q}{2}}\,\mathrm{d}x + \int_{B_{2R}} f(\varepsilon(u))\,\mathrm{d}x + O(m), \end{aligned} \tag{4.2}$$

where $O(m) \to 0$ as $m \to \infty$. By definition of δ_m we have

$$\delta_m \int_{B_{2R}} (1 + |\varepsilon(u_m)|^2)^{\frac{q}{2}}\,\mathrm{d}x \to 0$$

as $m \to \infty$, thus (4.2) implies

$$\liminf_{m\to\infty} \int_{B_{2R}} f(\varepsilon(v_m))\,\mathrm{d}x \le \int_{B_{2R}} f(\varepsilon(u))\,\mathrm{d}x. \tag{4.3}$$

By i) we get the existence of $\tilde u \in W^1_p(B_{2R};\mathbb{R}^n)$ s.t. $v_m \rightharpoondown \tilde u$ in $W^1_p(B_{2R};\mathbb{R}^n)$ at least for a subsequence. (4.3) gives

$$\int_{B_{2R}} f(\varepsilon(\tilde u))\,\mathrm{d}x \le \int_{B_{2R}} f(\varepsilon(u))\,\mathrm{d}x,$$

and since $\tilde u \in u + \overset{\circ}{W}{}^1_p(B_{2R};\mathbb{R}^n)$ together with $\operatorname{div}\tilde u = 0$ we find $\tilde u = u$, thus ii) holds. The remaining statements iii) and iv) follow from the chain of inequalities (4.2). □

Lemma 4.2. *Let $h_m := (1+|\varepsilon(v_m)|^2)^{p/4}$. Then we have $h_m \in W^1_{2,\mathrm{loc}}(B_{2R})$ together with*

$$\nabla h_m = \frac{p}{2}(1+|\varepsilon(v_m)|^2)^{\frac{p}{4}-1}|\varepsilon(v_m)|\nabla|\varepsilon(v_m)|.$$

In particular we have $\varepsilon(v_m) \in L^{p\chi}_{\mathrm{loc}}(B_{2R};\mathbb{S})$, where

$$\chi = \begin{cases} n/(n-2) & \text{if } n \ge 3,\\ \text{any number} & \text{if } n = 2 \end{cases}$$

Proof. Let us write

$$(1+|\varepsilon(v_m)|^2)^{\frac{p}{4}} = \Big\{(1+|\varepsilon(v_m)|^2)^{\frac{q}{4}}\Big\}^{\frac{p}{q}}.$$

Since $t \mapsto t^{p/q}$ is Lipschitz on $[1,\infty)$ and since by Lemma 3.1 we know $(1+|\varepsilon(v_m)|^2)^{q/4} \in W^1_{2,\mathrm{loc}}(B_{2R})$ (together with the formula for the derivative) we get the claim of Lemma 4.2. □

The next lemma contains a Caccioppoli-type inequality for the functions v_m.

Lemma 4.3. *For any $(n\times n)$-matrix Q (not necessarily from the space $\mathbb{S}$!) and for any $\eta \in C^\infty_0(B_{2R})$ we have the estimate*

$$\begin{aligned}\int_{B_{2R}} &\eta^2 D^2 f_m(\varepsilon(v_m))(\partial_k\varepsilon(v_m),\partial_k\varepsilon(v_m))\,\mathrm{d}x\\ &\le c\|\nabla\eta\|^2_{L^\infty(B_{2R})}\int_{\operatorname{spt}\nabla\eta}\Gamma_m^{\frac{q-2}{2}}|\nabla v_m - Q|^2\,\mathrm{d}x,\end{aligned} \tag{4.4}$$

where $\Gamma_m := 1+|\varepsilon(v_m)|^2$, and c denotes a positive constant independent of m and R.

Corollary 4.1. *For any radii $0 < r < r'$ such that $B_{r'}(\bar{x}) \Subset B_{2R}$ we have*

$$\int_{B_r(\bar{x})} D^2 f_m(\varepsilon(v_m))(\partial_k \varepsilon(v_m), \partial_k \varepsilon(v_m))\, \mathrm{d}x \leq c(r'-r)^{-2} \int_{B_{r'}(\bar{x})} \Gamma_m^{\frac{q}{2}}\, \mathrm{d}x. \tag{4.5}$$

Proof of Corollary 4.1. From (4.4) we get by Hölder's inequality

$$\begin{aligned} &\int_{B_{2R}} \eta^2 D^2 f_m(\varepsilon(v_m))(\partial_k \varepsilon(v_m), \partial_k \varepsilon(v_m))\, \mathrm{d}x \\ &\leq c\|\nabla \eta\|^2_{L^\infty(B_{2R})} \left(\int_{\operatorname{spt} \nabla\eta} \Gamma_m^{\frac{q}{2}}\, \mathrm{d}x \right)^{1-\frac{2}{q}} \left(\int_{\operatorname{spt}\nabla\eta} |\nabla v_m - Q|^q\, \mathrm{d}x \right)^{\frac{2}{q}}. \end{aligned}$$

Let us choose $\eta \equiv 1$ on $B_r(\bar{x})$, $\eta \equiv 0$ on $B_{2R} - B_{r'}(\bar{x})$ and such that $|\nabla \eta| \leq c(r'-r)^{-1}$. Consider a rigid motion $\gamma = Bx + a$ such that (see [FS1], Lemma 3.0.3)

$$\|v_m - \gamma\|_{L^q(B_{r'}(\bar{x}))} \leq c\|\varepsilon(v_m)\|_{L^q(B_{r'}(\bar{x}))}.$$

Since $\nabla(v_m - \gamma) = \nabla v_m - B$, we infer from Korn's inequality

$$\begin{aligned} \|\nabla v_m - B\|_{L^q(B_{r'}(\bar{x}))} &\leq c\big\{ \|v_m - \gamma\|_{L^q(B_{r'}(\bar{x}))} + \|\varepsilon(v_m)\|_{L^q(B_{r'}(\bar{x}))} \big\} \\ &\leq c\|\varepsilon(v_m)\|_{L^q(B_{r'}(\bar{x}))}, \end{aligned}$$

hence (4.5) follows by choosing $Q = B$. □

Proof of Lemma 4.3. Let $\sigma_m := Df_m(\varepsilon(v_m))$ which by Lemma 3.1 is of class $W^1_{q/(q-1),\mathrm{loc}}(B_{2R};\mathbb{S})$, moreover, the growth of Df_m implies $\sigma_m \in L^{q/(q-1)}(B_{2R};\mathbb{S})$. Therefore the mapping

$$\Phi : \overset{\circ}{W}{}^1_q(B_{2R};\mathbb{R}^n) \ni \varphi \mapsto \int_{B_{2R}} \sigma_m : \varepsilon(\varphi)\, \mathrm{d}x$$

belongs to the dual space $\overset{\circ}{W}{}^1_q(B_{2R};\mathbb{R}^n)^*$. The Euler equation satisfied by v_m shows $\Phi(\varphi) = 0$, if $\operatorname{div}\varphi = 0$, thus by a well known reasoning (see, e.g. [Ga], p. 180, Lemma 1.1, or [La], [LS]) there exists a pressure function $p_m \in L^{q/(q-1)}(B_{2R})$, $\int_{B_{2R}} p_m\, \mathrm{d}x = 0$, such that

$$\int_{B_{2R}} \sigma_m : \varepsilon(\varphi)\, \mathrm{d}x = \int_{B_{2R}} p_m \operatorname{div}\varphi\, \mathrm{d}x \tag{4.6}$$

for all $\varphi \in \overset{\circ}{W}{}^1_q(B_{2R};\mathbb{R}^n)$, hence

$$\nabla p_m = \operatorname{div}\sigma_m,$$

which means $p_m \in W^1_{q/(q-1),\mathrm{loc}}(B_{2R})$. Let us fix $\eta \in C_0^\infty(B_{2R})$, $0 \leq \eta \leq 1$, and denote by Δ_h the difference quotient in direction e_k, $k = 1, \dots, n$. From (4.6) we get

$$\int_{B_{2R}} \Delta_h \sigma_m : \varepsilon(\eta^2 \Delta_h[v_m - Qx])\, \mathrm{d}x = \int_{B_{2R}} \Delta_h p_m \operatorname{div}(\eta^2 \Delta_h[v_m - Qx])\, \mathrm{d}x. \tag{4.7}$$

Note, that at this stage we have to return to difference quotients again since we only know $\sigma_m \in W^1_{q/(q-1),\mathrm{loc}}(B_{2R};\mathbb{S})$ together with $\varepsilon(v_m) \in W^1_{2,\mathrm{loc}}(B_{2R};\mathbb{S})$ so that integrability of $\partial_k\sigma_m : \varepsilon(\partial_k v_m)$ is not immediate.

Observing $\varepsilon(\Delta_h[v_m - Qx]) = \varepsilon(\Delta_h v_m)$ and $\operatorname{div}\Delta_h(v_m - Qx) = 0$ we get from (4.7)

$$\begin{aligned}\int_{B_{2R}} \eta^2\Delta_h\sigma_m : \varepsilon(\Delta_h v_m)\,\mathrm{d}x &= \int_{B_{2R}} \eta^2\Delta_h\sigma_m : \varepsilon(\Delta_h[v_m - Qx])\,\mathrm{d}x\\ &= \int_{B_{2R}} \Delta_h\sigma_m : \varepsilon(\eta^2\Delta_h[v_m - Qx])\,\mathrm{d}x\\ &\quad -2\int_{B_{2R}} \Delta_h\sigma_m : (\nabla\eta \odot \Delta_h[v_m - Qx])\eta\,\mathrm{d}x\\ &= \int_{B_{2R}} \Delta_h p_m \operatorname{div}(\eta^2\Delta_h[v_m - Qx])\,\mathrm{d}x\\ &\quad -2\int_{B_{2R}} \eta\Delta_h\sigma_m : (\nabla\eta \odot \Delta_h[v_m - Qx])\,\mathrm{d}x,\end{aligned}$$

therefore

$$\begin{aligned}\int_{B_{2R}} \eta^2\Delta_h\sigma_m : \varepsilon(\Delta_h v_m)\,\mathrm{d}x &= 2\int_{B_{2R}} \Delta_h p_m \nabla\eta \cdot \Delta_h(v_m - Qx)\eta\,\mathrm{d}x\\ &\quad -2\int_{B_{2R}} \eta\Delta_h\sigma_m : (\nabla\eta \odot \Delta_h[v_m - Qx])\,\mathrm{d}x. \quad (4.8)\end{aligned}$$

Let $\tau_m := \sigma_m - p_m\mathbf{1}$. Since

$$\begin{aligned}&\Delta_h\sigma_m : (\nabla\eta \odot \Delta_h[v_m - Qx])\\ &\quad = \Delta_h\tau_m : (\nabla\eta \odot \Delta_h[v_m - Qx]) + \Delta_h p_m \nabla\eta \cdot \Delta_h(v_m - Qx),\end{aligned}$$

(4.8) implies

$$\int_{B_{2R}} \eta^2\Delta_h\sigma_m : \varepsilon(\Delta_h v_m)\,\mathrm{d}x = -2\int_{B_{2R}} \eta\Delta_h\tau_m : (\nabla\eta \odot \Delta_h[v_m - Qx])\,\mathrm{d}x. \quad (4.9)$$

The calculations carried out in the proof of Lemma 3.1 show $\Delta_h\sigma_m : \varepsilon(\Delta_h v_m) \geq 0$ and

$$\begin{aligned}\Delta_h\sigma_m : \varepsilon(\Delta_h v_m) &\to \partial_k\sigma_m : \varepsilon(\partial_k v_m)\\ &= D^2 f_m(\varepsilon(v_m))(\partial_k\varepsilon(v_m), \partial_k\varepsilon(v_m))\end{aligned}$$

pointwise a.e. as $h \to 0$ (on account of the weak differentiability of σ_m and $\varepsilon(v_m)$). Thus the Lemma of Fatou implies (summation over k)

$$\int_{B_{2R}} \eta^2\partial_k\sigma_m : \varepsilon(\partial_k v_m)\,\mathrm{d}x \leq \liminf_{h\to 0}\int_{B_{2R}} \eta^2\Delta_h\sigma_m : \varepsilon(\Delta_h v_m)\,\mathrm{d}x.$$

