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The effect of a penalty term involving higher order derivatives on the distribution of phases in an elastic medium with a two-well elastic potential

M. Bildhauer[1], M. Fuchs[2,*,†] and V. G. Osmolovskii[3]

[1] *Universität des Saarlandes, Fachrichtung 6.1 Mathematik, Postach 15 11 50, D-66041 Saarbrücken, Germany*
[2] *Universität des Saarlandes, Fachrichtung 6.1 Mathematik, Postach 15 11 50, D-66041 Saarbrücken, Germany*
[3] *Mathematics & Mechanics, Faculty, St. Petersburg State University, 198904 St. Petersburg, Russia*

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SUMMARY

We consider the problem of minimizing

$$I[u,\chi,h,\sigma]=\int_\Omega(\chi f_h^+(\varepsilon(u))+(1-\chi)f^-(\varepsilon(u)))\,\mathrm{d}x+\sigma\left(\int_\Omega|\Delta u|^2\,\mathrm{d}x\right)^{p/2}$$

$0<p<1,\ h\in\mathbb{R},\ \sigma>0$, among functions $u:\mathbb{R}^d\supset\Omega\to\mathbb{R}^d,\ u_{|\partial\Omega}=0$, and measurable characteristic functions $\chi:\Omega\to\mathbb{R}$. Here f_h^+, f^-, denote quadratic potentials defined on the space of all symmetric $d\times d$ matrices, h is the minimum energy of f_h^+ and $\varepsilon(u)$ denotes the symmetric gradient of the displacement field. An equilibrium state $\hat{u},\hat{\chi}$ of $I[\cdot,\cdot,h,\sigma]$ is termed one-phase if $\hat{\chi}\equiv 0$ or $\hat{\chi}\equiv 1$, two-phase otherwise. We investigate the way in which the distribution of phases is affected by the choice of the parameters h and σ.

KEY WORDS: elastic materials; phase transition; equilibrium states; regularization

1. INTRODUCTION

We consider an elastic medium which can exist in two different phases. If the medium occupies a bounded region $\Omega\subset\mathbb{R}^d$ (assumed to be of class C^2), then the energy density of the first (second) phase is given by

$$f_h^+(\varepsilon(u))=\langle A^+(\varepsilon(u)-\xi^+),\varepsilon(u)-\xi^+\rangle+h$$
$$(f^-(\varepsilon(u))=\langle A^-(\varepsilon(u)-\xi^-),\varepsilon(u)-\xi^-\rangle)$$

* Correspondence to: M. Fuchs, Universität des Saarlandes, Fachrichtung 6.1. Mathematik, Postach 15 11 50, D-66041 Saarbrücken, Germany
† E-mail: fuchs@math.uni-sb.de

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where $u=(u^1,\dots,u^d):\Omega\to\mathbb{R}^d$ is the field of displacements, $\varepsilon(u)=\frac{1}{2}(\partial_i u^j+\partial_j u^i)_{1\leqslant i,j\leqslant d}$ denotes the corresponding strain tensor, and $A^{\pm}:\mathbb{S}^d\to\mathbb{S}^d$ are linear, symmetric operators defined on the space $\mathbb{S}^d$ of all symmetric $d\times d$ matrices having the meaning of the tensors of elastic moduli of the first and the second phase. Finally, $\xi^{\pm}\in\mathbb{S}^d$ denote the stress-free strains of the ith phase, and we use the symbol $\langle\varepsilon,\varkappa\rangle:=\mathrm{tr}(\varepsilon\varkappa)$ for the scalar product in $\mathbb{S}^d$. Thus, the energy density of each phase is a quadratic function of the linear strain, where the energy density of the first phase depends in addition on the parameter $h\in\mathbb{R}$. Let us state the hypotheses imposed on the data: $A^{\pm}$ are assumed to be positive, i.e. for some number $v>0$ we have

$$v|\varepsilon|^2\leqslant\langle A^{\pm}\varepsilon,\varepsilon\rangle\leqslant v^{-1}|\varepsilon|^2\quad\text{for all }\varepsilon\in\mathbb{S}^d\tag{1}$$

hence, the parameter h measures the difference between the minima of f_h^+ and f^-. As a second condition concerning the tensors of elastic moduli we require that for some number $\mu\in(0,v)$

