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Two dimensional variational problems with linear growth

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Abstract. Suppose that $f\colon \mathbb{R}^{nN} \to \mathbb{R}$ is a strictly convex energy density of linear growth, $f(Z) = g(|Z|^2)$ if $N > 1$. If f satisfies an ellipticity condition of the form

$$D^2 f(Z)(Y, Y) \geq c(1 + |Z|^2)^{-\frac{\mu}{2}} |Y|^2 , \quad 1 < \mu \leq 3 ,$$

then, following [Bi3], there exists a unique (up to a constant) solution of the variational problem

$$\int_\Omega f(\nabla w)\,\mathrm{d}x + \int_{\partial\Omega} f_\infty((u_0 - w) \otimes \nu)\, d\mathcal{H}^{n-1} \to \min \text{ in } W_1^1(\Omega; \mathbb{R}^N) ,$$

provided that the given boundary data $u_0 \in W_1^1(\Omega; \mathbb{R}^N)$ are additionally assumed to be of class $L^\infty(\Omega; \mathbb{R}^N)$. Moreover, if $\mu < 3$, then the boundedness of u_0 yields local $C^{1,\alpha}$-regularity (and uniqueness up to a constant) of generalized minimizers of the problem

$$\int_\Omega f(\nabla w)\,\mathrm{d}x \to \min \text{ in } u_0 + \overset{\circ}{W}{}_1^1(\Omega; \mathbb{R}^N) .$$

In our paper we show that the restriction $u_0 \in L^\infty(\Omega; \mathbb{R}^N)$ is superfluous in the two dimensional case $n = 2$, hence we may prescribe boundary values from the energy class $W_1^1(\Omega; \mathbb{R}^N)$ and still obtain the above results.

1. Introduction

In the following we always consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ and a strictly convex energy density $f\colon \mathbb{R}^{nN} \to [0, \infty)$, which is of linear growth, i.e.

$$a|Z| - b \leq f(Z) \leq A|Z| + B \quad \text{for all } Z \in \mathbb{R}^{nN} \tag{1}$$

holds with suitable constants $a > 0$, $A > 0$, b, B. Moreover, we fix some boundary data u_0 of the Sobolev class $W_1^1(\Omega; \mathbb{R}^N)$. Then we are interested in the variational problem

$$J[w] := \int_\Omega f(\nabla w)\,\mathrm{d}x \to \min \quad \text{in } u_0 + \overset{\circ}{W}{}_1^1(\Omega; \mathbb{R}^N) , \tag{$\mathcal{P}$}$$

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which in general fails to have solutions. For this reason we introduce the set

$$\mathcal{M} = \Big\{ u \in \mathrm{BV}(\Omega;\mathbb{R}^N) : u \text{ is the } L^1\text{-limit of some}$$
$$J\text{-minimizing sequence } \{u_k\} \subset u_0 + \overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N) \Big\}$$

of generalized minimizers of problem $(\mathcal{P})$, which, by [BF3] (compare also the monograph [Gi] for the minimal surface case), coincides with the set of solutions of the relaxed problem

$$K[w] = \int_\Omega f(\nabla^a w)\,\mathrm{d}x + \int_\Omega f_\infty\Big(\frac{\nabla^s w}{|\nabla^s w|}\Big)\,\mathrm{d}|\nabla^s w| + \int_{\partial\Omega} f_\infty((u_0 - w)\otimes \nu)\,\mathrm{d}\mathcal{H}^{n-1}$$
$$\to \min \quad \text{in } \mathrm{BV}(\Omega;\mathbb{R}^N)\,, \tag{$\hat{\mathcal{P}}$}$$

where ν is the outward unit normal to $\partial\Omega$, f_∞ is the recession function of f, and $\nabla^a w$ and $\nabla^s w$ denote the regular and the singular part of ∇w w.r.t. the Lebesgue measure, respectively.

Our main concern is the study of the smoothness properties of generalized minimizers. To this purpose and in order to formulate what is known up to now, let us precisely state our general

Assumption 1. *The energy density* $f\colon \mathbb{R}^{nN} \to [0,\infty)$ *is supposed to satisfy the following set of hypotheses: there exist positive constants* ν_1, ν_2, ν_3 *and a real number* $1 < \mu \le 3$ *such that for any* $Z \in \mathbb{R}^{nN}$

i) $f \in C^2(\mathbb{R}^{nN})$*;*
ii) $|\nabla f(Z)| \le \nu_1$*;*
iii) for any $Y \in \mathbb{R}^{nN}$ *we have*

$$\nu_2(1+|Z|^2)^{-\frac{\mu}{2}}|Y|^2 \le D^2 f(Z)(Y,Y) \le \nu_3(1+|Z|^2)^{-\frac{1}{2}}|Y|^2\,.$$

Moreover, in the vector case $N > 1$ *we assume that*

$$f(Z) = g(|Z|^2) \tag{2}$$

for some function $g\colon [0,\infty) \to [0,\infty)$*, which is of class* C^2*.*

Remark 1. From Assumption 1 we easily obtain the following structure conditions (see [Bi2] or [Bi3] for a short proof).

i) There are real numbers $\nu_4 > 0$ and ν_5 such that for any $Z \in \mathbb{R}^{nN}$

$$\nabla f(Z) : Z \ge \nu_4(1+|Z|^2)^{\frac{1}{2}} - \nu_5\,,$$

where we use the symbol $Y : Z$ to denote the standard scalar-product in $\mathbb{R}^{nN}$.
ii) The integrand f is of linear growth in the sense that (1) holds.
iii) The energy density f satisfies a "balancing condition": there is a positive number ν_6 such that

$$|D^2 f(Z)||Z|^2 \le \nu_6(1+f(Z)) \quad \text{holds for any } Z \in \mathbb{R}^{nN}\,.$$

The most prominent (scalar) example satisfying Assumption 1 with the limit exponent $\mu = 3$ is the minimal surface integrand $f(Z) = \sqrt{1+|Z|^2}$ admitting only regular solutions (see, for instance, [Gi], [GMS] as well as the a priori gradient estimates for solutions of non-uniformly elliptic equations due to Ladyzhenskaya/Ural'tseva ([LU2]) and Simon ([Si])). These solutions are uniquely determined up to a constant. It should be emphasized that on account of the geometric structure of this example there is much better information in the minimal surface case than supposed in Assumption 1 (see Remark 2.3 of [Bi3]).

Remark 2. For the sake of completeness we should also mention the theory of perfect plasticity as a second significant example with a linear growth energy density. Here Assumption 1 of course no longer is valid, and we can only expect partial regularity results, which are mainly due to Seregin (compare [Se1]–[Se4]). Note that even in the two dimensional setting we just have some additional information on the singular set (see [Se4]).