Let us look at the r. h. s. of (4.9): by Lemma 3.1 we know $|\varepsilon(v_m)|^{q/2} \in L^{2n/(n-2)}_{\mathrm{loc}}(B_{2R})$ (suppose $n \geq 3$), thus there exists some $r > q$ such that $|\varepsilon(v_m)| \in$

$L^r_{\mathrm{loc}}(B_{2R})$ (which is immediate for $n = 2$), and from $v_m \in W^1_q(B_{2R};\mathbb{R}^n)$ we also deduce $|v_m| \in L^r(B_{2R})$, hence $v_m \in W^1_{r,\mathrm{loc}}(B_{2R};\mathbb{R}^n)$ by Korn's inequality. Finally, we recall $\tau_m \in W^1_{q/(q-1),\mathrm{loc}}(B_{2R};\mathbb{S})$ and use the estimate

$$\eta|\Delta_h\tau_m||\nabla\eta||\Delta_h(v_m - Qx)| \le c(\eta)\big\{|\Delta_h\tau_m|^{l_1} + |\Delta_h(v_m - Qx)|^{l_2}\big\}$$

with suitable exponents $l_1 < q/(q-1)$, $l_2 \in (q, r)$. Thus we have equi-integrability, and since

$$\Delta_h\tau_m : (\nabla\eta \odot \Delta_h[v_m - Qx]) \overset{h\to 0}{\to} \partial_k\tau_m : (\nabla\eta \odot \partial_k[v_m - Qx]) \text{ a.e.},$$

we see by Vitali's theorem that (4.9) turns into the estimate

$$\int_{B_{2R}} \eta^2\partial_k\sigma_m : \varepsilon(\partial_k v_m)\,\mathrm{d}x \le -2\int_{B_{2R}} \eta\partial_k\tau_m : (\nabla\eta \odot \partial_k[v_m - Qx])\,\mathrm{d}x, \tag{4.10}$$

in particular, the right-hand side is well defined.

Observing $|\nabla\tau_m| \le c|\nabla\sigma_m|$ and recalling $\Gamma_m = 1 + |\varepsilon(v_m)|^2$ we get

$$\begin{aligned}&\left|\int_{B_{2R}} \eta\partial_k\tau_m : (\nabla\eta \odot \partial_k[v_m - Qx])\,\mathrm{d}x\right|\\ &\le \left(\int_{B_{2R}} \eta^2|\nabla\tau_m|^2\Gamma_m^{\frac{2-q}{2}}\,\mathrm{d}x\right)^{\frac12}\left(\int_{B_{2R}} |\nabla\eta|^2\Gamma_m^{\frac{q-2}{2}}|\nabla v_m - Q|^2\,\mathrm{d}x\right)^{\frac12}\end{aligned} \tag{4.11}$$

together with (recall the formula for $\partial_k\sigma_m$ given in Lemma 3.1 as well as (2.1))

$$\begin{aligned}&\Gamma_m^{\frac{2-q}{2}}|\nabla\tau_m|^2 \le c\Gamma_m^{\frac{2-q}{2}}|\nabla\sigma_m|^2 = c\Gamma_m^{\frac{2-q}{2}}\partial_k\sigma_m : \partial_k\sigma_m\\ &= c\Gamma_m^{\frac{2-q}{2}}D^2f_m(\varepsilon(v_m))(\partial_k\varepsilon(v_m), \partial_k\sigma_m)\\ &\le c\Gamma_m^{\frac{2-q}{2}}\big(D^2f_m(\varepsilon(v_m))(\partial_k\varepsilon(v_m), \partial_k\varepsilon(v_m))\big)^{\frac12}\big(D^2f_m(\varepsilon(v_m))(\partial_k\sigma_m, \partial_k\sigma_m)\big)^{\frac12}\\ &\le c\Gamma_m^{\frac{2-q}{4}}\big(D^2f_m(\varepsilon(v_m))(\partial_k\varepsilon(v_m), \partial_k\varepsilon(v_m))\big)^{\frac12}|\nabla\sigma_m|,\end{aligned}$$

so that

$$|\nabla\sigma_m|\Gamma_m^{\frac{2-q}{4}} \le c\big(D^2f_m(\varepsilon(v_m))(\partial_k\varepsilon(v_m), \partial_k\varepsilon(v_m))\big)^{\frac12},$$

and we get the same bound for $|\nabla\tau_m|\Gamma_m^{\frac{2-q}{4}}$. Therefore we can replace (4.11) by

$$\begin{aligned}|r.h.s.\ of\ (4.10)| \le\ & c\left(\int_{B_{2R}} \eta^2D^2f_m(\varepsilon(v_m))(\partial_k\varepsilon(v_m), \partial_k\varepsilon(v_m))\,\mathrm{d}x\right)^{\frac12}\\ &\cdot\left(\int_{B_{2R}} |\nabla\eta|^2\Gamma_m^{\frac{q-2}{2}}|\nabla v_m - Q|^2\,\mathrm{d}x\right)^{\frac12}.\end{aligned} \tag{4.12}$$

Returning to (4.10) and using Young's inequality we have established (4.4). □

Using the corollary to Lemma 4.3 we next show uniform higher integrability of the gradient.

Lemma 4.4. *There exists an exponent* $\tilde{q} > q$ *(if* $n = 2$ *we can take any* $\tilde{q} < \infty$*) such that for any ball* $B_r(\bar{x}) \Subset B_{2R}$ *we have the estimate*

$$\int_{B_r(\bar{x})} (1 + |\varepsilon(v_m)|^2)^{\frac{\tilde{q}}{2}} \,\mathrm{d}x \le \mathrm{const}\left(n, p, q, r, \bar{x}, R, \int_{B_{2R}} f(\varepsilon(u)) \,\mathrm{d}x \right) < \infty.$$

Before giving the proof of Lemma 4.4 let us draw a few conclusions.

Corollary 4.2. *For any* $r < 2R$ *we have*

$$\sup_m \|v_m\|_{W^1_{\tilde{q}}(B_r)} < \infty, \tag{4.13}$$

thus u *belongs to the space* $W^1_{\tilde{q},\mathrm{loc}}(\Omega; \mathbb{R}^n)$.

Proof. Clearly (4.13) implies the second statement since we already know $v_m \rightharpoondown u$ in $W^1_p(B_{2R}; \mathbb{R}^n)$. To prove (4.13) we observe that $\sup_m \|v_m\|_{W^1_p(B_{2R})} < \infty$ implies

$$\sup_m \|v_m\|_{L^{p_1}(B_{2R})} < \infty, \tag{4.14}$$

where we define

$$p_1 := \begin{cases} \tilde{q}, & \text{if } p \ge n, \\ np/(n-p), & \text{if } p < n. \end{cases}$$

In case $p_1 \ge \tilde{q}$ we may use Lemma 4.4, (4.14) and Korn's inequality to get (4.13). If $p_1 < \tilde{q}$ holds, then (4.14) and Korn's inequality show (again using Lemma 4.4)

$$\sup_m \|v_m\|_{W^1_{p_1}(B_r)} < \infty.$$

Thus $\sup_m \|v_m\|_{L^{p_2}(B_r)} < \infty$ for a suitable exponent p_2, and after a finite number of iterations we reach (4.13). □

Proof of Lemma 4.4. Let $n \ge 3$, the necessary adjustments needed in case $n = 2$ can be found in [BF1]. We will mainly use (4.5), and since the integral on the right-hand side of (4.5) is not supported on $B_{r'}(\bar{x}) - B_r(\bar{x})$ we have to change the arguments presented in [BF1] for the standard variational case.

Let $\chi := n/(n-2)$, $\alpha := p\chi$ and define as before $\Gamma_m := 1 + |\varepsilon(v_m)|^2$. From (4.1) it follows that $q < \alpha$. W.l.o.g. we may assume $p < q$. If $p = q$, then we replace p in the following by a slightly smaller number p^* such that still $q < p^*(1 + 2/n)$. Thus we may find $\theta \in (0, 1)$ such that

$$\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{\alpha},$$

and it is easy to see that (4.1) is equivalent to

$$\frac{q}{p}(1-\theta) < 1. \tag{4.15}$$

Let $B_r(\bar{x}) \Subset B_{2R}$ and consider $0 \le \eta \in C_0^\infty(B_{2R})$ with $\eta \equiv 1$ on $B_r(\bar{x})$. We have the following estimates

$$\begin{aligned}\int_{B_r(\bar{x})} \Gamma_m^{\frac{\alpha}{2}} \,\mathrm{d}x &\le \int_{B_{2R}} (\eta h_m)^{2\chi} \,\mathrm{d}x \le c\left(\int_{B_{2R}} |\nabla(\eta h_m)|^2 \,\mathrm{d}x\right)^\chi \\ &\le c\left\{\int_{B_{2R}} |\nabla\eta|^2 h_m^2 \,\mathrm{d}x + \int_{B_{2R}} \eta^2 |\nabla h_m|^2 \,\mathrm{d}x\right\}^\chi \\ &=: c\{T_1 + T_2\}^\chi,\end{aligned}$$

where h_m is taken from Lemma 4.2. Obviously

$$T_1 \le c\|\nabla\eta\|^2_{L^\infty(B_{2R})} \int_{B_{2R}} (1 + |\varepsilon(v_m)|^2)^{\frac{p}{2}} \,\mathrm{d}x,$$

whereas

$$T_2 \le c \int_{B_{2R}} \eta^2 D^2 f_m(\varepsilon(v_m))(\partial_k \varepsilon(v_m), \partial_k \varepsilon(v_m)) \,\mathrm{d}x. \tag{4.16}$$

Here we used the formula for ∇h_m stated in Lemma 4.2 combined with the lower bound for $D^2 f_m$. We now specify η: if $r' > r$ is such that $B_{r'}(\bar{x}) \Subset B_R$, then we let $\eta \equiv 0$ outside of $B_{(r+r')/2}(\bar{x})$, hence $|\nabla\eta| \le c/(r' - r)$.