$$|\langle(A^+-A^-)\varepsilon,\varepsilon\rangle|\leqslant\mu|\varepsilon|^2\quad\text{for all }\varepsilon\in\mathbb{S}^d\tag{2}$$

is satisfied. Finally, we suppose that

$$A^+\xi^+\neq A^-\xi^-\tag{3}$$

is valid. Clearly, (2) holds in case $A^+=A^-$ for which (3) reduces to the condition $\xi^+\neq\xi^-$. If χ denotes the characteristic function of the set occupied by the first phase, then it is natural to take the functional

$$J[u,\chi,h]:=\int_\Omega(\chi f_h^+(\varepsilon(u))+(1-\chi)f^-(\varepsilon(u)))\,\mathrm{d}x\tag{4}$$

as the total deformation energy of the medium and to define an equilibrium state of J as a minimizing pair $(\hat{u},\hat{\chi})$ consisting of a deformation $\hat{u}$ and a measurable characteristic function $\hat{\chi}$. Following standard convention, we say that the equilibrium state is one-phase if $\hat{\chi}\equiv 0$ or $\hat{\chi}\equiv 1$, two-phase otherwise. Let us consider displacement fields u vanishing on $\partial\Omega$. Then the domain of definition of the functional $J[\cdot,\cdot,h]$ is the space of all pairs (u,χ) with $u\in X:=\overset{\circ}{W}{}_2^1(\Omega;\mathbb{R}^d)$ (equipped with the norm $\|u\|_X:=\|\varepsilon(u)\|_{L^2(\Omega;\mathbb{S}^d)}$) and χ denoting an arbitrary measurable characteristic function $\Omega\to\mathbb{R}$. Unfortunately, the variational problem $J[\cdot,\cdot,h]\to\min$ may fail to have solutions as it is shown by an example in Reference [1]. One way to overcome this difficulty is to introduce the quasiconvex envelope $\tilde{f}_h$ of the integrand $f_h:=\min\{f_h^+,f^-\}\leqslant\chi f_h^++(1-\chi)f^-$ (see Reference [2] for a definition) and to pass to the relaxed problem

$$\int_\Omega\tilde{f}_h(\varepsilon(u))\,\mathrm{d}x\to\min\quad\text{in }X$$

(note that by Dacorogna's formula $u\equiv 0$ is a solution; non-trivial solutions were produced in Reference [3]), we refer the reader to References [2, 4, 5] for a more detailed outline of this approach and for further references. From the physical point of view (compare Reference [6]), it is also reasonable to consider a regularization of the functional J from (4), taking the area of the separating surface between the different phases into account, i.e. we replace J by the energy

$$J[u,\chi,h,\sigma]=J[u,\chi,h]+\sigma\int_\Omega|\nabla\chi| \tag{5}$$

where $\sigma>0$ denotes a parameter, and the characteristic function χ is required to be an element of the space $BV(\Omega)$ of all functions having bounded variation (see for example [7] for definitions). This model was investigated in References [8, 3, 9] establishing various existence results for the functional from (5), in particular, in Reference [9] we showed how the distribution of phases depends on the choices for the parameters h and σ.

In the present note we regularize $J[u,\chi,h]$ by adding a penalty term involving higher order derivatives of the displacement field. In principal, this model was proposed by Kohn and Müller in References [10, 11]. To be precise, suppose that a number $0<p<1$ is fixed, and for $\sigma>0$ let

$$I[u,\chi,h,\sigma]:=J[u,\chi,h]+\sigma\left(\int_\Omega|\Delta u|^2\,\mathrm{d}x\right)^{p/2} \tag{6}$$

where now $u\in H:=W_2^2(\Omega;\mathbb{R}^d)\cap X$ and (as in (4))

$$\chi\in M:=\{\text{measurable characteristic functions }\Omega\to\mathbb{R}\}$$

With a slight abuse of notation we sometimes only assume $\chi\in L^\infty(\Omega)$, $0\leqslant\chi\leqslant1$ a.e., equilibrium states of I however are always defined w.r.t. $H\times M$. Note, that on account of $\partial\Omega\in C^2$, the quantity

$$\|u\|_H:=\|\Delta u\|_{L^2(\Omega;\mathbb{R}^d)}$$

introduces a norm on the space H being equivalent to the W_2^2-norm which is a consequence of the Calderon–Zygmund regularity results. Our main result now concerns the analysis of the effect of the parameters $h\in\mathbb{R}$ and $\sigma>0$ on the distribution of phases, we have