The discussion of μ-elliptic integrands satisfying Assumption 1 without an additional geometric structure condition started in [BF2]. Here the one parameter family

$$\Phi_\mu(Z) := \int_0^{|Z|} \int_0^s (1+t^2)^{-\frac{\mu}{2}} \,\mathrm{d}t\,\mathrm{d}s\,, \qquad 1 < \mu \le 3\,,$$

serves as a typical example. Note that in the case $\Phi_{\mu=3}$ we exactly recover the minimal surface integrand. For a detailed discussion of examples with the limit exponent $\mu = 3$ of ellipticity, which are not of minimal surface type, we refer to [Bi2], [BF5] (for instance, we may consider integrands which are not depending on $|Z|$ but on $\mathrm{dist}(Z, C)$, where C denotes a suitable convex set).

However, smoothness of generalized minimizers was proved in [BF2] under the quite restrictive assumption $1 < \mu < 1 + 2/n$. Even in two dimensions the reasoning of [BF2] is limited to the case $\mu < 2$.

The considerable improvement to ellipticity exponents $1 < \mu \le 3$ then was given in [Bi2] and [Bi3] by imposing an additional L^∞-bound on the data u_0. Here we observe that, on account of the counterexample given in [Bi2] and [BF5], we do not expect to get an extension of Theorem 1 below to the case $\mu > 3$.

Remark 3. Before we are going to discuss Theorem 1 below, we like to include some short remarks on analogous results for functionals with (p, q)-growth conditions. Here the energy density is supposed to be of superlinear growth satisfying for some positive constants λ, Λ and for all $Z, Y \in \mathbb{R}^{nN}$

$$\lambda(1+|Z|^2)^{\frac{p-2}{2}}|Y|^2 \le D^2 f(Z)(Y,Y) \le \Lambda(1+|Z|^2)^{\frac{q-2}{2}}|Y|^2\,, \qquad 1 < p \le q\,.$$

i) This ellipticity condition formally coincides with Assumption 1, iii), by letting $\mu = 2 - p$ and $q = 1$.

ii) The first results on the smoothness of solutions for problems with (p, q)-growth are due to Marcellini (see [Ma1], [Ma2] and a series of subsequent papers). His assumptions on p and q are similar to the condition $\mu < 1 + 2/n$.

iii) Closely related is the paper [FM] of Fuchs and Mingione, where the authors study energy densities with nearly linear growth conditions. Here the assumption $\mu < 1 + 2/n$ was introduced (if we again formally let $q = 1$ in [FM]).
iv) The analogue to $\mu < 3$ in the (p, q)-case reads as $q < 2+p$. This condition first appeared in [ELM], where higher integrability up to a certain limit exponent is proved in the superquadratic case.
v) For a discussion of full $C^{1,\alpha}_{loc}$-regularity in the case $q < 2 + p$ and for a more detailed overview on the known results we refer to [Bi2].

Let us turn our attention back to variational problems with linear growth. Here it is known that we have

Theorem 1 ([Bi2], [Bi3]). *Suppose that Assumption 1 holds in the limit case $\mu = 3$ and that we have in addition $u_0 \in L^\infty \cap W^1_1(\Omega; \mathbb{R}^N)$. Then there is a generalized minimizer $u^* \in \mathcal{M}$ such that*

i) $\nabla^s u^* = 0$.
ii) For any $\Omega' \Subset \Omega$ we have

$$\int_{\Omega'} |\nabla u^*| \ln(1 + |\nabla u^*|)\, \mathrm{d}x < \infty\,.$$

iii) u^* is (up to a constant) the unique solution of the problem

$$\int_\Omega f(\nabla w)\, \mathrm{d}x + \int_{\partial\Omega} f_\infty\big((u_0 - w) \otimes \nu\big)\, \mathrm{d}\mathcal{H}^{n-1} \to \min \quad in \quad W^1_1(\Omega; \mathbb{R}^N)\,. \tag{$\mathcal{P}'$}$$

If ellipticity is slightly better, i.e. if $\mu < 3$, then full regularity is obtained in the sense of

Theorem 2 ([Bi2], [Bi3]). *Suppose that Assumption 1 holds with $\mu < 3$ and that we again have $u_0 \in L^\infty \cap W^1_1(\Omega; \mathbb{R}^N)$. In the vector-valued case we assume in addition to (2) that there are real numbers $\beta \in (0, 1]$, $K > 0$, such that for all Z, $\tilde{Z} \in \mathbb{R}^{nN}$*

$$|D^2 f(Z) - D^2 f(\tilde{Z})| \le K|Z - \tilde{Z}|^\beta\,. \tag{3}$$

Then we have:

i) each generalized minimizer $u \in \mathcal{M}$ is an element of the space $C^{1,\alpha}(\Omega; \mathbb{R}^N)$ for any $0 < \alpha < 1$;
ii) for $u, v \in \mathcal{M}$ we have $\nabla u = \nabla v$, i.e. up to a constant uniqueness of generalized minimizers holds true.

In the following we study the question whether at least in two dimensions the assumption $u_0 \in L^\infty(\Omega; \mathbb{R}^n)$ can be dropped, i.e. we are going to discuss the Dirichlet boundary value problem $(\mathcal{P})$ with data u_0 from the energy class $W^1_1(\Omega; \mathbb{R}^N)$. In fact, it turns out that:

Theorem 3. *In the two dimensional case $n = 2$, Theorems 1 and 2 remain valid without the requirement $u_0 \in L^\infty(\Omega; \mathbb{R}^N)$.*

From now on we restrict our considerations to the two dimensional case $n = 2$ and proceed as follows: after introducing some suitable (and well known) regularization, we will prove in Section 3 uniform local higher integrability in the limit case $\mu = 3$. Using this result, we complete the proof of Theorem 3 in Section 4 by reducing the problem to the setting discussed in [Bi3].