The right-hand side of (4.16) is bounded by

$$c \int_{B_{(r'+r)/2}(\bar{x})} D^2 f_m(\varepsilon(v_m))(\partial_k \varepsilon(v_m), \partial_k \varepsilon(v_m)) \,\mathrm{d}x,$$

and on account of (4.5) this quantity is controlled by

$$c(r' - r)^{-2} \int_{B_{r'}(\bar{x})} \Gamma_m^{\frac{q}{2}} \,\mathrm{d}x.$$

Thus we have shown

$$\left(\int_{B_r(\bar{x})} \Gamma_m^{\frac{\alpha}{2}} \,\mathrm{d}x\right)^{\frac{1}{\chi}} \le c(r' - r)^{-2}\left\{\int_{B_{2R}} \Gamma_m^{\frac{p}{2}} \,\mathrm{d}x + \int_{B_{r'}(\bar{x})} \Gamma_m^{\frac{q}{2}} \,\mathrm{d}x\right\}, \tag{4.17}$$

the constant c being independent of the balls and m. Now we apply the interpolation inequality for Lebesgue spaces, i.e.

$$\|\sqrt{\Gamma_m}\|_{L^q(B_{r'}(\bar{x}))} \le \|\sqrt{\Gamma_m}\|^\theta_{L^p(B_{r'}(\bar{x}))} \|\sqrt{\Gamma_m}\|^{1-\theta}_{L^\alpha(B_{r'}(\bar{x}))}$$

and in conclusion

$$\int_{B_{r'}(\bar{x})} \Gamma_m^{\frac{q}{2}} \,\mathrm{d}x \le \left(\int_{B_{r'}(\bar{x})} \Gamma_m^{\frac{p}{2}} \,\mathrm{d}x\right)^{\theta\frac{q}{p}} \left(\int_{B_{r'}(\bar{x})} \Gamma_m^{\frac{\alpha}{2}} \,\mathrm{d}x\right)^{(1-\theta)\frac{q}{\alpha}}. \tag{4.18}$$

We have $(1 - \theta)q/\alpha = (1 - \theta)q/(p\chi) < 1/\chi$ by (4.15), so the right-hand side of (4.18) is estimated by $(0 < \delta < 1)$

$$c\delta\left(\int_{B_{r'}(\bar{x})} \Gamma_m^{\frac{\alpha}{2}} \,\mathrm{d}x\right)^{\frac{1}{\chi}} + c\delta^{-\gamma}\left(\int_{B_{r'}(\bar{x})} \Gamma_m^{\frac{p}{2}} \,\mathrm{d}x\right)^\beta$$

for suitable exponents γ, β. Next we combine (4.17), (4.18) and the latter inequality to get (with the choice $\delta = \delta'(r'-r)^2$, $\delta' \in (0,1)$):

$$\left(\int_{B_r(\bar{x})} \Gamma_m^{\frac{\alpha}{2}}\,\mathrm{d}x\right)^{\frac{1}{\chi}} \leq c(r'-r)^{-2}\int_{B_{2R}} \Gamma_m^{\frac{p}{2}}\,\mathrm{d}x$$
$$+c(\delta')(r'-r)^{-\tilde{\gamma}}\left(\int_{B_{2R}} \Gamma_m^{\frac{p}{2}}\,\mathrm{d}x\right)^{\beta} + c\delta'\left(\int_{B_{r'}(\bar{x})} \Gamma_m^{\frac{\alpha}{2}}\,\mathrm{d}x\right)^{\frac{1}{\chi}}.$$

Here $\tilde{\gamma}$ denotes another positive exponent. If we let $c\delta' = 1/2$, then we obtain

$$\left(\int_{B_r(\bar{x})} \Gamma_m^{\frac{\alpha}{2}}\,\mathrm{d}x\right)^{\frac{1}{\chi}} \leq \frac{1}{2}\left(\int_{B_{r'}(\bar{x})} \Gamma_m^{\frac{\alpha}{2}}\,\mathrm{d}x\right)^{\frac{1}{\chi}} + c(r'-r)^{-\tilde{\gamma}}\left(\int_{B_{2R}} \Gamma_m^{\frac{p}{2}}\,\mathrm{d}x\right)^{\beta}. \quad (4.19)$$

This is exactly the situation of Lemma 3.1, p. 161, of [Gi], thus (4.19) implies the growth estimate

$$\left(\int_{B_r(\bar{x})} \Gamma_m^{\frac{\alpha}{2}}\,\mathrm{d}x\right)^{\frac{1}{\chi}} \leq c(r'-r)^{-\tilde{\gamma}}\left(\int_{B_{2R}} \Gamma_m^{\frac{p}{2}}\,\mathrm{d}x\right)^{\beta},$$

for all $0 < r < r'$, $B_{r'}(\bar{x}) \Subset B_{2R}$. Recalling $\alpha > q$, the proof of Lemma 4.4 is complete, since by Lemma 4.1, iv), we know that for $m \gg 1$ the quantity $1 + \int_{B_{2R}} f(\varepsilon(u))\,\mathrm{d}x$ is a bound for $\int_{B_{2R}} f(\varepsilon(v_m))\,\mathrm{d}x$, and by the growth of f it follows that $\int_{B_{2R}} \Gamma_m^{p/2}\,\mathrm{d}x$ can be bounded in terms of $\int_{B_{2R}} f(\varepsilon(v_m))\,\mathrm{d}x$. □

According to Lemma 4.2 and using the lower bound for $D^2 f$, we obtain from (4.4) of Lemma 4.3 the inequality

$$\int_{B_{2R}} \eta^2|\nabla h_m|^2\,\mathrm{d}x \leq c\|\nabla\eta\|^2_{L^\infty(B_{2R})}\int_{\mathrm{spt}\,\nabla\eta} \Gamma_m^{\frac{q-2}{2}}|\nabla v_m - Q|^2\,\mathrm{d}x, \quad (4.20)$$

and our purpose is to establish a limit version of (4.20). First we show

Lemma 4.5. *Let $h := (1+|\varepsilon(u)|^2)^{p/4}$. Then we have:*

a) $h \in W^1_{2,\mathrm{loc}}(\Omega)$;
b) $h_m \rightharpoondown h$ *in* $W^1_{2,\mathrm{loc}}(B_{2R})$ *as* $m \to \infty$;
c) $\varepsilon(v_m) \to \varepsilon(u)$ *a.e. on* B_{2R} *as* $m \to \infty$.

Proof. As demonstrated in the proof of Corollary 4.1, (4.20) gives local bounds on $\|\nabla h_m\|_{L^2}$ in terms of local bounds for the quantity $\|\varepsilon(v_m)\|_{L^q}$, and the latter bounds just have been established in Lemma 4.4. Therefore we find a function $\tilde{h} \in W^1_{2,\mathrm{loc}}(B_{2R})$ such that $h_m \rightharpoondown \tilde{h}$ in $W^1_{2,\mathrm{loc}}(B_{2R})$ and also $h_m \to \tilde{h}$ a.e. on B_{2R} at least for a subsequence. Suppose that c) is true. Then $h_m \to h$ a.e., hence

$\tilde{h} = h$, thus we get b) for the whole sequence. For proving c) let us write

$$\int_{B_{2R}} f(\varepsilon(v_m))\,\mathrm{d}x - \int_{B_{2R}} f(\varepsilon(u))\,\mathrm{d}x = \int_{B_{2R}} Df(\varepsilon(u)) : (\varepsilon(v_m) - \varepsilon(u))\,\mathrm{d}x \tag{4.21}$$
$$+ \int_{B_{2R}} \int_0^1 D^2 f(\varepsilon(u) + t[\varepsilon(v_m) - \varepsilon(u)])(\varepsilon(v_m) - \varepsilon(u), \varepsilon(v_m) - \varepsilon(u))(1-t)\,\mathrm{d}t\,\mathrm{d}x.$$

Note that on account of $u \in W^1_{q,\mathrm{loc}}(\Omega;\mathbb{R}^n)$ and $v_m \in W^1_q(B_{2R};\mathbb{R}^n)$ all quantities on the right-hand side of (4.21) are well defined. Recall that $v_m \in u_m + \overset{\circ}{W}{}^1_q(B_{2R};\mathbb{R}^n)$, where u_m was a regularization of the function u, in particular we have

$$\|u - u_m\|_{W^1_q(\tilde{\Omega})} \to 0 \quad \text{for all } \tilde{\Omega} \Subset \Omega. \tag{4.22}$$

We have

$$\int_{B_{2R}} Df(\varepsilon(u)) : (\varepsilon(v_m) - \varepsilon(u))\,\mathrm{d}x = \int_{B_{2R}} Df(\varepsilon(u)) : (\varepsilon(v_m) - \varepsilon(u_m))\,\mathrm{d}x$$
$$+ \int_{B_{2R}} Df(\varepsilon(u)) : (\varepsilon(u_m) - \varepsilon(u))\,\mathrm{d}x\,;$$

the first term on the right-hand side is 0 due to the Euler equation satisfied by u, the second one vanishes as $m \to \infty$ on account of (4.22). By Lemma 4.1 the same is true for the left-hand side of (4.21), thus

$$\int_{B_{2R}} \int_0^1 D^2 f(\varepsilon(u) + t[\varepsilon(v_m) - \varepsilon(u)])(\varepsilon(v_m) - \varepsilon(u), \varepsilon(v_m) - \varepsilon(u))(1-t)\,\mathrm{d}t\,\mathrm{d}x \to 0$$

as $m \to \infty$. In case $p \geq 2$ the lower bound for $D^2 f$ immediately shows $\varepsilon(v_m) \to \varepsilon(u)$ a.e. on B_{2R}. If $p < 2$, then we observe

$$\int_{B_{2R}} \int_0^1 D^2 f(\dots)(\varepsilon(v_m) - \varepsilon(u), \varepsilon(v_m) - \varepsilon(u))(1-t)\,\mathrm{d}t\,\mathrm{d}x$$
$$\geq \int_{B_{2R}} \int_0^1 c(1 + |\varepsilon(u) + t[\varepsilon(v_m) - \varepsilon(u)]|^2)^{\frac{p-2}{2}} |\varepsilon(v_m) - \varepsilon(u)|^2 (1-t)\,\mathrm{d}t\,\mathrm{d}x$$
$$\geq \int_{B_{2R}} c(1 + [|\varepsilon(u)| + |\varepsilon(v_m)|]^2)^{\frac{p-2}{2}} |\varepsilon(v_m) - \varepsilon(u)|^2\,\mathrm{d}x.$$

Recall that $h_m(x) \to \tilde{h}(x)$ for a.a. $x \in B_{2R}$, which means that $|\varepsilon(v_m)|$ has a finite limit almost everywhere. On the other hand, by the above estimate

$$(1 + [|\varepsilon(u)| + |\varepsilon(v_m)|]^2)^{\frac{p-2}{2}} |\varepsilon(v_m) - \varepsilon(u)|^2 \to 0 \quad \text{a.e.},$$

altogether we have established c). □

Lemma 4.6. *Let $h := (1 + |\varepsilon(u)|^2)^{p/4}$. Then, for any $(n \times n)$-matrix Q and for all balls $B_r(\bar{x}) \subset B_{r'}(\bar{x}) \Subset B_{2R}$ we have*

$$\int_{B_r(\bar{x})} |\nabla h|^2\,\mathrm{d}x \leq c(r' - r)^{-2} \int_{B_{r'}(\bar{x})} \Gamma^{\frac{q-2}{2}} |\nabla u - Q|^2\,\mathrm{d}x, \quad \Gamma := 1 + |\varepsilon(u)|^2. \tag{4.23}$$

Remark 4.1. Since $\nabla u \in L^q_{\rm loc}(\Omega;\mathbb{R}^{n\times n})$, we can deduce from (4.23) that

$$\int_{B_{tR}} |\nabla h|^2 \,\mathrm{d}x \le c(1-t)^{-2} R^{-2} \int_{B_R} \Gamma^{\frac{q-2}{2}} |\nabla u - Q|^2 \,\mathrm{d}x \tag{4.24}$$

being valid for any ball $B_R \Subset \Omega$ and all $t \in (0,1)$

Proof of Lemma 4.6. From (4.20) it is immediate that

$$\int_{B_r(\bar{x})} |\nabla h_m|^2 \,\mathrm{d}x \le c(r'-r)^{-2} \int_{B_{r'}(\bar{x})} \Gamma_m^{\frac{q-2}{2}} |\nabla v_m - Q|^2 \,\mathrm{d}x, \tag{4.25}$$

and Lemma 4.5, b), shows that

$$\int_{B_r(\bar{x})} |\nabla h|^2 \,\mathrm{d}x \le \liminf_{m\to\infty} \big\{ l.h.s.\ of\ (4.25) \big\}.$$

Let us recall that (see (4.13))

$$\sup_m \big\{ \|v_m\|_{W^1_{\tilde q}(B_T)} + \|u\|_{W^1_{\tilde q}(B_{2R})} \big\} < \infty \tag{4.26}$$

for each $T < 2R$. Korn's inequality implies

$$\begin{aligned} \int_{B_{r'}(\bar{x})} |\nabla v_m - \nabla u|^p \,\mathrm{d}x \ &\le\ c \bigg\{ \int_{B_{r'}(\bar{x})} |v_m - u|^p \,\mathrm{d}x + \int_{B_{r'}(\bar{x})} |\varepsilon(v_m) - \varepsilon(u)|^p \,\mathrm{d}x \bigg\} \\ &=:\ c\{I_1 + I_2\}, \end{aligned}$$