Theorem 1.1. Let (1)–(3) hold. Then, for each $h\in\mathbb{R}$ and all $\sigma>0$, the functional $I[\cdot,\cdot,h,\sigma]$ attains its minimum on the set $H\times M$. There are two bounded, continuous functions $h^\pm(\sigma)$, $\sigma>0$, and a number $\sigma^*>0$ with the following properties:

$$h^+(\sigma)>\hat h \quad \text{on } (0,\sigma^*), \quad h^+(\sigma)\equiv\hat h \quad \text{for } \sigma\geqslant\sigma^*$$

$$h^-(\sigma)<\hat h \quad \text{on } (0,\sigma^*), \quad h^-(\sigma)\equiv\hat h \quad \text{for } \sigma\geqslant\sigma^*$$

$$\hat h:=\langle A^-\xi^-,\xi^-\rangle-\langle A^+\xi^+,\xi^+\rangle$$

h^+ strictly decreases on $(0,\sigma^*)$, h^- is strictly increasing on $(0,\sigma^*)$

The graphs of $h^\pm$ divide the half-plane of parameters $\sigma>0$, $h\in\mathbb{R}$, into three open regions

$$A:=\{(\sigma,h):\sigma>0,h>h^+(\sigma)\}$$

$$B:=\{(\sigma,h):0<\sigma<\sigma^*,h^-(\sigma)<h<h^+(\sigma)\}$$

$$C:=\{(\sigma,h):\sigma>0,h<h^-(\sigma)\}$$

in which we have the following distribution of phases:
(i) for $(\sigma, h) \in A$ we only have the one-phase equilibrium $\hat{u} \equiv 0$, $\hat{\chi} \equiv 0$;
(ii) for $(\sigma, h) \in C$ only the one-phase equilibrium $\hat{u} \equiv 0$, $\hat{\chi} \equiv 1$ exists;
(iii) for $(\sigma, h) \in B$ only two-phase states of equilibria exist.
On the graphs of $h^{\pm}$ we have the following distribution of equilibrium states:
(iv) for $h = h^{+}(\sigma)$, $0 < \sigma < \sigma^{*}$, we have the one-phase equilibrium state $\hat{u} \equiv 0$, $\hat{\chi} \equiv 0$ and at least one two-phase equilibrium;
(v) for $h = h^{-}(\sigma)$, $0 < \sigma < \sigma^{*}$, we have the one-phase equilibrium state $\hat{u} \equiv 0$, $\hat{\chi} \equiv 1$ and at least one two-phase equilibrium;
(vi) for $h = \hat{h}$, $\sigma > \sigma^{*}$, the equilibrium states consist of the pairs $\hat{u} \equiv 0$, $\hat{\chi} \equiv$ *any measurable characteristic function*;
(vii) for $h = \hat{h}$, $\sigma = \sigma^{*}$, there exist the equilibrium states $\hat{u} \equiv 0$, $\hat{\chi} \equiv$ *arbitary measurable characteristic function* and at least one two-phase equilibrium state with $\hat{u} \not\equiv 0$.

Remark 1.2. (a) Except for the behaviour at $h = \hat{h}$ together with $\sigma \geqslant \sigma^{*}$ (see (vi) and (vii)) Theorem 1.1 corresponds in a qualitative sense to Theorem 2.1 in Reference [9]. Of course we do not claim that the functions $h^{\pm}$ as well as the numbers σ^{*} are the same in both cases.

(b) The different behaviour for the choice $h = \hat{h}$, $\sigma \geqslant \sigma^{*}$ originates from the fact that in this case the penalty term $\sigma(\int_{\Omega} |\Delta u|^2 \, dx)^{p/2}$ does not create a formation of phases.

(c) In Reference [3] the reader will find further comments on the above model, moreover, the choice $p < 1$ is explained.

Concerning the regularity of solutions, we have the following.