2. Regularization

We start with a well known regularization procedure. However, we focus on the discussion of boundary data from the energy class $W_1^1(\Omega; \mathbb{R}^N)$, and, in contrast to [Bi3], we now include a precise approximation argument w.r.t. the boundary data as sketched, for instance, in [BF1]. To this purpose let us consider a sequence $\{u_0^m\}$, $u_0^m \in C^\infty(\overline{\Omega}; \mathbb{R}^N)$ such that

$$u_0^m \to u_0 \quad \text{in } \; W_1^1(\Omega; \mathbb{R}^N) \; \text{ as } \; m \to \infty \, . \tag{4}$$

We then denote by u_δ^m, $0 < \delta < 1$, the unique solution of the variational problem

$$J_\delta[w] := \frac{\delta}{2} \int_\Omega |\nabla w|^2 \, \mathrm{d}x + J[w] \to \min \qquad \text{in } \; u_0^m + \overset{\circ}{W}{}_2^1(\Omega; \mathbb{R}^N) \tag{$\mathcal{P}_\delta^m$}$$

and abbreviate $f_\delta = \frac{\delta}{2} |\cdot|^2 + f$. If $\delta = \delta(m)$ is chosen sufficiently small (see the proof of Lemma 1, i) and ii), for the precise conditions) and if we write for short $u_\delta = u_{\delta(m)}^m$, then the main properties of the regularization are summarized in the following lemma.

Lemma 1. *i) There is a real number c, independent of δ, such that*

$$\delta \int_\Omega |\nabla u_\delta|^2 \, \mathrm{d}x \le c \, , \qquad \int_\Omega |\nabla u_\delta| \le c \, ;$$

ii) each L^1-cluster point u^* of the sequence $\{u_\delta\}$ is a generalized minimizer in the sense that $u^* \in \mathcal{M}$ holds;

iii) u_δ is of class $W_{2,loc}^2 \cap W_{\infty,loc}^1(\Omega; \mathbb{R}^N)$;

iv)

$$\int_\Omega \nabla f_\delta(\nabla u_\delta) : \nabla \varphi \, \mathrm{d}x = 0 \quad \textit{for all } \; \varphi \in C_0^\infty(\Omega; \mathbb{R}^N) \, ;$$

v) for $\gamma = 1, 2$ we have

$$\int_\Omega D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \nabla \varphi) \, \mathrm{d}x = 0 \quad \textit{for all } \; \varphi \in C_0^\infty(\Omega; \mathbb{R}^N) \, .$$

Proof. ad i). The minimality of u_δ implies $J_\delta[u_\delta] = J_{\delta(m)}[u^m_{\delta(m)}] \le J_{\delta(m)}[u^m_0]$, and if $\delta(m)$ is chosen sufficiently small, then

$$J_{\delta(m)}[u^m_0] = \frac{\delta(m)}{2}\int_\Omega |\nabla u^m_0|^2\,\mathrm{d}x + \int_\Omega f(\nabla u^m_0)\,\mathrm{d}x \le \frac{1}{m} + \int_\Omega f(\nabla u^m_0)\,\mathrm{d}x\,.$$

If we recall in addition the convergence (4) and the linear growth of f (see Assumption 1, ii)), i.e.

$$\left|\int_\Omega \big(f(\nabla u^m_0) - f(\nabla u_0)\big)\,\mathrm{d}x\right| \le c\int_\Omega |\nabla u^m_0 - \nabla u_0|\,\mathrm{d}x \to 0 \quad \text{as } m\to\infty\,, \tag{5}$$

then the existence of a positive number c, independent of δ, is established such that i) holds.

ad ii). As shown in [BF1], Lemma 3.1, (see also [Se3], Lemma 2, and [Bi2] Remark II.1.8), we have for any fixed $m\in\mathbb{N}$

$$J[u^m_\delta] \to \inf_{w\in u^m_0+\overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)} J[w] \quad \text{as } \delta\to 0\,,$$

in particular it is possible to choose $\delta(m)$ sufficiently small such that for all $m\in\mathbb{N}$

$$J[u^m_{\delta(m)}] \le \inf_{w\in u^m_0+\overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)} J[w] + \frac{1}{m}\,. \tag{6}$$

We then fix $\varepsilon > 0$, and similar to (5) we can choose $m_0\in\mathbb{N}$ sufficiently large such that for all $m\ge m_0$

$$|J[w] - J[w - u^m_0 + u_0]| \le c\int_\Omega |\nabla u^m_0 - \nabla u_0|\,\mathrm{d}x \le \varepsilon \quad \text{for all } w\in W^1_1(\Omega;\mathbb{R}^N)\,. \tag{7}$$

As an immediate consequence we see that

$$\left|\inf_{w\in u^m_0+\overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)} J[w] - \inf_{w\in u_0+\overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)} J[w]\right| \le \varepsilon\,,$$

whenever $m\ge m_0$. This, together with the choice of $\delta(m)$ (recall (6)), implies (w.l.o.g. $m^{-1}\le\varepsilon$ for all $m\ge m_0$)

$$J[u^m_{\delta(m)}] \le \inf_{w\in u^m_0+\overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)} J[w] + \varepsilon \le \inf_{w\in u_0+\overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)} J[w] + 2\varepsilon \tag{8}$$

for all $m\ge m_0$. Finally we let $w^m_{\delta(m)} = u^m_{\delta(m)} + u_0 - u^m_0$ and by (7) and (8) the sequence $\{w^m_{\delta(m)}\}$ is seen to be a J-minimizing sequence from $u_0 + \overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^N)$. Since the sequences $\{w^m_{\delta(m)}\}$ and $\{u^m_{\delta(m)}\}$ generate the same L^1-cluster points, assertion ii) is proved.

ad iii)–v). iv) is the Euler equation for u_δ which, in the scalar case, implies iii) by Theorem 5.2, Chapter 4 of [LU1]. In the vector-valued setting, we refer to [Uh] (compare [GM], Theorem 3.1) which, together with the standard difference quotient technique, gives iii). Finally, on account of iii), the Euler equation iv) may be differentiated with v) as a result. □

As a corollary of v) we obtain the following Caccioppoli-type inequality.

Corollary 1. *If $\{u_\delta\}$ denotes the regularization introduced above, then there are positive numbers c_1, c_2, such that for any $\eta \in C_0^\infty(\Omega)$, $0 \le \eta \le 1$, and for any δ as above*

$$\begin{aligned}\int_\Omega D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\eta^2 \,\mathrm{d}x &\le c_1 \int_\Omega |D^2 f_\delta(\nabla u_\delta)||\nabla u_\delta|^2 |\nabla \eta|^2 \,\mathrm{d}x \\ &\le c_2 \max_\Omega |\nabla \eta|^2 . \end{aligned} \tag{9}$$

Here and in the following we always take the sum w.r.t. repeated Greek indices $\gamma = 1, 2$ and w.r.t. repeated Latin indices $i = 1, \dots, N$.

Proof. From iii) of Lemma 1 and a standard density argument we see that for $\gamma = 1, 2$ the choice $\varphi = \eta^2 \partial_\gamma u_\delta$ is admissible in the differentiated form v), Lemma 1, of the Euler equation. Using Young's inequality, the left-hand inequality of (9) is immediate. The uniform bound on the right-hand side of (9) follows from Remark 1, iii). □

3. Local higher integrability in the limit case

Here we are going to establish uniform local higher integrability of the sequence $\{\nabla u_\delta\}$ in the limit case $\mu = 3$.