$I_1 \to 0$ as $m \to \infty$ by Lemma 4.1, ii), and the same is true for I_2. To see the convergence of I_2 we note $\varepsilon(v_m) \to \varepsilon(u)$ a.e., thus $|\varepsilon(v_m) - \varepsilon(u)|^p \to 0$ a.e., whereas by (4.26) $|\varepsilon(v_m) - \varepsilon(u)|^p \rightharpoondown: \vartheta$ weakly in $L^{\tilde q/p}_{\rm loc}(B_{2R})$, thus $\vartheta = 0$ and therefore $\lim_{m\to\infty} \int_{B_{r'}(\bar{x})} |\varepsilon(v_m) - \varepsilon(u)|^p \,\mathrm{d}x = 0$. This implies $\int_{B_{r'}(\bar{x})} |\nabla v_m - \nabla u|^p \,\mathrm{d}x \to 0$, $m \to \infty$, in particular we may assume

$$\nabla v_m \to \nabla u \quad \text{a.e. on } B_{r'}(\bar{x}). \tag{4.27}$$

Finally it is clear that

$$\Gamma_m^{\frac{q-2}{2}} |\nabla v_m - Q|^2 \le c(Q)[|\nabla v_m|^q + 1],$$

hence $\Gamma_m^{(q-2)/2} |\nabla v_m - Q|^2$ (by (4.26)) is bounded in $L^{\tilde q/q}(B_{r'}(\bar{x}))$, so that

$$\Gamma_m^{\frac{q-2}{2}} |\nabla v_m - Q|^2 \rightharpoondown: \tilde\vartheta \quad \text{in } L^{\tilde q/q}(B_{r'}(\bar{x})).$$

But by (4.27) $\tilde\vartheta = \Gamma^{(q-2)/2} |\nabla u - Q|^2$ and we obtain

$$\lim_{m\to\infty} \int_{B_{r'}(\bar{x})} \Gamma_m^{\frac{q-2}{2}} |\nabla v_m - Q|^2 \,\mathrm{d}x = \int_{B_{r'}(\bar{x})} \Gamma^{\frac{q-2}{2}} |\nabla u - Q|^2 \,\mathrm{d}x,$$

so that (4.23) is established. □

5. Partial regularity via blow-up

In this section we adjust the well-known blow-up arguments (implying partial regularity for vector-valued problems in standard variational calculus) to the situation studied here. We like to mention that the blow-up technique was originated by Evans and Gariepy ([EG]) and that similar techniques were used earlier in the setting of Geometric Measure Theory.

So, assume that we are in the situation of Theorem 2.1, a), and define the excess of u w.r.t. a ball $B_r(x) \Subset \Omega$

$$E(u, B_r(x)) := \fint_{B_r(x)} |\varepsilon(u) - (\varepsilon(u))_{x,r}|^2 \,\mathrm{d}z + \fint_{B_r(x)} |\varepsilon(u) - (\varepsilon(u))_{x,r}|^q \,\mathrm{d}z,$$

$(\dots)_{x,r}$ and $\fint_{B_r(x)} \dots$ denoting mean values. Recall that on account of Corollary 4.2 $E(u, B_r(x))$ is well defined.

We will make use of a Campanato-type estimate, which can be traced in [GM1], a proof is also given in [FS1], Lemma 3.0.5, v).

Lemma 5.1. *Consider a matrix $A \in \mathbb{S}$ such that $|A| \leq L$. Let $w \in W^1_2(B_1;\mathbb{R}^n)$,* $\operatorname{div} w = 0$*, satisfy*

$$\int_{B_1} D^2 f(A)(\varepsilon(w), \varepsilon(\varphi)) \,\mathrm{d}z = 0$$

for all $\varphi \in \overset{\circ}{W}{}^1_2(B_1;\mathbb{R}^n)$, $\operatorname{div} \varphi = 0$. Then there is a constant $C^* = C(n,p,q,L)$ such that

$$\fint_{B_\tau} |\varepsilon(w) - (\varepsilon(w))_\tau|^2 \,\mathrm{d}z \leq C^* \tau^2 \fint_{B_1} |\varepsilon(w) - (\varepsilon(w))_1|^2 \,\mathrm{d}z$$

for any $\tau \in (0,1)$.

Remark 5.1. The constant C^* – according to [FS1], Lemma 3.0.5, v) – depends on the ellipticity constants of the form $D^2 f(A)$. Since

$$\lambda(1+|A|^2)^{\frac{p-2}{2}} |\varepsilon|^2 \leq D^2 f(A)(\varepsilon,\varepsilon) \leq \Lambda(1+|A|^2)^{\frac{q-2}{2}} |\varepsilon|^2$$

we deduce in case $p \geq 2$

$$\lambda|\varepsilon|^2 \leq D^2 f(A)(\varepsilon,\varepsilon) \leq \Lambda(1+L^2)^{\frac{q-2}{2}} |\varepsilon|^2,$$

whereas for $p < 2$ we get

$$\lambda(1+L^2)^{\frac{p-2}{2}} |\varepsilon|^2 \leq D^2 f(A)(\varepsilon,\varepsilon) \leq \Lambda(1+L^2)^{\frac{q-2}{2}} |\varepsilon|^2,$$

thus C^* is independent of the particular matrix A.

Lemma 5.2. (Blow-Up Lemma) *Given $L > 0$ we let $C_* := 2C^*$. Then, for any $\tau \in (0, 1/4)$, there exists a number $\varepsilon = \varepsilon(L, \tau)$ with the following property: if for some ball $B_r(x) \Subset \Omega$ we have*

$$|(\varepsilon(u))_{x,r}| \le L, \quad E(x,r) := E(u, B_r(x)) < \varepsilon,$$

then

$$E(x, \tau r) \le C_* \tau^2 E(x, r).$$

By iterating this result we obtain:

Corollary 5.1. *Let*

$$\Omega_0 := \big\{ x \in \Omega : \sup_{r>0} |(\varepsilon(u))_{x,r}| < \infty \text{ and } \liminf_{r \downarrow 0} E(x,r) = 0 \big\}.$$

Then Ω_0 is an open set of full Lebesgue measure and $\varepsilon(u)$ is of class $C^{0,\alpha}$ on Ω_0 for any $0 < \alpha < 1$.

From this we immediately obtain Theorem 2.1, a): let $\omega \Subset \Omega_0$ denote an open set and observe $\varepsilon(u) \in C^{0,\alpha}(\overline{\omega})$. Fix $\varphi \in C_0^1(\omega)$, $\operatorname{div} \varphi = 0$; then for $|h| \ll 1$ we get

$$\int_\omega \Delta_h \big\{ DF(\varepsilon(u)) \big\} : \varepsilon(\varphi)\, \mathrm{d}x = 0. \tag{5.1}$$

Recall

$$\Delta_h \big\{ DF(\varepsilon(u)) \big\} = \int_0^1 D^2 f(\varepsilon(u) + th\varepsilon(\Delta_h u))(\varepsilon(\Delta_h u), \cdot) \tag{5.2}$$

and observe that on account of the regularity of $\varepsilon(u)$ we may repeat the arguments of the proof of Lemma 3.1 (replace s by q there) with the result that $\varepsilon(\Delta_h u)$ is locally bounded in $L^2(\omega; \mathbb{S})$. In fact, this follows from the corresponding version of (3.9) where now $\omega(r)$ is seen to be an upper bound for $|\varepsilon(\Delta_h u)|^2$ by observing that $D^2 f(\xi)$ is evaluated on a bounded set of matrices ξ. Thus $u \in W^2_{2,\mathrm{loc}}(\omega; \mathbb{R}^n)$ and from (5.2) we get

$$\Delta_h \big\{ Df(\varepsilon(u)) \big\} \overset{h \to 0}{\to} D^2 f(\varepsilon(u))(\varepsilon(\partial_k u), \cdot) \quad \text{in } L^2_{\mathrm{loc}}(\omega; \mathbb{S}),$$

since obviously

$$\int_0^1 D^2 f(\varepsilon(u) + th\varepsilon(\Delta_h u))\, \mathrm{d}t \overset{h \to 0}{\to} D^2 f(\varepsilon(u)) \quad \text{in } L^\infty_{\mathrm{loc}}(\omega; \mathbb{S}).$$

Altogether we obtain the limit version of (5.1), i.e.

$$\int_\omega D^2 f(\varepsilon(u))(\varepsilon(\partial_k u), \varepsilon(\varphi))\, \mathrm{d}x = 0 \tag{5.3}$$

for all $\varphi \in C_0^1(\omega; \mathbb{R}^n)$, $\operatorname{div} \varphi = 0$, and any $k = 1, \ldots, n$. (5.3) can be seen as an elliptic system with continuous matrix $D^2 f(\varepsilon(u))$ for the function $\partial_k u$, thus

Theorem 1.1 of [GM1] implies $\partial_k u \in C^{0,\alpha}(\omega)$, which gives the claim of Theorem 2.1, a).

We now come to the

Proof of Lemma 5.2. To argue by contradiction, assume that $L > 0$ is fixed and that for some $\tau \in (0, 1/4)$ there exists a sequence of balls $B_{r_m}(x_m) \Subset \Omega$ such that

$$|(\varepsilon(u))_{x_m,r_m}| \le L, \quad E(x_m, r_m) =: \lambda_m^2 \overset{m\to\infty}{\to} 0 \tag{5.4}$$

and

$$E(x_m, \tau r_m) > C_* \tau^2 \lambda_m^2. \tag{5.5}$$

(Of course we could also replace u by a sequence of local minimizers in order to get the statements independent of the particular local minimizer.) We let $A_m := (\varepsilon(u))_{x_m,r_m}$ and

$$u_m(z) := \frac{1}{\lambda_m r_m}\big[u(x_m + r_m z) - r_m A_m z - \gamma_m(z)\big], \quad |z| < 1,$$

where γ_m is a rigid motion such that

$$\int_{B_1} |u_m|^2 \,\mathrm{d}z \le c \int_{B_1} |\varepsilon(u_m)|^2 \,\mathrm{d}z. \tag{5.6}$$

Writing $\gamma_m(z) = B_m z + a_m$ with B_m skew-symmetric and $a_m \in \mathbb{R}^n$ we have

$$\begin{aligned} \nabla u_m(z) &= \frac{1}{\lambda_m}\big[\nabla u(x_m + r_m z) - A_m - \frac{1}{r_m} B_m\big], \\ \varepsilon(u_m)(z) &= \frac{1}{\lambda_m}\big[\varepsilon(u)(x_m + r_m z) - A_m\big], \end{aligned}$$

and (5.4) implies

$$\fint_{B_1} |\varepsilon(u_m)|^2 \,\mathrm{d}z + \lambda_m^{q-2} \fint_{B_1} |\varepsilon(u_m)|^q \,\mathrm{d}z = 1. \tag{5.7}$$

By (5.6), (5.7) we see

$$\sup_m \big\{ \|u_m\|_{L^2(B_1)} + \|\varepsilon(u_m)\|_{L^2(B_1)} \big\} < \infty,$$

and Korn's inequality gives boundedness of $\{u_m\}$ in $W_2^1(B_1; \mathbb{R}^n)$, thus we may assume (for a subsequence)

$$u_m \rightharpoonup: \hat{u} \quad \text{in } W_2^1(B_1; \mathbb{R}^n) \tag{5.8}$$

for a function $\hat{u}$ from this space which satisfies $\operatorname{div} \hat{u} = 0$ (note that u_m is a solenoidal field). Clearly (5.8) gives

$$\lambda_m \nabla u_m \to 0 \quad \text{in } L^2(B_1; \mathbb{R}^{n\times n}) \text{ and a.e.} \tag{5.9}$$