Theorem 1.3. With the above notation let $(\hat{u}, \hat{\chi}) \in H \times M$ denote on equilibrium state of $I[\cdot, \cdot, h, \sigma], \sigma > 0$. Then $\hat{u}$ is of class $C^{2,\alpha}(\Omega; \mathbb{R}^d)$ for any $0 < \alpha < 1$.

Remark 1.4. For $h \in \mathbb{R}$, $\sigma > 0$ and $u \in H$ let (recall $f_h = \min\{f_h^{+}, f^{-}\}$)

$$\tilde{I}[u, h, \sigma] = \int_{\Omega} f_h(\varepsilon(u)) \, dx + \sigma \|\Delta u\|^2_{L^2(\Omega; \mathbb{R}^d)}$$

Clearly, the variational problem

$$\tilde{I} \to \min \quad \text{on } H$$

has at least one solution $\hat{u}$ (compare also Lemma 2.2 and Theorem 2.3 below). For $u \in H$ let

$$\chi_u := \begin{cases} 0 & \text{if } f_h^{+}(\varepsilon(u)) \geqslant f^{-}(\varepsilon(u)), \\ 1 & \text{otherwise.} \end{cases}$$

Then we have

$$I[u, \chi, h, \sigma] \geqslant \tilde{I}[u, h, \sigma] \geqslant \tilde{I}[\hat{u}, h, \sigma] = I[\hat{u}, \chi_{\hat{u}}, h, \sigma]$$

for any $u \in H$ and any measurable characteristic function χ. Thus $\hat{u}$ generates a minimizing pair $(\hat{u}, \chi_{\hat{u}})$ of $I[\cdot, \cdot, h, \sigma]$. Conversely, consider an equilibrium state $(\check{u}, \check{\chi})$ of $I[\cdot, \cdot, h, \sigma]$. Observing (recall $f_h \leqslant \check{\chi} f_h^{+} + (1 - \check{\chi}) f^{-}$)

$$\tilde{I}[\check{u}, h, \sigma] \leqslant I[\check{u}, \check{\chi}, h, \sigma] \leqslant I[u, \chi_u, h, \sigma] = \tilde{I}[u, h, \sigma] \quad \text{for all } u \in H$$

we deduce $\tilde{I}[\cdot, h, \sigma]$-minimality of $\check{u}$. So there is a one-to-one correspondence between the minimizing deformation fields of both functionals. But the deformation field u alone does not serve the complete information, for example, in case $u \equiv 0$ there exist various possibilities for the distribution of phases as described in Theorem 1.1.

As an alternative to the model proposed in Theorem 1.1 we may associate to each $\tilde{I}[\cdot, h, \sigma]$-minimizing deformation field $\hat{u}$ the function $\chi_{\hat{u}}$ and introduce the notion of one (two)-phase equilibrium states $(\hat{u}, \chi_{\hat{u}})$ as before. Then again we get the statements of Theorem 1.1 where in part (vi) and (vii) the phrase "$\hat{\chi}=$ any measurable characteristic function" has to be replaced by the requirement $\hat{\chi} = \chi_0$. Obviously the number of equilibrium states $(\hat{u}, \chi_{\hat{u}})$ generated by $\tilde{I}[\cdot, h, \sigma]$-minimizers $\hat{u}$ is in general much smaller than the number of equilibria considered in the first model: if $\hat{\chi}$ is a measurable characteristic function satisfying

$$\int_\Omega f_h(\varepsilon(\hat{u}))\,\mathrm{d}x = \int_\Omega (\hat{\chi} f_h^+(\varepsilon(\hat{u})) + (1-\hat{\chi}) f^-(\varepsilon(\hat{u})))\,\mathrm{d}x$$

then $(\hat{u}, \hat{\chi})$ is a minimizing pair for $I[\cdot,\cdot,h,\sigma]$. But since we are mainly interested in the qualitative behaviour of the distribution of phases depending on h and σ, we do not see any principal difference between both models except for the different behaviour at $h = \hat{h}$, $\sigma \geqslant \sigma^*$.