Let us, for a moment, concentrate on the scalar case $N = 1$. Then we have the following assertion.

Lemma 2. *Suppose that Assumption 1 holds in the two dimensional scalar case $n = 2$, $N = 1$, and let $\{u_\delta\}$ denote the regularization introduced above. Moreover, fix a ball $B_r(x_0)$ satisfying $B_{2r}(x_0) \Subset \Omega$. Then there is a positive number $c = c(r)$, independent of δ, such that for any $\eta \in C_0^\infty(B_{2r}(x_0))$, $0 \le \eta \le 1$,*

$$\begin{aligned}&\int_{B_{2r}(x_0)} (1 + |\nabla u_\delta|^2)^{\frac{1}{2}} |u_\delta - (u_\delta)_{2r}|^2 \eta^2 \,\mathrm{d}x \\ &\quad + \delta \int_{B_{2r}(x_0)} |\nabla u_\delta|^2 |u_\delta - (u_\delta)_{2r}|^2 \eta^2 \,\mathrm{d}x \le c .\end{aligned}$$

Here $(u_\delta)_{2r}$ denotes the mean value of u_δ on $B_{2r}(x_0)$.

Remark 4. i) Following the proof of Theorem 4 below, it becomes obvious that this estimate is exactly the one which is needed to reach the limit case $\mu = 3$.

ii) Inequality (10) given below is the main reason why the results in two dimensions are better than the ones stated in [Bi3] for arbitrary dimensions.

Proof of Lemma 2. Note that in the two dimensional case $n = 2$ we have by Sobolev-Poincarè's inequality

$$\left(\int_{B_{2r}(x_0)} |u_\delta - (u_\delta)_{2r}|^2 \, \mathrm{d}x\right)^{\frac{1}{2}} \leq c_1 \int_{B_{2r}(x_0)} |\nabla u_\delta| \, \mathrm{d}x \leq c_2 \tag{10}$$

for some positive constants c_1, c_2, which are not depending on δ (recall Lemma 1, i)). Moreover, as a result of Lemma 1, iii), and a standard density argument, $\varphi = (u_\delta - (u_\delta)_{2r})^3 \eta^2$, $\eta \in C_0^\infty(B_{2r}(x_0))$, $0 \leq \eta \leq 1$, is seen to be admissible in the Euler equation iv) of Lemma 1, thus we obtain

$$\begin{aligned} 3 &\int_{B_{2r}(x_0)} \nabla f(\nabla u_\delta) \cdot \nabla u_\delta |u_\delta - (u_\delta)_{2r}|^2 \eta^2 \, \mathrm{d}x \\ &\quad + 3\delta \int_{B_{2r}(x_0)} |\nabla u_\delta|^2 |u_\delta - (u_\delta)_{2r}|^2 \eta^2 \, \mathrm{d}x \\ &= -2 \int_{B_{2r}(x_0)} \nabla f_\delta(\nabla u_\delta) \cdot \nabla \eta \eta (u_\delta - (u_\delta)_{2r})^3 \, \mathrm{d}x \, . \end{aligned}$$

From this equality we arrive at (recalling Remark 1, i), (10) and the boundedness of $|\nabla f|$)

$$\begin{aligned} &\int_{B_{2r}(x_0)} (1 + |\nabla u_\delta|^2)^{\frac{1}{2}} |u_\delta - (u_\delta)_{2r}|^2 \eta^2 \, \mathrm{d}x \\ &\quad + \delta \int_{B_{2r}(x_0)} |\nabla u_\delta|^2 |u_\delta - (u_\delta)_{2r}|^2 \eta^2 \, \mathrm{d}x \\ &\leq c(1 + I_1 + I_2) \, , \end{aligned} \tag{11}$$

where the constant c again is not depending on δ, and I_1, I_2 are given by

$$\begin{aligned} I_1 &= \int_{B_{2r}(x_0)} |u_\delta - (u_\delta)_{2r}|^3 \eta |\nabla \eta| \, \mathrm{d}x \, , \\ I_2 &= \delta \int_{B_{2r}(x_0)} |\nabla u_\delta| |u_\delta - (u_\delta)_{2r}|^3 \eta |\nabla \eta| \, \mathrm{d}x \, . \end{aligned}$$

Estimating I_1 we observe that (using (10), Hölder's inequality, Sobolev-Poincarè's inequality and Young's inequality for some sufficiently small number $\varepsilon > 0$)

$$\begin{aligned} I_1 &\leq \left(\int_{B_{2r}(x_0)} |u_\delta - (u_\delta)_{2r}|^4 \eta^2 \, \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{B_{2r}(x_0)} |u_\delta - (u_\delta)_{2r}|^2 |\nabla \eta|^2 \, \mathrm{d}x\right)^{\frac{1}{2}} \\ &\leq c \int_{B_{2r}(x_0)} \left|\nabla(|u_\delta - (u_\delta)_{2r}|^2 \eta)\right| \mathrm{d}x \\ &\leq c \left(1 + \int_{B_{2r}(x_0)} |u_\delta - (u_\delta)_{2r}| |\nabla u_\delta| \eta \, \mathrm{d}x\right) \\ &\leq c \Bigg(1 + \int_{B_{2r}(x_0)} \Big\{\varepsilon |u_\delta - (u_\delta)_{2r}|^2 (1 + |\nabla u_\delta|^2)^{\frac{1}{2}} \eta^2 \\ &\qquad + \varepsilon^{-1} (1 + |\nabla u_\delta|^2)^{\frac{1}{2}} \Big\} \, \mathrm{d}x\Bigg) \, . \end{aligned} \tag{12}$$

Here again c denotes some positive local constant which is not depending on δ. Note that the "ε"-part on the right-hand side of (12) can be absorbed (for $\varepsilon > 0$ sufficiently small) on the left-hand side of (11), whereas the remaining integral is uniformly bounded w.r.t. δ.