By (5.7) $\lambda_m^{1-2/q}\varepsilon(u_m)$ is bounded in $L^q(B_1;\mathbb{S})$, thus weakly convergent towards a tensor field $\bar{\varepsilon}$. By (5.8) $\varepsilon(u_m) \rightharpoondown \varepsilon(\hat{u})$ in $L^2(B_1;\mathbb{S})$, thus

$$\lambda_m^{1-\frac{2}{q}}\varepsilon(u_m) \overset{m\to\infty}{\rightharpoondown} 0 \quad \text{in } L^q(B_1;\mathbb{S}), \text{ if } q>2. \tag{5.10}$$

From (5.5) we deduce after scaling

$$\fint_{B_\tau} |\varepsilon(u_m) - (\varepsilon(u_m))_\tau|^2 \,\mathrm{d}z + \lambda_m^{q-2} \fint_{B_\tau} |\varepsilon(u_m) - (\varepsilon(u_m))_\tau|^q \,\mathrm{d}z > C_* \tau^2. \tag{5.11}$$

Finally (compare (5.4)), we have (for some subsequence)

$$A_m \overset{m\to\infty}{\to:} A \in \mathbb{S}, \quad |A| \le L. \tag{5.12}$$

We claim

$$\lambda_m^{1-\frac{2}{q}}\nabla u_m \overset{m\to\infty}{\rightharpoondown} 0 \quad \text{in } L^q(B_1;\mathbb{R}^{n\times n}), \text{ if } q>2 \tag{5.13}$$

together with

$$\int_{B_1} \lambda_m^{q-2}|u_m|^q \,\mathrm{d}z \overset{m\to\infty}{\to} 0, \quad \text{if } q>2. \tag{5.14}$$

To see this we use the interpolation inequality Lemma 3.0.2 of [FS1], being valid on account of $u_m \in W^1_q(B_1;\mathbb{R}^n)$:

$$\|u_m\|_{L^q(B_1)} \le c\big\{\|u_m\|_{L^2(B_1)} + \|\varepsilon(u_m)\|_{L^q(B_1)}. \tag{5.15}$$

Korn's inequality then implies

$$\|\nabla u_m\|_{L^q(B_1)} \le c\big\{\|u_m\|_{L^q(B_1)} + \|\varepsilon(u_m)\|_{L^q(B_1)}\big\} \le c\big\{1 + \|\varepsilon(u_m)\|_{L^q(B_1)}\big\}$$

on account of $\sup_m \|u_m\|_{L^2(B_1)} < \infty$. This shows

$$\lambda_m^{q-2}\int_{B_1} |\nabla u_m|^q \,\mathrm{d}z \le c\left\{1 + \lambda_m^{q-2}\int_{B_1} |\varepsilon(u_m)|^q \,\mathrm{d}z\right\}$$

and by (5.7) we get boundedness of $\lambda_m^{q-2}\int_{B_1}|\nabla u_m|^q \,\mathrm{d}x$. Now the same reasoning leading to (5.10) (using (5.8) again) implies (5.13). For (5.14) we observe (recalling (5.15))

$$\lambda_m^{q-2}\int_{B_1} |u_m|^q \,\mathrm{d}z \le c\left\{1 + \lambda_m^{q-2}\int_{B_1} |\varepsilon(u_m)|^q \,\mathrm{d}z\right\},$$

thus $\bar{u}_m := \lambda_m^{1-2/q}u_m$ is bounded in $W^1_q(B_1;\mathbb{R}^n)$, and we may assume $\bar{u}_m \rightharpoondown: \bar{u}$ in $W^1_q(B_1;\mathbb{R}^n)$, $\bar{u}_m \to \bar{u}$ in $L^q(B_1;\mathbb{R}^n)$. Let $q>2$. Then $\bar{u}_m \to \bar{u}$ in $L^2(B_1;\mathbb{R}^n)$, hence $\bar{u}=0$ on account of (5.8).

After these preparations we now show that the limit $\hat{u}$ from (5.8) satisfies a nice equation, i.e. we claim

Proposition 5.1. *We have*

$$\int_{B_1} D^2 f(A)(\varepsilon(\hat{u}), \varepsilon(\varphi))\, \mathrm{d}z = 0$$

being valid for all $\varphi \in C_0^1(B_1; \mathbb{R}^n)$, φ *being solenoidal.*

Proof of Proposition 5.1. We have

$$\int_{B_{r_m}(x_m)} Df(\varepsilon(u)) : \varepsilon(\psi)\, \mathrm{d}x = 0 \quad \text{for all } \psi \in C_0^1(B_{r_m}(x_m); \mathbb{R}^n),\ \operatorname{div} \psi = 0,$$

thus after scaling

$$\int_{B_1} Df(\lambda_m \varepsilon(u_m) + A_m) : \varepsilon(\varphi)\, \mathrm{d}z = 0 \quad \text{for all } \varphi \in C_0^1(B_1; \mathbb{R}^n),\ \operatorname{div} \varphi = 0.$$

We rewrite this in the form

$$\lambda_m^{-1} \int_{B_1} \big[Df(\lambda_m \varepsilon(u_m) + A_m) - Df(A_m)\big] : \varepsilon(\varphi)\, \mathrm{d}z = 0$$

and observe

$$\begin{aligned} &\lambda_m^{-1}\big[Df(\lambda_m \varepsilon(u_m) + A_m) - Df(A_m)\big] \\ &= \int_0^1 D^2 f(A_m + t\lambda_m \varepsilon(u_m))(\varepsilon(u_m), \cdot)\, \mathrm{d}t \end{aligned}$$

to obtain

$$\begin{aligned} &\int_{B_1} D^2 f(A_m)(\varepsilon(u_m), \varepsilon(\varphi))\, \mathrm{d}z \qquad (5.16) \\ &= -\int_{B_1} \Bigg\{ \int_0^1 D^2 f(A_m + t\lambda_m \varepsilon(u_m))(\varepsilon(u_m), \varepsilon(\varphi))\, \mathrm{d}t \\ &\quad - D^2 f(A_m)(\varepsilon(u_m), \varepsilon(\varphi)) \Bigg\}\, \mathrm{d}z. \end{aligned}$$

By (5.6) and (5.8) we see

$$l.h.s.\ of\ (5.16) \overset{m\to\infty}{\to} \int_{B_1} D^2 f(A)(\varepsilon(\hat{u}), \varepsilon(\varphi))\, \mathrm{d}z. \qquad (5.17)$$

Given $\delta > 0$ we use (5.9) and Egoroff's theorem to find a subset M of B_1 such that $\mathcal{L}^n(M) < \delta$ and

$$\lambda_m \nabla u_m \overset{m\to\infty}{\to} 0 \quad \text{uniformly on } B_1 - M.$$

Since $\varepsilon(u_m)$ is bounded in $L^2(B_1; \mathbb{S})$, we deduce

$$\int_{B_1 - M} \Bigg\{ \int_0^1 D^2 f(A_m + t\lambda_m \varepsilon(u_m))(\varepsilon(u_m), \varepsilon(\varphi))\, \mathrm{d}t - D^2 f(A_m)(\varepsilon(u_m), \varepsilon(\varphi)) \Bigg\} \mathrm{d}z \overset{m\to\infty}{\to} 0,$$

in particular there exists m_δ such that

$$\left| \int_{B_1 - M} \left\{ \int_0^1 \dots \right\} \mathrm{d}z \right| \le \delta \quad \text{for all } m \ge m_\delta. \tag{5.18}$$

On M we argue as follows:

$$\begin{aligned} \left| \int_M \left\{ \int_0^1 \dots \right\} \mathrm{d}z \right| \;&\le\; \int_M \int_0^1 |D^2 f(A_m + t\lambda_m \varepsilon(u_m))||\varepsilon(u_m)||\varepsilon(\varphi)| \,\mathrm{d}t\,\mathrm{d}z \\ &\quad + \sup_m |D^2 f(A_m)| \|\varepsilon(\varphi)\|_{L^\infty(B_1)} \int_M |\varepsilon(u_m)| \,\mathrm{d}z \\ &=: I + \mathit{II}, \end{aligned}$$

$$\begin{aligned} \mathit{II} \;&\le\; \sup_m |D^2 f(A_m)| \|\varepsilon(\varphi)\|_{L^\infty(B_1)} \mathcal{L}^n(M)^{\frac{1}{2}} \|\varepsilon(u_m)\|_{L^2(B_1)} \le c(\varphi)\sqrt{\delta}, \\ I \;&\le\; c \int_M \int_0^1 (1 + |A_m + t\lambda_m \varepsilon(u_m)|^2)^{\frac{q-2}{2}} |\varepsilon(\varphi)||\varepsilon(u_m)| \,\mathrm{d}t\,\mathrm{d}z \\ &\le\; c(\varphi) \left\{ \int_M |\varepsilon(u_m)| \,\mathrm{d}z + \int_M \lambda_m^{q-2} |\varepsilon(u_m)|^{q-1} \,\mathrm{d}z \right\} \\ &\le\; c(\varphi) \left\{ \sqrt{\delta}\, \|\varepsilon(u_m)\|_{L^2(B_1)} + \lambda_m^{q-2} \int_M |\varepsilon(u_m)|^{q-1} \,\mathrm{d}z \right\}, \end{aligned}$$

where in case $q = 2$ the quantity I is seen to be bounded by $\sqrt{\delta}\, \|\varepsilon(u_m)\|_{L^2(B_1)}$. If $q > 2$, then we estimate (recalling (5.7)

$$\begin{aligned} \lambda_m^{q-2} \int_M |\varepsilon(u_m)|^{q-1} \,\mathrm{d}z \;&\le\; \delta \lambda_m^{q-2} \int_{B_1} |\varepsilon(u_m)|^q \,\mathrm{d}z + \lambda_m^{q-2} c(\delta) \mathcal{L}^n(M) \\ &\le\; c\delta + c(\delta)\lambda_m^{q-2}. \end{aligned}$$

For $m \ge \tilde{m}_\delta$ we obtain $c(\delta)\lambda_m^{q-2} \le \delta$, thus $I + \mathit{II} \le c\sqrt{\delta}$ (w.l.o.g. $\delta < 1$) for all $m \ge \tilde{m}_\delta$. Taking into account (5.18), we see that the r.h.s. of (5.16) vanishes as $m \to \infty$. This together with (5.17) proves Proposition 5.1. □

By Proposition 5.1 we may apply Lemma 5.1 to the function $\hat{u}$ with the result

$$⨍_{B_\tau} |\varepsilon(\hat{u}) - (\varepsilon(\hat{u}))_\tau|^2 \,\mathrm{d}z \le C^* \tau^2 ⨍_{B_1} |\varepsilon(\hat{u}) - (\varepsilon(\hat{u}))_1|^2 \,\mathrm{d}z. \tag{5.19}$$

From $(\varepsilon(u_m))_1 = 0$ and $\varepsilon(u_m) \rightharpoonup \varepsilon(\hat{u})$ in $L^2(B_1, \mathbb{S})$ (see (5.8)) we find $(\varepsilon(\hat{u}))_1 = 0$ and therefore

$$⨍_{B_1} |\varepsilon(\hat{u}) - (\varepsilon(\hat{u}))_1|^2 \,\mathrm{d}z = ⨍_{B_1} |\varepsilon(\hat{u})|^2 \,\mathrm{d}z \le \liminf_{m\to\infty} ⨍_{B_1} |\varepsilon(u_m)|^2 \,\mathrm{d}z \le 1,$$

where the last inequality is due to (5.7). Suppose that we can show $\varepsilon(u_m) \to \varepsilon(\hat{u})$ strongly in $L^2_{\mathrm{loc}}(B_1;\mathbb{S})$. Then we get from (5.19)

$$\lim_{m\to\infty} \fint_{B_\tau} |\varepsilon(u_m) - (\varepsilon(u_m))_\tau|^2 \,\mathrm{d}z \le C^* \tau^2,$$

and if in addition for $q > 2$ we can also establish (compare (5.10)) $\lambda_m^{1-2/q}\varepsilon(u_m) \to 0$ strongly in $L^q_{\mathrm{loc}}(B_1;\mathbb{S})$, then the latter inequality implies

$$\lim_{m\to\infty} \left\{ \fint_{B_\tau} |\varepsilon(u_m) - (\varepsilon(u_m))_\tau|^2 \,\mathrm{d}z + \lambda_m^{q-2} \fint_{B_\tau} |\varepsilon(u_m) - (\varepsilon(u_m))_\tau|^q \,\mathrm{d}z \right\} \le C^* \tau^2$$

contradicting (5.11) on account of $C_* = 2C^*$.