Remark 1.5. At the end, let us briefly discuss some situations for which the non-uniqueness w.r.t. the function χ can be removed. Let $(\hat{u}, \hat{\chi})$ denote an equilibrium state of $I[u, \chi, h, \sigma]$ with $\hat{\chi} := \chi_{\hat{u}}$. We introduce the sets

$$E^{+(-)} := [\, f_h^+(\varepsilon(\hat{u})) > (<) f^-(\varepsilon(\hat{u}))]$$
$$E^0 := [\, f_h^+(\varepsilon(\hat{u})) = f^-(\varepsilon(\hat{u}))]$$

and consider $\chi \in L^\infty(\Omega)$, $0 \leqslant \chi \leqslant 1$. Then

$$I[\hat{u}, \hat{\chi}, h, \sigma] = I[\hat{u}, \chi, h, \sigma] \tag{7}$$

if and only if

$$\int_{E^+} (\hat{\chi} - \chi)(f_h^+(\varepsilon(\hat{u})) - f^-(\varepsilon(\hat{u})))\,\mathrm{d}x + \int_{E^-} (\hat{\chi} - \chi)(f_h^+(\varepsilon(\hat{u})) - f^-(\varepsilon(\hat{u})))\,\mathrm{d}x = 0$$

Since $\hat{\chi} = \chi_{\hat{u}} = \begin{cases} 0 & \text{on } E^+ \\ 1 & \text{on } E^- \end{cases}$, we see

$$\chi = \hat{\chi} \quad \text{on } E^+ \cup E^- \tag{8}$$

and the 'non-uniqueness' can be excluded for the case that E_0 is a set of Lebesgue measure zero. In order to find a sufficient condition for $|E_0| = 0$ let us assume that $\hat{u} \not\equiv 0$. Then $\|\Delta\hat{u}\|_{L^2(\Omega;\mathbb{R}^d)} > 0$ and for any $v \in H$ the expression $\|\Delta\hat{u} + t\Delta v\|_{L^2(\Omega;\mathbb{R}^d)} > 0$ is differentiable at $t = 0$. For $\chi \in L^\infty(\Omega)$, $0 \leqslant \chi \leqslant 1$, with (8) and all $v \in H$ we have according to (7)

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} I[\hat{u} + tv, \chi, h, \sigma] = 0, \quad \text{i.e.}$$

$$2\int_\Omega \chi\langle A^+(\varepsilon(\hat u)-\xi^+)-A^-(\varepsilon(\hat u)-\xi^-),\varepsilon(v)\rangle\,\mathrm{d}x$$
$$+2\int_\Omega \langle A^-\varepsilon(v),\varepsilon(\hat u)-\xi^-\rangle\,\mathrm{d}x+p\sigma\left(\int_\Omega|\Delta\hat u|^2\right)^{p/2-1}\int_\Omega \Delta\hat u\cdot\Delta v\,\mathrm{d}x=0 \tag{9}$$

Let $|E_0|>0$. Then we use (9) with $\chi=0$ on E_0 and with $\chi=\Phi$ on E_0, where $\Phi\in L^\infty(E_0)$, $0\leqslant\Phi\leqslant 1$. Subtracting the results we get

$$\int_{E_0}\Phi\langle A^+(\varepsilon(\hat u)-\xi^+)-A^-(\varepsilon(\hat u)-\xi^-),\varepsilon(v)\rangle\,\mathrm{d}x=0$$

and since Φ can be chosen arbitrarily, this turns into

$$\langle A^+(\varepsilon(\hat u)-\xi^+)-A^-(\varepsilon(\hat u)-\xi^-),\varepsilon(v)\rangle=0$$

a.e. on E_0. Consider a Lebesgue point $x_0\in E_0$ of $\varepsilon(\hat u)$ and let $v(x)=\eta(x)x_kE^l$ where $\eta\in C_0^\infty(\Omega)$, $\eta\equiv 1$ near x_0, and E^l is the lth standard unit-vector in $\mathbb{R}^d$. Then $\varepsilon(v)(x_0)=(\delta_{ik}\delta^{jl})_{1\leqslant i,j\leqslant d}$ and the above identity implies

$$A^+(\varepsilon(\hat u)-\xi^+)-A^-(\varepsilon(\hat u)-\xi^-)=0$$

on E_0, hence

$$(A^+-A^-)\varepsilon(\hat u)=A^+\xi^+-A^-\xi^-$$

and we get a contradiction if we assume that

$$A^+\xi^+-A^-\xi^-\notin \operatorname{Im}(A^+-A^-) \tag{10}$$

holds. For example we have (10) in case $A^+=A^-$ together with $\xi^+\neq\xi^-$. Thus, the assumption $\hat u\not\equiv 0$ combined with (10) shows $|E_0|=0$ and we can associate to $\hat u$ a unique function χ such that (7) is valid.