To find an estimate for I_2, we recall the uniform bound for $\delta \int_\Omega |\nabla u_\delta|^2 \,\mathrm{d}x$. In fact, it can be easily seen that this quantity converges to zero if $\delta \to 0$ (see [BF1]), but here we merely need i) of Lemma 1. As a consequence, we have with local constants and for $\varepsilon > 0$ sufficiently small

$$\begin{aligned}
I_2 &\le c\delta \left(\int_{B_{2r}(x_0)} |\nabla u_\delta|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{B_{2r}(x_0)} |u_\delta - (u_\delta)_{2r}|^6 \eta^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\le c\delta^{\frac{1}{2}} \int_{B_{2r}(x_0)} \left|\nabla(|u_\delta - (u_\delta)_{2r}|^3 \eta)\right| \mathrm{d}x \\
&\le c\delta^{\frac{1}{2}} \left(\int_{B_{2r}(x_0)} |u_\delta - (u_\delta)_{2r}|^2 |\nabla u_\delta| \eta \,\mathrm{d}x + \int_{B_{2r}(x_0)} |u_\delta - (u_\delta)_{2r}|^3 \,\mathrm{d}x\right) \\
&\le c\delta^{\frac{1}{2}} \Bigg(\int_{B_{2r}(x_0)} \left(\varepsilon\delta^{\frac{1}{2}} |u_\delta - (u_\delta)_{2r}|^2 |\nabla u_\delta|^2 \eta^2 + \varepsilon^{-1}\delta^{-\frac{1}{2}} |u_\delta - (u_\delta)_{2r}|^2\right) \mathrm{d}x \\
&\qquad + \int_{B_{2r}(x_0)} |u_\delta - (u_\delta)_{2r}|^3 \,\mathrm{d}x\Bigg) \\
&=: c\sum_{i=1}^{3} I_2^i \,. \qquad (13)
\end{aligned}$$

Now I_2^1 can be absorbed on the left-hand side of (11), whereas the second integral I_2^2 is uniformly bounded w.r.t. δ. I_2^3 is estimated with the help of (10), Hölder's and Sobolev-Poincarè's inequality

$$\begin{aligned}
I_2^3 &= \delta^{\frac{1}{2}} \int_{B_{2r}(x_0)} |u_\delta - (u_\delta)_{2r}|^3 \,\mathrm{d}x \\
&\le \delta^{\frac{1}{2}} \left(\int_{B_{2r}(x_0)} |u_\delta - (u_\delta)_{2r}|^4 \,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{B_{2r}(x_0)} |u_\delta - (u_\delta)_{2r}|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\le c\delta^{\frac{1}{2}} \int_{B_{2r}(x_0)} \left|\nabla |u_\delta - (u_\delta)_{2r}|^2\right| \mathrm{d}x \\
&\le c\delta^{\frac{1}{2}} \left(\int_{B_{2r}(x_0)} |\nabla u_\delta|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{B_{2r}(x_0)} |u_\delta - (u_\delta)_{2r}|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\le c \,. \qquad (14)
\end{aligned}$$

If we recall that the ε-terms occurring on the right-hand side of (12) and (13) can be absorbed on the left-hand side of (11), then Lemma 2 follows from the uniform estimates for the remaining terms on the right-hand side of (12), (13) and (14), respectively. □

Remark 5. Going through the proof of Lemma 2 we see that the assertion is not depending on the exponent μ of ellipticity.

Instead of the assumption $u_0 \in L^\infty(\Omega)$ used [Bi3], Lemma 2 now is the main tool yielding uniform local higher integrability of $|\nabla u_\delta|$ in the scalar case.

Theorem 4. *Consider the two dimensional scalar case $n = 2$, $N = 1$, together with the general Assumption 1. If $B_{2r}(x_0) \Subset \Omega$, then there exists a local constant c, independent of δ, such that the regularizing sequence $\{u_\delta\}$ satisfies*

$$\int_{B_r(x_0)} (1 + |\nabla u_\delta|^2)^{\frac{1}{2}} \ln(1 + |\nabla u_\delta|^2)\, \mathrm{d}x \leq c\,.$$

Proof. We let $\omega_\delta = \ln(1 + |\nabla u_\delta|^2)$ and choose $\varphi = (u_\delta - (u_\delta)_{2r})\omega_\delta \eta^2$, $\eta \in C_0^\infty(B_{2r}(x_0))$, $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_r(x_0)$. Again φ is easily seen to be admissible in the Euler equation iv), Lemma 1, and we obtain

$$\begin{aligned}
&\int_{B_{2r}(x_0)} \nabla f(\nabla u_\delta) \cdot \nabla u_\delta \omega_\delta \eta^2\, \mathrm{d}x + \delta \int_{B_{2r}(x_0)} |\nabla u_\delta|^2 \omega_\delta \eta^2\, \mathrm{d}x \\
&\quad = - \int_{B_{2r}(x_0)} \nabla f(\nabla u_\delta) \cdot \nabla \omega_\delta (u_\delta - (u_\delta)_{2r}) \eta^2\, \mathrm{d}x \\
&\qquad -2 \int_{B_{2r}(x_0)} \nabla f(\nabla u_\delta) \cdot \nabla \eta \eta (u_\delta - (u_\delta)_{2r}) \omega_\delta\, \mathrm{d}x \\
&\qquad -\delta \int_{B_{2r}(x_0)} \nabla u_\delta \cdot \nabla \omega_\delta (u_\delta - (u_\delta)_{2r}) \eta^2\, \mathrm{d}x \\
&\qquad -2\delta \int_{B_{2r}(x_0)} \nabla u_\delta \cdot \nabla \eta \eta (u_\delta - (u_\delta)_{2r}) \omega_\delta\, \mathrm{d}x \\
&\quad =: \sum_{i=1}^{4} I_i\,.
\end{aligned}$$

Similar to the proof of Lemma 2, a lower bound for the first integral on the left-hand side is given by Remark 1, i), thus

$$\begin{aligned}
&\int_{B_{2r}(x_0)} (1 + |\nabla u_\delta|^2)^{\frac{1}{2}} \omega_\delta \eta^2\, \mathrm{d}x + \delta \int_{B_{2r}(x_0)} |\nabla u_\delta|^2 \omega_\delta \eta^2\, \mathrm{d}x \\
&\qquad \leq c \left(\int_{B_{2r}(x_0)} \omega_\delta \eta^2\, \mathrm{d}x + \sum_{i=1}^{4} |I_i| \right). \qquad (15)
\end{aligned}$$

Clearly $\int_{B_{2r}(x_0)} \omega_\delta \eta^2\, \mathrm{d}x$ is uniformly bounded w.r.t. δ, and in order to find an estimate for I_1 we observe

$$|\nabla \omega_\delta|^2 \leq \frac{4}{1 + |\nabla u_\delta|^2} |\nabla^2 u_\delta|^2\,.$$