We therefore have to show

Proposition 5.2. *The weak convergences as stated in* (5.8) *and* (5.10) *can be improved to local strong convergence, i.e. we have as $m \to \infty$*

i) $\varepsilon(u_m) \to \varepsilon(\hat{u})$ *in* $L^2_{\mathrm{loc}}(B_1;\mathbb{S})$*;*

ii) $\lambda_m^{1-\frac{2}{q}}\varepsilon(u_m) \to 0$ *in* $L^q_{\mathrm{loc}}(B_1;\mathbb{S})$*, if* $q > 2$.

Proof of Proposition 5.2. We start with the proof of the following identity ($w_m := u_m - \hat{u}$)

$$\lim_{m\to\infty} \int_{B_\rho}\int_0^1 (1+|A_m+\lambda_m\varepsilon(\hat{u})+t\lambda_m\varepsilon(w_m)|^2)^{\frac{p-2}{2}} |\varepsilon(w_m)|^2 (1-t)\,\mathrm{d}t\,\mathrm{d}z = 0 \quad (5.20)$$

for any $0 < \rho < 1$. Let $\varphi \in C^1_0(B_1)$ be non-negative with $\varphi \equiv 1$ on B_ρ and $\varphi \equiv 0$ on $B_1 - B_r$, where $\rho < r < 1$, and consider a sequence $\varphi_m \in \overset{\circ}{W}{}^1_q(B_r;\mathbb{R}^n)$ such that

$$\operatorname{div}\varphi_m = \operatorname{div}(u_m + \varphi[\hat{u} - u_m])$$

together with

$$\|\nabla\varphi_m\|_{L^q(B_r)} \le c\|\nabla\varphi\cdot(\hat{u} - u_m)\|_{L^q(B_r)}.$$

Minimality of u gives (by scaling)

$$\int_{B_r} f(A_m + \lambda_m\varepsilon(u_m))\,\mathrm{d}z \le \int_{B_r} f(A_m + \lambda_m\varepsilon(u_m + \varphi[\hat{u} - u_m] - \varphi_m))\,\mathrm{d}z. \quad (5.21)$$

Let us write (by Taylor expansion)

$$\begin{aligned} &\int_{B_r}\int_0^1 (1-t)\varphi D^2 f(A_m + \lambda_m\varepsilon(\hat{u}) + t\lambda_m\varepsilon(w_m))(\varepsilon(w_m),\varepsilon(w_m))\,\mathrm{d}t\,\mathrm{d}z \\ &= \lambda_m^{-2}\int_{B_r}\varphi\big\{f(A_m+\lambda_m\varepsilon(u_m)) - f(A_m+\lambda_m\varepsilon(\hat{u})\big\}\,\mathrm{d}z \\ &\quad -\lambda_m^{-1}\int_{B_r}\varphi Df(A_m+\lambda_m\varepsilon(\hat{u})) : \varepsilon(w_m)\,\mathrm{d}z. \end{aligned} \quad (5.22)$$

The first integral on the right-hand side of (5.22) can be rewritten as (using (5.21))

$$\begin{aligned}
&\int_{B_r} f(A_m + \lambda_m \varepsilon(u_m))\,\mathrm{d}z \\
&\quad - \int_{B_r} \big[(1-\varphi) f(A_m + \lambda_m \varepsilon(u_m)) + \varphi f(A_m + \lambda_m \varepsilon(\hat{u}))\big]\,\mathrm{d}z \\
&\leq \int_{B_r} f(A_m + \lambda_m \varepsilon(u_m + \varphi[\hat{u} - u_m] - \varphi_m))\,\mathrm{d}z \\
&\quad - \int_{B_r} f(A_m + \lambda_m[(1-\varphi)\varepsilon(u_m) + \varphi\varepsilon(\hat{u})])\,\mathrm{d}z,
\end{aligned}$$

where we also used the convexity of f. We obtain

$$\begin{aligned}
\text{r.h.s. of (5.22)} \ \leq\ & \lambda_m^{-2}\bigg\{ \int_{B_r} f(A_m + \lambda_m \varepsilon(u_m + \varphi[\hat{u} - u_m] - \varphi_m))\,\mathrm{d}z \\
& - \int_{B_r} f(A_m + \lambda_m[(1-\varphi)\varepsilon(u_m) + \varphi\varepsilon(\hat{u})])\,\mathrm{d}z \bigg\} \\
& - \lambda_m^{-1} \int_{B_r} \varphi D f(A_m + \lambda_m \varepsilon(\hat{u})) : \varepsilon(w_m)\,\mathrm{d}z \\
=: \ & \lambda_m^{-2}\{I_1 - I_2\} - \lambda_m^{-1} I_3. \qquad (5.23)
\end{aligned}$$

Let $\varepsilon_m := A_m + \lambda_m((1-\varphi)\varepsilon(u_m) + \varphi\varepsilon(\hat{u}))$. Then

$$\begin{aligned}
\lambda_m^{-2}\{I_1 - I_2\} \ =\ & \lambda_m^{-2}\bigg\{ \int_{B_r} f(\varepsilon_m + \lambda_m((\hat{u} - u_m)\odot\nabla\varphi - \varepsilon(\varphi_m)))\,\mathrm{d}z - \int_{B_r} f(\varepsilon_m)\,\mathrm{d}z \bigg\} \\
=\ & \lambda_m^{-1} \int_{B_r} D f(\varepsilon_m) : \{(\hat{u} - u_m)\odot\nabla\varphi - \varepsilon(\varphi_m)\}\,\mathrm{d}z \\
& + \int_{B_r} \int_0^1 D^2 f(\varepsilon_m + t\lambda_m[(\hat{u} - u_m)\odot\nabla\varphi - \varepsilon(\varphi_m)]) \\
& (\nabla\varphi\odot[\hat{u} - u_m] - \varepsilon(\varphi_m), \nabla\varphi\odot[\hat{u} - u_m] - \varepsilon(\varphi_m))(1-t)\,\mathrm{d}t\,\mathrm{d}z \\
=: \ & \lambda_m^{-1} I_4 + I_5,
\end{aligned}$$

and for I_5 we obtain due to the growth of $D^2 f$:

$$\begin{aligned}
I_5 \ \leq\ & c \int_{B_r} \big(1 + |\varepsilon_m|^2 + \lambda_m^2 |\hat{u} - u_m|^2 |\nabla\varphi|^2 + \lambda_m^2 |\varepsilon(\varphi_m)|^2\big)^{\frac{q-2}{2}} \\
& \cdot\big(|\nabla\varphi|^2 |\hat{u} - u_m|^2 + |\varepsilon(\varphi_m)|^2\big)\,\mathrm{d}z \\
\leq\ & c \int_{B_r} \big(1 + |\varepsilon_m|^{q-2} + \lambda_m^{q-2} |\hat{u} - u_m|^{q-2} |\nabla\varphi|^{q-2} + \lambda_m^{q-2} |\varepsilon(\varphi_m)|^{q-2}\big) \\
& \cdot\big(|\nabla\varphi|^2 |\hat{u} - u_m|^2 + |\varepsilon(\varphi_m)|^2\big)\,\mathrm{d}z
\end{aligned}$$

$$
\begin{aligned}
&= c\bigg\{ \int_{B_r} (|\nabla\varphi|^2|\hat{u}-u_m|^2 + |\varepsilon(\varphi_m)|^2)\,\mathrm{d}z + \int_{B_r} \lambda_m^{q-2}|\nabla\varphi|^q|\hat{u}-u_m|^q\,\mathrm{d}z \\
&\quad + \int_{B_r} |\varepsilon(\varphi_m)|^2\lambda_m^{q-2}|\hat{u}-u_m|^{q-2}|\nabla\varphi|^{q-2}\,\mathrm{d}z \\
&\quad + \int_{B_r} |\nabla\varphi|^2|\hat{u}-u_m|^2|\varepsilon_m|^{q-2}\,\mathrm{d}z + \int_{B_r} |\varepsilon_m|^{q-2}|\varepsilon(\varphi_m)|^2\,\mathrm{d}z \\
&\quad + \lambda_m^{q-2}\int_{B_r} |\varepsilon(\varphi_m)|^{q-2}|\nabla\varphi|^2|\hat{u}-u_m|^2\,\mathrm{d}z + \lambda_m^{q-2}\int_{B_r}|\varepsilon(\varphi_m)|^q\,\mathrm{d}z\bigg\} \\
&=: c\{K_1+\cdots+K_7\}.
\end{aligned}
$$

Let us recall

$$\int_{B_r} |\nabla\varphi_m|^q\,\mathrm{d}z \le c\int_{B_r}|\nabla\varphi|^q|u_m-\hat{u}|^q\,\mathrm{d}z,$$

and according to Lemma 3.0.4 of [FS1] this estimate also holds with $q=2$. This together with (5.8) shows

$$K_1 \to 0 \quad \text{as } m\to\infty.$$

If $q>2$, we deduce from (5.14) and $\hat{u}\in L^\infty_{\mathrm{loc}}(B_1;\mathbb{R}^n)$ that

$$K_2 \to 0 \quad \text{as } m\to\infty,$$

which is obvious for $q=2$. For K_3 we again let $q>2$ and get

$$K_3 \le \lambda_m^{q-2}\bigg\{\int_{B_1}|\hat{u}-u_m|^q|\nabla\varphi|^q\,\mathrm{d}z\bigg\}^{1-\frac{2}{q}}\bigg\{\int_{B_1}|\varepsilon(\varphi_m)|^q\,\mathrm{d}z\bigg\}^{\frac{2}{q}},$$

thus by (5.14) $K_3\to 0$ as $m\to\infty$. The same reasoning applies to K_6 and K_7. By definition

$$
\begin{aligned}
|\varepsilon_m| &\le c\big(1+\lambda_m|\varepsilon(u_m)|+\lambda_m\varphi|\varepsilon(u_m-\hat{u})|\big), \text{ i.e.}\\
|\varepsilon_m|^{q-2} &\le c\big(1+\lambda_m^{q-2}|\varepsilon(u_m)|^{q-2}+\lambda_m^{q-2}\varphi^{q-2}|\varepsilon(u_m-\hat{u})|^{q-2}\big),
\end{aligned}
$$

and we obtain

$$
\begin{aligned}
K_4 &\le c\bigg\{\int_{B_1}|\nabla\varphi|^2|\hat{u}-u_m|^2\,\mathrm{d}z + \lambda_m^{q-2}\int_{B_1}|\nabla\varphi|^2|\varepsilon(u_m)|^{q-2}|\hat{u}-u_m|^2\,\mathrm{d}z \\
&\quad + \lambda_m^{q-2}\int_{B_1}|\nabla\varphi|^2\varphi^{q-2}|\varepsilon(u_m-\hat{u})|^{q-2}|\hat{u}-u_m|^2\,\mathrm{d}z\bigg\}.
\end{aligned}
$$

From this estimate it follows that $K_4\to 0$ as $m\to\infty$, since for example in case

$q > 2$

$$\lambda_m^{q-2} \int_{B_1} |\nabla\varphi|^2 \varphi^{q-2} |\varepsilon(u_m - \hat{u})|^{q-2} |\hat{u} - u_m|^2 \, \mathrm{d}z$$
$$\leq c(\varphi) \lambda_m^{q-2} \left\{ \int_{B_1 \cap \operatorname{spt} \varphi} |\varepsilon(u_m - \hat{u})|^q \, \mathrm{d}z \right\}^{1-\frac{2}{q}} \left\{ \int_{B_1 \cap \operatorname{spt} \varphi} |u_m - \hat{u}|^q \, \mathrm{d}z \right\}^{\frac{2}{q}}.$$

Then we can use (5.14) and the L^q-boundedness of $\lambda_m^{1-2/q} \varepsilon(u_m)$ (see (5.7)). We leave the discussion of K_5 to the reader.