Our paper is organized as follows: in Section 2, we prove some existence and lower semicontinuity results concerning the functional I from (6). Section 3 contains a series of lemmata which are used in Section 4 and Section 5 to prove statements (i)–(vii) of Theorem 1.1. In the last section we prove Theorem 1.3.

2. SOME EXISTENCE RESULTS

From now on we assume that all the conditions stated in Section 1 are valid.

Lemma 2.1. Let $h\in\mathbb{R}$, $\sigma\geqslant 0$ be given. Then we have for any $(u,\chi)\in H\times M$

$$\frac{v}{2}\|u\|_X^2+\sigma\|u\|_H^p\leqslant I[u,\chi,h,\sigma]+h|\Omega|+\frac{4+v^2}{v^3}(|\xi^+|^2+|\xi^-|^2)$$

Proof. Assumption (1) implies

$$I[u,\chi,h,\sigma] \geqslant v\int_\Omega |\varepsilon(u)|^2\,\mathrm{d}x - |h|\,|\Omega| + \sigma\|u\|_H^p$$
$$-\frac{1}{v}\int_\Omega (|\xi^+|^2 + |\xi^-|^2)\,\mathrm{d}x$$
$$-2\int_\Omega (|\langle A^+\varepsilon(u),\xi^+\rangle| + |\langle A^-\varepsilon(u),\xi^-\rangle|)\,\mathrm{d}x$$

The lemma is proved by combining this inequality with

$$|\langle A^\pm\varepsilon,\tilde\varepsilon\rangle| \leqslant \sqrt{\langle A^\pm\varepsilon,\varepsilon\rangle}\sqrt{\langle A^\pm\tilde\varepsilon,\tilde\varepsilon\rangle}. \qquad \square$$

Next we establish a lower semicontinuity result.

Lemma 2.2. Consider sequences $\{u_n\}$, $\{\chi_n\}$, $\{h_n\}$ and $\{\sigma_n\}$, $u_n\in H$, $\chi_n\in L^\infty(\Omega)$, $0\leqslant\chi_n\leqslant 1$, $h_n\in\mathbb{R}$, $\sigma_n\geqslant 0$ such that $u_n \rightharpoonup u$ in H, $\chi_n\rightharpoonup\chi$ in $L^2(\Omega)$, $h_n\to h$ and $\sigma_n\to\sigma$ as $n\to\infty$. Then we have

$$I[u,\chi,h,\sigma]\leqslant \liminf_{n\to\infty} I[u_n,\chi_n,h_n,\sigma_n]$$

Proof. The uniform L^∞-bound together with the weak L^2-convergence of the sequence $\{\chi_n\}$ yields

$$\chi_n \overset{n\to\infty}{\rightharpoonup} \chi \text{ in } L^s(\Omega) \text{ for any } s<\infty, \quad 0\leqslant\chi\leqslant 1 \text{ a.e.}$$

The weak H-convergence of the sequence $\{u_n\}$ gives in addition

$$\varepsilon(u_n) \overset{n\to\infty}{\to} \varepsilon(u) \text{ in } L^r(\Omega;\mathbb{S}^d) \quad \text{for some } r>2$$

thus

$$I[u_n,\chi_n,h_n,0]\to I[u,\chi,h,0] \quad \text{as } n\to\infty$$

Moreover, again by weak convergence of the sequence $\{u_n\}$,

$$\|u\|_H^p \leqslant \liminf_{n\to\infty}\|u_n\|_H^p$$

i.e. we get the estimate

$$I[u,\chi,h,\sigma] = I[u,\chi,h,0] + \sigma\|u\|_H^p \leqslant \liminf_{n\to\infty} I[u_n,\chi_n,h_n,0] + \liminf_{n\to\infty}(\sigma_n\|u_n\|_H^p)$$
$$\leqslant \liminf_{n\to\infty}(I[u_n,\chi_n,h_n,0] + \sigma_n\|u_n\|_H^p) = \liminf_{n\to\infty} I[u_n,\chi_n,h_n,\sigma_n] \qquad \square$$

As a consequence we obtain the following existence theorem

Theorem 2.3. The functional $I[\cdot,\cdot,h,\sigma]$, $h\in\mathbb{R}$, $\sigma>0$, attains its minimum on the set $H\times M$.