This, together with Lemma 2, implies (again we make use of the fact that $|\nabla f|$ is bounded)

$$
\begin{aligned}
|I_1| &\le c \int_{B_{2r}(x_0)} |u_\delta - (u_\delta)_{2r}| |\nabla \omega_\delta| \eta^2 \, \mathrm{d}x \\
&\le c \left(\int_{B_{2r}(x_0)} (1 + |\nabla u_\delta|^2)^{\frac{1}{2}} |u_\delta - (u_\delta)_{2r}|^2 \eta^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \\
&\qquad \cdot \left(\int_{B_{2r}(x_0)} (1 + |\nabla u_\delta|^2)^{-\frac{1}{2}} |\nabla \omega_\delta|^2 \eta^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \\
&\le c \left(\int_{B_{2r}(x_0)} (1 + |\nabla u_\delta|^2)^{-\frac{3}{2}} |\nabla^2 u_\delta|^2 \eta^2 \, \mathrm{d}x \right)^{\frac{1}{2}} .
\end{aligned}
$$

Here the right-hand side is bounded through the Caccioppoli-type inequality (9) of Corollary 1. Note that we exactly reach the limit case $\mu = 3$. Next,

$$
|I_2| \le c \int_{B_{2r}(x_0)} \left(|u_\delta - (u_\delta)_{2r}|^2 + \eta^2 |\nabla \eta|^2 \omega_\delta^2 \right) \mathrm{d}x \le c
$$

is immediately verified,

$$
\begin{aligned}
|I_3| &\le c\delta \int_{B_{2r}(x_0)} \left(|\nabla u_\delta|^2 |u_\delta - (u_\delta)_{2r}|^2 \eta^2 + |\nabla \omega_\delta|^2 \eta^2 \right) \mathrm{d}x \\
&\le c \left(1 + \delta \int_{B_{2r}(x_0)} (1 + |\nabla u_\delta|^2)^{-1} |\nabla^2 u_\delta|^2 \eta^2 \, \mathrm{d}x \right) \\
&\le c
\end{aligned}
$$

again follows from Lemma 2 and Corollary 1. Thus, together with

$$
|I_4| \le c\delta \int_{B_{2r}(x_0)} \left(|\nabla u_\delta|^2 |u_\delta - (u_\delta)_{2r}|^2 \eta^2 + |\nabla \eta|^2 \omega_\delta^2 \right) \mathrm{d}x \le c \, ,
$$

the Theorem is proved recalling (15) and since the constants occurring above are not depending on δ. □

Let us turn our attention to the vectorial setting $N > 1$.

Theorem 5. *Theorem 4 extends to the two dimensional vector-valued case $n = 2$, $N > 1$.*

Proof. The theorem is established once the following claims are verified (we keep the notation introduced above)

i) $\varphi = |u_\delta - (u_\delta)_{2r}|^2 (u_\delta - (u_\delta)_{2r}) \eta^2$ is admissible in the Euler equation iv) of Lemma 1 (this test-function is used to prove Lemma 2).
ii) This choice of φ implies (11).
iii) $\varphi = (u_\delta - (u_\delta)_{2r}) \omega_\delta \eta^2$ also is admissible (this is necessary to follow the arguments given in the proof of Theorem 4).

If i)–iii) are verified, then the remaining arguments given in the proofs of Lemma 2 and Theorem 4 can be carried over to the vectorial setting without any changes.

ad i) & iii). We already have noted (see Lemma 1, iii)) that u_δ is of class $W^2_{2,loc} \cap W^1_{\infty,loc}(\Omega;\mathbb{R}^N)$. This immediately gives i) and iii).

ad ii). Here we first observe that the representation $f(Z) = g(|Z|^2)$ implies

$$\nabla f(0) = 0\,.$$

In particular we have

$$\nabla f(Z) : Z = \int_0^1 D^2 f(\theta Z)(Z,Z)\,\mathrm{d}\theta \ge 0\,,$$

thus, with the notation $f_\delta(Z) = g_\delta\big(|Z|^2\big)$,

$$g'_\delta(|Z|^2) \ge 0 \quad \text{for any} \quad Z \in \mathbb{R}^{2N}\,. \tag{16}$$

We next let $\psi = |u_\delta - (u_\delta)_{2r}|^2(u_\delta - (u_\delta)_{2r})$, and with the help of (16) we obtain a.e.

$$\begin{aligned}
\nabla f_\delta(\nabla u_\delta) : \nabla\psi &= 2g'_\delta(|\nabla u_\delta|^2)\nabla u_\delta : \nabla\psi\\
&= 2g'_\delta(|\nabla u_\delta|^2)\Big[\partial_\alpha u^i_\delta \partial_\alpha u^i_\delta |u_\delta - (u_\delta)_{2r}|^2\\
&\quad +\big(\partial_\alpha u^i_\delta(u^i_\delta - (u_\delta)^i_{2r})\big)2\big(\partial_\alpha u^j_\delta(u^j_\delta - (u_\delta)^j_{2r}\big)\Big]\\
&\ge 2g'_\delta(|\nabla u_\delta|^2)\partial_\alpha u^i_\delta \partial_\alpha u^i_\delta |u_\delta - (u_\delta)_{2r}|^2\\
&= \nabla f_\delta(\nabla u_\delta) : \nabla u_\delta |u_\delta - (u_\delta)_{2r}|^2\,.
\end{aligned}$$

Of course this implies (11) exactly in the same way as above, and Theorem 5 is proved. □

Before we are going to discuss the case $\mu < 3$, let us complete the

Proof of Theorem 3 in the case $\mu = 3$. We fix the regularization $\{u_\delta\}$ as introduced above. Then, if $n = 2$, Theorem 4 and Theorem 5, respectively, together with the de la Vallèe Poussin criterion yield a subsequence (which is not relabelled) such that $u_\delta \rightharpoondown : u^*$ in $W^1_{1,loc}(\Omega;\mathbb{R}^N)$ (recall that Lemma 1, ii), gives $u^* \in \mathcal{M}$). Lower semicontinuity w.r.t. weak W^1_1-convergence then proves the assertions i) and ii) as stated in Theorem 1, where we now (in contrast to [Bi3]) merely have to assume that $u_0 \in W^1_1(\Omega;\mathbb{R}^N)$. The last claim is a consequence of the following Lemma given in [BF3] (compare [Bi2]). □

Lemma 3. *Suppose that the variational integrand $f\colon \mathbb{R}^{nN} \to [0,\infty)$ is strictly convex, of linear growth, i.e.*

$$a|Z| - b \le f(Z) \le A|Z| + B$$

with some positive constants $a > 0$, $A > 0$, b, B, and satisfies $f(0) = 0$. Moreover, we assume that there exists

$$u^* \in \mathcal{M}' := \Big\{ u \in \mathcal{M} : \, u \in W^1_{1,loc}(\Omega; \mathbb{R}^N) \Big\} = \mathcal{M} \cap W^1_1(\Omega; \mathbb{R}^N) \, .$$

Then we have

i) The elements of $\mathcal{M}'$ are solutions of problem $(\mathcal{P}')$ and vice versa.
ii) The set $\mathcal{M}'$ is uniquely determined up to constants.