Putting together our estimates and going back to (5.23) we have shown that

$$r.h.s.\ of\ (5.22) \leq \lambda_m^{-1}[I_4 - I_3] + O(m), \tag{5.24}$$

where $O(m) \to 0$ as $m \to \infty$. Let us look at $\lambda_m^{-1}[I_4 - I_3]$:

$$\begin{aligned}
\lambda_m^{-1}[I_4 - I_3] &= \lambda_m^{-1} \left\{ \int_{B_r} Df(\varepsilon_m) : [(\hat{u} - u_m) \odot \nabla\varphi - \varepsilon(\varphi_m)] \, \mathrm{d}z \right. \\
&\quad \left. - \int_{B_r} \varphi Df(A_m + \lambda_m \varepsilon(\hat{u})) : \varepsilon(w_m) \, \mathrm{d}z \right\} \\
&= \lambda_m^{-1} \left\{ \int_{B_r} \left\{ Df(\varepsilon_m) - Df(A_m + \lambda_m \varepsilon(\hat{u})) \right\} : (\hat{u} - u_m) \odot \nabla\varphi \, \mathrm{d}z \right. \\
&\quad \left. - \int_{B_r} Df(A_m + \lambda_m \varepsilon(\hat{u})) : \varepsilon(\varphi w_m) \, \mathrm{d}z - \int_{B_r} Df(\varepsilon_m) : \varepsilon(\varphi_m) \, \mathrm{d}z \right\} \\
&= \lambda_m^{-1} \left\{ \int_{B_r} [Df(\varepsilon_m) - Df(A_m + \lambda_m \varepsilon(\hat{u}))] : (\hat{u} - u_m) \odot \nabla\varphi \, \mathrm{d}z \right. \\
&\quad - \int_{B_r} [Df(\varepsilon_m) - Df(A_m + \lambda_m \varepsilon(\hat{u}))] : \varepsilon(\varphi_m) \, \mathrm{d}z \\
&\quad \left. - \int_{B_r} Df(A_m + \lambda_m \varepsilon(\hat{u})) : (\varepsilon(\varphi_m) + \varepsilon(\varphi w_m)) dz \right\} \\
&=: \lambda_m^{-1} \{ J_1 - J_2 - J_3 \},
\end{aligned}$$

$$\begin{aligned}
|J_1| &= \lambda_m \left| \int_{B_r} \int_0^1 D^2 f(A_m + \lambda_m \varepsilon(\hat{u}) + t\lambda_m (1 - \varphi) \varepsilon(u_m - \hat{u})) \right. \\
&\quad \left. (\varepsilon(u_m) - \varepsilon(\hat{u}), (\hat{u} - u_m) \odot \nabla\varphi)(1 - \varphi) \, \mathrm{d}t \, \mathrm{d}z \right|, \\
|J_2| &= \lambda_m \left| \int_{B_r} \int_0^1 D^2 f(\dots)(\varepsilon(u_m) - \varepsilon(\hat{u}), \varepsilon(\varphi_m))(1 - \varphi) \, \mathrm{d}t \, \mathrm{d}z \right|,
\end{aligned}$$

and similar to the previous discussion of I_5 we see

$$\frac{1}{\lambda_m} J_{1,2} \to 0, \quad \text{as } m \to \infty.$$

(Here we also use $\varepsilon(\hat{u}) \in L^\infty_{\mathrm{loc}}(B_1;\mathbb{S})$, thus $\varepsilon(\hat{u})$ is bounded on $B_r(0)$.) Since φ_m and φw_m are of class $\overset{\circ}{W}{}^1_q(B_r;\mathbb{R}^n)$, we clearly have

$$\int_{B_r} Df(A_m) : (\varepsilon(\varphi_m) + \varepsilon(\varphi w_m))\,\mathrm{d}z = 0,$$

therefore

$$\begin{aligned} \lambda_m^{-1}|J_3| &= \lambda_m^{-1}\left|\int_{B_r} (Df(A_m + \lambda_m\varepsilon(\hat{u})) - Df(A_m)) : (\varepsilon(\varphi_m) + \varepsilon(\varphi w_m))\,\mathrm{d}z\right| \\ &= \left|\int_{B_r}\int_0^1 D^2 f(A_m + t\lambda_m\varepsilon(\hat{u}))(\varepsilon(\hat{u}), \varepsilon(\varphi_m) + \varepsilon(\varphi w_m)\,\mathrm{d}t\,\mathrm{d}z\right|, \end{aligned}$$

and as usual $\varepsilon(\hat{u}) \in L^\infty(B_r;\mathbb{S})$ together with the known convergences implies $\lambda_m^{-1}|J_3| \to 0$ as $m \to \infty$. Remembering (5.24) we have finally established that the right-hand side of (5.22) is dominated by a quantity which vanishes as $m \to \infty$, thus

$$0 = \lim_{m\to\infty}\int_{B_r}\int_0^1 (1-t)\varphi D^2 f(A_m + \lambda_m\varepsilon(\hat{u}) + t\lambda_m\varepsilon(w_m))(\varepsilon(w_m), \varepsilon(w_m))\,\mathrm{d}t\,\mathrm{d}z.$$

Now the lower bound for $D^2 f$ immediately gives (5.20).

The proof of Proposition 5.2 splits into two cases.

Case 1. Let $p \geq 2$. Then (5.20) obviously implies i) of Proposition 5.2. We further have

$$A_m + \lambda_m\varepsilon(\hat{u}) + t\lambda_m\varepsilon(w_m) = ta_m + (1-t)b_m,$$

$$a_m := A_m + \lambda_m\varepsilon(\hat{u}) + \lambda_m\varepsilon(w_m), \quad b_m = A_m + \lambda_m\varepsilon(\hat{u}),$$

thus (analogous to [GM2], inequality (2.2))

$$\int_0^1 (1 + |ta_m + (1-t)b_m|^2)^{\frac{p-2}{2}}\,\mathrm{d}t \geq c(1 + |a_m|^2 + |b_m - a_m|^2)^{\frac{p-2}{2}} \geq c|b_m - a_m|^{p-2},$$

and we get from (5.20)

$$\int_{B_\rho} \lambda_m^{p-2}|\varepsilon(w_m)|^p\,\mathrm{d}z \to 0 \quad \text{as } m \to \infty. \tag{5.25}$$

Let

$$\psi_m := \lambda_m^{-1}\big[(1 + |A_m + \lambda_m\varepsilon(u_m)|^2)^{\frac{p}{4}} - (1 + |A_m|^2)^{\frac{p}{4}}\big].$$

By Lemma 4.5 we know $\psi_m \in W^1_2(B_1)$, and we get

$$\begin{aligned}\int_{B_\rho} |\nabla\psi_m|^2\,\mathrm{d}z &= \lambda_m^{-2} r_m^{2-n} \int_{B_{\rho r_m}(x_m)} |\nabla h|^2\,\mathrm{d}x\\ &\le c(\rho) r_m^{-n}\lambda_m^{-2}\int_{B_{r_m}(x_m)} (1+|\varepsilon(u)|^2)^{\frac{q-2}{2}}|\nabla u - Q|^2\,\mathrm{d}x.\end{aligned}$$

For the last estimate we used inequality (4.24), h being defined in Lemma 4.5, and Q representing any matrix from $\mathbb{R}^{n\times n}$. If we choose $Q = A_m + r_m^{-1}B_m$ (recall the definition of u_m) and observe

$$\nabla u(x_m + r_m z) = \lambda_m \nabla u_m(z) + A_m + \frac{1}{r_m}B_m,$$

we obtain

$$\int_{B_\rho} |\nabla\psi_m|^2\,\mathrm{d}z \le c(\rho)\int_{B_1} (1+|A_m + \lambda_m\varepsilon(u_m)|^2)^{\frac{q-2}{2}}|\nabla u_m|^2\,\mathrm{d}z, \tag{5.26}$$

and this inequality also holds if $1 < p < 2$. Writing $\theta(\varepsilon) = (1+|\varepsilon|^2)^{p/4}$ we get

$$\begin{aligned}|\psi_m| &= \lambda_m^{-1}\left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\theta(A_m + t\lambda_m\varepsilon(u_m))\,\mathrm{d}t\right|\\ &= \left|\int_0^1 \varepsilon(u_m) : \nabla\theta(A_m + t\lambda_m\varepsilon(u_m))\,\mathrm{d}t\right|\\ &\le c\int_0^1 |\varepsilon(u_m)|(1+|A_m + t\lambda_m\varepsilon(u_m)|^2)^{\frac{p-2}{4}}\,\mathrm{d}t\\ &\le c\big\{|\varepsilon(u_m)| + \lambda_m^{\frac{p-2}{2}}|\varepsilon(u_m)|^{\frac{p}{2}}\big\},\end{aligned}$$

and (5.25) together with (5.8) implies (since $w_m = u_m - \hat{u}$ and $\hat{u} \in C^\infty(\overline{B}_\rho;\mathbb{R}^n)$)

$$\int_{B_\rho} |\psi_m|^2\,\mathrm{d}z \le c(\rho). \tag{5.27}$$

If we look at the right-hand side of (5.26), then by (5.8) and (5.13) we see that also

$$\int_{B_\rho} |\nabla\psi_m|^2\,\mathrm{d}z \le c(\rho). \tag{5.28}$$

Again it should be noted that (5.28) is valid also in case $1 < p < 2$.

Now we proceed as follows assuming $q > 2$: consider a number $M \gg 1$ and let

$$U_m := U_m(M,\rho) := \big\{z \in B_\rho : \lambda_m|\varepsilon(u_m)| \le M\big\}.$$

Then

$$\begin{aligned}\int_{U_m} \lambda_m^{q-2}|\varepsilon(u_m)|^q \,\mathrm{d}z &\le c\left\{\int_{U_m} \lambda_m^{q-2}|\varepsilon(w_m)|^q \,\mathrm{d}z + \int_{U_m} \lambda_m^{q-2}|\varepsilon(\hat u)|^q \,\mathrm{d}z\right\}\\ &\le c\left\{\int_{U_m} \lambda_m^{q-2}(|\varepsilon(u_m)|^{q-2} + |\varepsilon(\hat u)|^{q-2})|\varepsilon(w_m)|^2 \,\mathrm{d}z\right.\\ &\qquad \left. + \int_{U_m} \lambda_m^{q-2}|\varepsilon(\hat u)|^q \,\mathrm{d}z\right\} \qquad (5.29)\end{aligned}$$

and the right-hand side of (5.29) vanishes as $m \to \infty$ by definition of U_m and on account of $\varepsilon(w_m) \to 0$ in $L^2(B_\rho;\mathbb{S})$.