Proof. Recalling the fact that $\mathcal{M}$ coincides with the set of solutions of the variational problem $(\hat{\mathcal{P}})$, we shortly sketch the proof for the sake of completeness.

ad i). Fix $u^* \in \mathcal{M}'$. On account of the K-minimizing property of u^* and since $\nabla^s u^* \equiv 0$, the representation of K clearly implies that $u^* \in \mathcal{M}'$ is a solution of $(\mathcal{P}')$.

Conversely, consider a solution v^* of problem $(\mathcal{P}')$ and a J-minimizing sequence $\{u_m\}$ from $u_0 + \overset{\circ}{W}{}_1^1(\Omega; \mathbb{R}^N)$. The minimality of v^* gives

$$K[v^*] = \int_\Omega f(\nabla v^*) \, \mathrm{d}x + \int_{\partial\Omega} f_\infty((u_0 - v^*) \otimes \nu) \, d\mathcal{H}^{n-1} \le \int_\Omega f(\nabla u_m) \, \mathrm{d}x \, ,$$

and i) follows from $\inf\{J[w] \, : \, w \in u_0 + \overset{\circ}{W}{}_1^1(\Omega; \mathbb{R}^N)\} = \inf\{K[w] \, : \, w \in BV(\Omega; \mathbb{R}^N)\}$ and the above mentioned identification of solutions.

ad ii). To prove uniqueness up to a constant, we just observe that f_∞ is convex, whereas f is strictly convex. This immediately gives $\nabla u^* = \nabla u^{**}$ a.e. for any two generalized minimizers u^*, $u^{**} \in \mathcal{M}'$, hence the lemma is proved. □

4. The Case $\mu < 3$

Proof of Theorem 3 in the case $\mu < 3$. We proceed in three steps:

we first fix a L^1-cluster point $u^* \in \mathcal{M}$ of the regularizing sequence $\{u_\delta\}$ and use the higher integrability established in the last section to define a suitable local auxiliary variational problem. Here we find uniform local gradient estimates according to Theorem 6.1 of [Bi3].

Next, the auxiliary solutions are modified and extended to the whole domain Ω. We obtain a sequence $\{w_m\}$, where it turns out that the L^1-cluster points w^* are generalized minimizers of the original problem, hence elements of the set $\mathcal{M}$.

Finally, the duality relation holds a.e. both for u^* and for w^*, which completes the proof of Theorem 3.

Step 1. From now on suppose that Assumption 1 holds with $n = 2$ and $\mu < 3$. We fix a L^1-cluster point u^* of the regularizing sequence $\{u_\delta\}$ (introduced in Section 2), and recall that u^* is already known to be of class $W^1_1(\Omega; \mathbb{R}^N)$. We fix $x_0 \in \Omega$

and write with a slight abuse of notation $u^*(r,\theta) = u^*(x_0 + re^{i\theta})$. Moreover, let us assume that $B_{2R_0}(x_0) \Subset \Omega$ and observe that

$$\int_0^{R_0}\int_0^{2\pi} \left|\frac{\partial u^*}{\partial\theta}\right| \,\mathrm{d}\theta\,\mathrm{d}r \le \int_0^{R_0}\int_0^{2\pi} |\nabla u^*|\,\mathrm{d}\theta\, r\,\mathrm{d}r \le c < \infty\,.$$

Hence there exists a radius $R_0/2 \le R \le R_0$ such that

$$\int_0^{2\pi} \left|\frac{\partial u^*(R,\theta)}{\partial\theta}\right| \mathrm{d}\theta \le c < \infty\,. \tag{17}$$

Next, we pass to a smooth sequence $\{u_m\}$, $u_m \in C^\infty(\Omega;\mathbb{R}^N)$, with the property

$$u_m \to u^* \quad \text{in } \; W_1^1(\Omega;\mathbb{R}^N) \; \text{ as } \; m \to \infty\,, \tag{18}$$

hence it is possible to estimate

$$\begin{aligned}\int_0^{R_0} h_m(r)\,\mathrm{d}r &:= \int_0^{R_0}\int_0^{2\pi} \left|\frac{\partial(u_m - u^*)}{\partial\theta}\right| \mathrm{d}\theta\,\mathrm{d}r \\ &\le \int_0^{R_0}\int_0^{2\pi} |\nabla(u_m - u^*)|\,\mathrm{d}\theta\, r\,\mathrm{d}r \overset{m\to\infty}{\to} 0\,.\end{aligned}$$

Thus, $h_m(r) \to 0$ in $L^1((0,R_0))$ as $m \to \infty$, and we may assume in addition to (17) that R is chosen to satisfy

$$\int_0^{2\pi} \left|\frac{\partial u_m(R,\theta)}{\partial\theta}\right| \mathrm{d}\theta \le c < \infty\,, \tag{19}$$

where the constant c does not depend on m. As a consequence of (19) it is finally established: there is a radius $R \in (R_0/2, R_0)$ and real number $K > 0$ such that for all $m \in \mathbb{N}$

$$\left|u_{m|\partial B_R(x_0)}\right| \le K\,, \tag{20}$$

and we have found suitable boundary data to consider the variational problem

$$\begin{aligned} J_\delta[w, B_R(x_0)] &:= \int_{B_R(x_0)} f(\nabla w)\,\mathrm{d}x + \frac{\delta}{2}\int_{B_R(x_0)} |\nabla w|^2\,\mathrm{d}x \\ &\to \min \quad \text{in } \; u_m + \overset{\circ}{W}{}_2^1(B_R(x_0);\mathbb{R}^N)\,. \end{aligned} \tag{$\mathcal{P}_\delta^m$}$$