On the other hand, if we choose M large enough, then on $B_\rho - U_m$ we get

$$\psi_m \ge c\lambda_m^{-1}\lambda_m^{\frac{p}{2}}|\varepsilon(u_m)|^{\frac{p}{2}}, \quad \text{i.e.}$$

$$\lambda_m^{q-2+\frac{2-p}{p}q}\psi_m^{\frac{2q}{p}} \ge c\lambda_m^{q-2}|\varepsilon(u_m)|^q.$$

By (4.1) we have $q < p(1+2/n)$, thus $2q/p < 2n/(n-2)$, and by (5.27) and (5.28) we get

$$\sup_m \int_{B_\rho} \psi_m^{\frac{2q}{p}} \,\mathrm{d}z < \infty.$$

Moreover, for proving ii) of Proposition 5.2, we may assume $p < q$, since for $p = q$ the assertion ii) is contained in (5.25). But, if $p < q$, then $q - 2 + q(2-p)/p > 0$, and the above estimate shows

$$\lim_{m\to\infty} \int_{B_\rho - U_m} \lambda_m^{q-2}|\varepsilon(u_m)|^q \,\mathrm{d}z = 0,$$

hence

$$\lim_{m\to\infty} \int_{B_\rho} \lambda_m^{q-2}|\varepsilon(u_m)|^q \,\mathrm{d}z = 0.$$

Case 2. Let $1 < p < 2$. Using the boundedness of $\{A_m\}$ together with the boundedness of $\lambda_m\varepsilon(\hat u)$ on B_ρ, the integral in (5.20) is easily seen to be bounded from below by $c(a + \lambda_m^2|\varepsilon(w_m)|^2)^{(p-2)/2}|\varepsilon(w_m)|^2$, where $a > 0$ is a suitable constant, so that by (5.20)

$$\int_{B_\rho} (a + \lambda_m^2|\varepsilon(w_m)|^2)^{\frac{p-2}{2}}|\varepsilon(w_m)|^2 \,\mathrm{d}z \to 0$$

as $m \to \infty$, thus

$$\int_{U_m} |\varepsilon(w_m)|^2 \,\mathrm{d}z \to 0 \quad \text{as } m \to \infty. \qquad (5.30)$$

As remarked earlier (5.28) extends to the case $p < 2$, and (5.27) follows since

$$|\psi_m| \le c\int_0^1 |\varepsilon(u_m)|(1 + |A_m + t\lambda_m\varepsilon(u_m)|^2)^{\frac{p-2}{4}} \,\mathrm{d}t \le c|\varepsilon(u_m)|.$$

For M large we have

$$|\psi_m|^{\frac{4}{p}}\lambda_m^{\frac{2(2-p)}{p}} \geq |\varepsilon(u_m)|^2 \quad \text{on } B_\rho - U_m,$$

and $2(2-p)/p > 0$ together with $4/p \leq 2n/(n-2)$ (for the latter inequality observe $2 \leq q < p(1+2/n)$, thus $p > 2n/(n+2)$ and therefore $4/p < 4(n+2)/2n = 2+4/n$; thus $4/p < 2n/(n-2)$ follows from $2 + 4/n < 2n/(n-2)$) implies

$$\int_{B_\rho - U_m} |\varepsilon(u_m)|^2 \,\mathrm{d}z \to 0 \quad \text{as } m \to \infty,$$

hence

$$\int_{B_\rho - U_m} |\varepsilon(w_m)|^2 \,\mathrm{d}z \to 0 \quad \text{as } m \to \infty. \tag{5.31}$$

For (5.31) we have to discuss $\int_{B_\rho - U_m} |\varepsilon(\hat{u})|^2 \,\mathrm{d}z$, which means by the local boundedness of $\varepsilon(\hat{u})$ that we have to show $\mathcal{L}^n(B_\rho - U_m) \to 0$. But

$$\mathcal{L}^n(B_\rho - U_m) \leq \int_{B_\rho} \frac{1}{M}\lambda_m|\varepsilon(u_m)|\,\mathrm{d}z \overset{m\to\infty}{\to} 0$$

by (5.9). Combining (5.30) and (5.31) Proposition 5.2, i), is shown for the case $1 < p < 2$.

Next let us assume $q > 2$. Then, for proving Proposition 5.2, ii), we may exactly follow the lines after (5.28): as remarked before, $|\psi_m| \leq c|\varepsilon(u_m)|$ holds if $1 < p < 2$, thus (5.27) and (5.28) are available, the calculation in (5.29) does not involve p, and the estimate of $\lambda_m^{q-2}|\varepsilon(u_m)|^q$ on $B_\rho - U_m$ remains valid. This completes the proof of Proposition 5.2. □

Now, by the comments given after the proof of Proposition 5.1, we get the desired contradiction, the blow-up lemma is established, and Theorem 2.1, a), is proved. □

6. The case $n = 2$: proof of Theorem 2.1, b)

We here use a technique due to Frehse and Seregin (see [FrS]) which has also been applied in [FO], [FS2], [FR] and [BF4]. From now on assume that we are in the situation of Theorem 2.1, b). Using the notation and the results from Section 4 we recall inequality (4.10)

$$\int_{B_{2R}(x_0)} \eta^2 \partial_k\sigma_m : \varepsilon(\partial_k v_m)\,\mathrm{d}x \leq -2\int_{B_{2R}(x_0)} \eta\partial_k\tau_m : (\nabla\eta \odot \partial_k[v_m - Qx])\,\mathrm{d}x \tag{6.1}$$

being valid for any $\eta \in C_0^1(B_{2R}(x_0))$ and for arbitrary matrices $Q \in \mathbb{R}^{2\times 2}$. Let

$$H_m := \left(D^2 f_m(\varepsilon(v_m))(\partial_k\varepsilon(v_m), \partial_k\varepsilon(v_m))\right)^{\frac{1}{2}}.$$

In (4.5) we showed $H_m \in L^2_{\mathrm{loc}}(B_{2R}(x_0))$, and if we combine (4.5) with Lemma 4.4, it follows that

$$\int_{B_r(x_0)} H_m^2 \,\mathrm{d}x \le c(r) < \infty \quad \text{for all } r < 2R. \tag{6.2}$$

Let $B_{2r}(\bar{x}) \Subset B_{2R}(x_0)$ and consider $\eta \in C^1_0(B_{2r}(\bar{x}))$, $\eta \equiv 1$ on $B_r(\bar{x})$, $|\nabla\eta| \le c/r$. The gradient of η is supported on $T_r(\bar{x}) = B_{2r}(\bar{x}) - B_r(\bar{x})$, thus we get from (6.1)

$$\int_{B_r(\bar{x})} H_m^2 \,\mathrm{d}x \le cr^{-1}\left(\int_{T_r(\bar{x})} |\nabla\tau_m|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{T_r(\bar{x})} |\nabla v_m - Q|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}. \tag{6.3}$$

For the left-hand side we used the identity

$$\partial_k\sigma_m : \varepsilon(\partial_k v_m) = H_m^2.$$

Next observe

$$|\nabla\tau_m|^2 \le c|\nabla\sigma_m|^2 \le cH_m^2,$$

where the second inequality has been proved before formula (4.12) (note that we use $q = 2$ here). Let us specify Q: we define $B_m := \fint_{T_r(\bar{x})} \varepsilon(v_m)\,\mathrm{d}x$ and let $\tilde{v}_m := v_m - B_m x$. Next we choose a rigid motion γ_m according to

$$\|\tilde{v}_m - \gamma_m\|_{L^2(T_r(\bar{x}))} \le c\|\varepsilon(\tilde{v}_m)\|_{L^2(T_r(\bar{x}))}.$$

Then Korn's inequality implies

$$\|\nabla(\tilde{v}_m - \gamma_m)\|_{L^2(T_r(\bar{x}))} \le c\|\varepsilon(\tilde{v}_m)\|_{L^2(T_r(\bar{x}))},$$

thus

$$\|\nabla v_m - (\nabla\gamma_m + B_m)\|_{L^2(T_r(\bar{x}))} \le c\|\varepsilon(v_m) - B_m\|_{L^2(T_r(\bar{x}))}.$$

We therefore choose $Q = B_m + \nabla\gamma_m$ in (6.3) to get

$$\int_{B_r(\bar{x})} H_m^2 \,\mathrm{d}x \le \frac{c}{r}\left(\int_{T_r(\bar{x})} H_m^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{T_r(\bar{x})} |\varepsilon(v_m) - B_m|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}. \tag{6.4}$$

We know $v_m \in W^2_{2,\mathrm{loc}}(B_{2R}(x_0))$ (apply Lemma 3.1, a), for the case $s = 2$), thus we can use Sobolev–Poincaré's inequality to estimate

$$\left(\int_{T_r(\bar{x})} |\varepsilon(v_m) - B_m|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \le c\int_{T_r(\bar{x})} |\nabla\varepsilon(v_m)| \,\mathrm{d}x.$$

By Lemma 4.2 the function $h_m = (1+|\varepsilon(v_m)|^2)^{p/4}$ is in the space $W^1_{2,\mathrm{loc}}(B_{2R}(x_0))$, and we clearly have

$$|\nabla\varepsilon(v_m)| \le ch_m H_m,$$

since $h_m H_m \ge ch_m|\nabla\varepsilon(v_m)|(1+|\varepsilon(v_m)|^2)^{(p-2)/4}$ by the ellipticity of $D^2 f$. Going back to (6.4), we obtain

$$\int_{B_r(\bar{x})} H_m^2 \,\mathrm{d}x \le \frac{c}{r}\left(\int_{T_r(\bar{x})} H_m^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \int_{T_r(\bar{x})} h_m H_m \,\mathrm{d}x. \tag{6.5}$$

Let us also recall that (see (4.20))

$$\int_{B_\rho(x_0)} |\nabla h_m|^2 \,\mathrm{d}x \le c(\rho) < \infty \quad \text{for all } \rho < 2R. \tag{6.6}$$

Combining (6.5) with (6.2) and (6.6) we find using Lemma 4.1 of [FrS]: if $\omega \Subset B_{2R}(x_0)$ is a fixed subdomain and if t is some number > 1, then there is a constant K depending on t, ω and the local bounds from (6.2) and (6.6) such that

$$\int_{B_r(\bar{x})} H_m^2 \,\mathrm{d}x \le K|\ln r|^{-t} \quad \text{for all } B_r(\bar{x}) \subset \omega,$$

hence

$$\int_{B_r(\bar{x})} |\nabla \sigma_m|^2 \,\mathrm{d}x \le K|\ln r|^{-t} \quad \text{for all } B_r(\bar{x}) \subset \omega,$$

and if we choose $t > 2$, the version of the Dirichlet-growth theorem given in [Fre], p. 287, shows that σ_m is continuous with modulus of continuity not depending on m. Therefore there exists a continuous function $\tilde{\sigma}$ such that $\sigma_m \to \tilde{\sigma}$ locally uniform. In Lemma 4.5 we showed $\varepsilon(v_m) \to \varepsilon(u)$ a.e. , thus $\tilde{\sigma} = \sigma := Df(\varepsilon(u))$. But Df is a homeomorphism $\mathbb{S} \to \mathbb{S}$ and so we deduce continuity of $\varepsilon(u)$. In this case the partial regularity criterion holds at each point, hence $u \in C^{1,\alpha}(\Omega;\mathbb{R}^2)$. □

Let us now look at the case $q > 2$. Clearly, (6.1) holds and we have

$$|\nabla \tau_m| \Gamma_m^{\frac{2-q}{4}} \le c H_m$$

(compare the calculations after (4.11)). If we use Hölder's inequality on the right-hand side of (6.1), we get

$$\int_{B_r(\bar{x})} H_m^2 \,\mathrm{d}x \le \frac{c}{r} \left(\int_{T_r(\bar{x})} H_m^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{T_r(\bar{x})} \Gamma_m^{\frac{q-2}{2}} |\nabla v_m - Q|^2 \,\mathrm{d}x \right)^{\frac{1}{2}},$$

but we did not succeed in applying the arguments of [BF4] to the second term on the right-hand side of the above inequality. This would be possible if we could replace ∇v_m by the symmetric derivative $\varepsilon(v_m)$.

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