If $\delta = \delta(m)$ is chosen sufficiently small (analogous arguments are given in Section 2) and if we denote by v_m the unique solution of problem $(\mathcal{P}_\delta^m)$, then

$$J_{\delta(m)}[v_m, B_R(x_0)] \le J_{\delta(m)}[u_m, B_R(x_0)] \le c \tag{21}$$

follows with a constant c not depending of m. Moreover, by (20), we find (citing for example the maximum principle given in [DLM] or the convex hull property shown in [BF4])

$$\|v_m\|_{L^\infty(B_R(x_0);\mathbb{R}^N)} \le K\,. \tag{22}$$

At this point we observe that the a priori gradient estimates established in Theorem 6.1 of [Bi3] only depend on the data and the constants occurring on the right-hand side of (21) and (22), respectively. As a result, a real number $c > 0$, independent of m, is found such that

$$\|\nabla v_m\|_{L^\infty(B_{R/2}(x_0);\mathbb{R}^{2N})} \le c\,. \tag{23}$$

Step 2. Given u^*, u_m and v_m as above we choose $\eta \in C^\infty(B_R(x_0))$, $\eta \equiv 1$ on $B_R(x_0) - B_{3R/4}(x_0)$, $\eta \equiv 0$ on $B_{R/2}(x_0)$, and let w_m^1: $B_R(x_0) \to \mathbb{R}^N$,

$$w_m^1 := v_m + \eta(u^* - u_m)\,, \qquad \text{hence} \qquad w^1_{m\,|\partial B_R(x_0)} = u^*_{|\partial B_R(x_0)}\,.$$

We then claim that w_m^1 provides a $J_{|B_R(x_0)}$-minimizing sequence w.r.t. the boundary data $u^*_{|B_R(x_0)}$: in fact, (18) implies as $m \to \infty$

$$\left|\int_{B_R(x_0)} \big(f(\nabla u_m) - f(\nabla u^*)\big)\,\mathrm{d}x\right| \le c \int_{B_R(x_0)} |\nabla u_m - \nabla u^*|\,\mathrm{d}x \to 0\,,$$

and if we decrease δ (if necessary), then we obtain from the minimality of v_m

$$\begin{aligned}\int_{B_R(x_0)} f(\nabla v_m)\,\mathrm{d}x &\le J_{\delta(m)}[v_m, B_R(x_0)] \le J_{\delta(m)}[u_m, B_R(x_0)]\\ &\overset{m\to\infty}{\to} \int_{B_R(x_0)} f(\nabla u^*)\,\mathrm{d}x\,.\end{aligned} \tag{24}$$

Moreover, we have

$$\begin{aligned}\left|\int_{B_R(x_0)} \big(f(\nabla w_m^1) - f(\nabla v_m)\big)\,\mathrm{d}x\right| \quad &\le \quad c\int_{B_R(x_0)} \big|\nabla(\eta(u^* - u_m))\big|\,\mathrm{d}x\\ &\overset{m\to\infty}{\to} 0\,,\end{aligned}$$

which, together with (24) and the minimality of u^* (recall that $u^* \in W_1^1(\Omega;\mathbb{R}^N)$ is a local J-minimizer) implies

$$\int_{B_R(x_0)} f(\nabla w_m^1)\,\mathrm{d}x \overset{m\to\infty}{\to} \int_{B_R(x_0)} f(\nabla u^*)\,\mathrm{d}x\,, \tag{25}$$

i.e. the assertion is proved.

Next we claim that the sequence $\{w_m^1\}$ can be extended to a J-minimizing sequence from $u_0 + \overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^N)$.

To this purpose we recall that, according to the previous sections, there exists a J-minimizing sequence from $u_0 + \overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^N)$, which we now denote by $\{u_k'\}$, such that we even have $u_k' \rightharpoondown u^*$ in $W^1_{1,loc}(\Omega;\mathbb{R}^N)$ as $k \to \infty$. With [BF1], Lemma 7.1 on local comparison functions, we find a J-minimizing sequence $\{u_k''\}$ from $u_0 + \overset{\circ}{W}{}_1^1(\Omega;\mathbb{R}^N)$ such that for any $k \in \mathbb{N}$ and for a suitable ball $B_{R'}(x_0)$, $R < R'$, the identity

$$u''_{k\,|B_{R'}(x_0)} \equiv u^*_{|B_{R'}(x_0)}\,, \qquad \text{in particular} \qquad u''_{k\,|\partial B_R(x_0)} \equiv u^*_{|\partial B_R(x_0)}\,,$$

holds true. On the other hand, for all $m \in \mathbb{N}$ we also have $w^1_{m|\partial B_R(x_0)} \equiv u^*_{|\partial B_R(x_0)}$, hence, on account of (25), it is possible to extend the sequence $\{w^1_m\}$ to a J-minimizing sequence $\{w_m\}$ from $u_0 + \overset{\circ}{W}{}^1_1(\Omega; \mathbb{R}^N)$. Summarizing these remarks, it is proved in the second step that L^1-cluster points w^* of the extended sequence $\{w_m\}$ are generalized minimizers in the sense that $w^* \in \mathcal{M}$.

Step 3. Finally we recall that partial regularity for u^* follows from [AG] (compare [BF1] and [Bi2]), i.e. there is an open set $\Omega_0 \subset \Omega$ of full Lebesgue measure, $|\Omega - \Omega_0| = 0$, such that

$$u^* \in C^{1,\alpha}(\Omega_0; \mathbb{R}^N)\,.$$

As a consequence, the duality relation

$$\sigma = \nabla f(\nabla u^*) \quad \text{ in } \ \Omega_0$$

is derived in [BF1]. Here σ denotes the solution of the dual variational problem (see [ET] for precise definitions and a detailed discussion). Let us just note that σ is uniquely determined (see [Bi1]) and that on the open set Ω_0 it is admissible to perform the variation of σ as described in [BF2], Lemma 5.1 (compare [Se4] for an earlier discussion of this minimax inequality). As a result, any generalized minimizer $v^* \in \mathcal{M}$ is also seen to satisfy

$$\sigma = \nabla f(\nabla v^*) \quad \text{ in } \ \Omega_0\,.$$

Since $w^* \in \mathcal{M}$ was proved in Step 2, we obtain

$$\nabla w^* = \nabla u^* \quad \text{ a.e.}$$

On the other hand, recall that

$$w_{m|B_{R/2}(x_0)} = w^1_{m|B_{R/2}(x_0)} = v_{m|B_{R/2}(x_0)}\,,$$

hence the a priori estimate (23) yields

$$\|\nabla u^*\|_{L^\infty(B_{R/2}(x_0); \mathbb{R}^{2N})} \leq c\,.$$

Note that we really have local Lipschitz continuity of u^*, since $u^* \in W^1_1(\Omega; \mathbb{R}^N)$, in particular $\nabla^s u^* \equiv 0$, was already shown in the last section.

Once we have established local a priori gradient estimates, local $C^{1,\alpha}$-regularity follows in a standard way (see [GT] for the scalar case and [GM], [MS] in the vector-valued setting, some details are given in [Bi2]). Note that in the vector case $N > 1$ condition (3) is chosen in accordance to [GM]. To complete the proof of Theorem 3 in the case $\mu < 3$, we finally observe that uniqueness up to a constant follows with the help of the above mentioned variation of σ (details are given in [BF2] and [Se4]). $\square$

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