Proof. Lemma 2.1 immediately gives

$$\gamma := \inf_{(u,\chi)\in H\times M} I[u,\chi,h,\sigma] > -\infty$$

and we may consider a minimizing sequence (u_n,χ_n) s.t. (again recall Lemma 2.1)

$$u_n \rightharpoondown: \hat{u} \text{ in } H, \quad \chi_n \rightharpoondown: \tilde{\chi} \text{ in } L^2(\Omega) \quad \text{as } n\to\infty$$

We do not know that $\tilde{\chi}$ is an element of M, however $0\leqslant\tilde{\chi}\leqslant 1$ and, by Lemma 2.2,

$$I[\hat{u},\tilde{\chi},h,\sigma] \leqslant \liminf_{n\to\infty} I[u_n,\chi_n,h,\sigma] \tag{11}$$

Therefore, if $\hat{\chi}$ is defined via

$$\hat{\chi} := \begin{cases} 0 & \text{on the set } [f_h^+(\varepsilon(\hat{u}))\geqslant f^-(\varepsilon(\hat{u}))] \\ 1 & \text{on the set } [f_h^+(\varepsilon(\hat{u}))< f^-(\varepsilon(\hat{u}))] \end{cases}$$

and if we observe (11) together with

$$\begin{aligned} \tilde{\chi} f_h^+(\varepsilon(\hat{u})) + (1-\tilde{\chi}) f^-(\varepsilon(\hat{u})) &= \tilde{\chi}(f_h^+(\varepsilon(\hat{u})) - f^-(\varepsilon(\hat{u}))) + f^-(\varepsilon(\hat{u})) \\ &\geqslant \hat{\chi}(f_h^+(\varepsilon(\hat{u})) - f^-(\varepsilon(\hat{u}))) + f^-(\varepsilon(\hat{u})) \end{aligned}$$

$(\hat{u},\hat{\chi})\in H\times M$ is seen to be an equilibrium state of I. □

Next, consider the energies of one-phase deformations, i.e. we let

$$I^+[u,h,\sigma] := I[u,1,h,\sigma] = \int_\Omega f_h^+(\varepsilon(u))\,\mathrm{d}x + \sigma\|u\|_H^p$$

$$I^-[u,\sigma] := I[u,0,h,\sigma] = \int_\Omega f^-(\varepsilon(u))\,\mathrm{d}x + \sigma\|u\|_H^p, \quad u\in H$$

Lemma 2.4. On H the functionals $I^\pm$ attain their unique minima at $u^\pm\equiv 0$.

Proof. For any $u\in H$ we have

$$\begin{aligned} I^+[u,h,\sigma] &= \int_\Omega [\langle A^+(\varepsilon(u)-\xi^+), \varepsilon(u)-\xi^+\rangle + h]\,\mathrm{d}x + \sigma\|u\|_H^p \\ &= \int_\Omega \langle A^+\varepsilon(u),\varepsilon(u)\rangle\,\mathrm{d}x + |\Omega|\langle A^+\xi^+,\xi^+\rangle + h|\Omega| + \sigma\|u\|_H^p \\ &\geqslant |\Omega|\langle A^+\xi^+,\xi^+\rangle + h|\Omega| \end{aligned}$$

where equality holds if and only if $u\equiv 0$. An analogous inequality is true for I^- and the lemma is proved. □

We finish this section by introducing the quantity $I_0(h) := \min\{I^+[0,h,\sigma], I^-[0,\sigma]\}$, i.e.

$$I_0(h) = \begin{cases} |\Omega|(\langle A^+\xi^+, \xi^+\rangle + h), & h \leqslant \hat{h} \\ |\Omega|\langle A^-\xi^-, \xi^-\rangle, & h \geqslant \hat{h} \end{cases}$$

$$\hat{h} := \langle A^-\xi^-, \xi^-\rangle - \langle A^+\xi^+, \xi^+\rangle$$

which measures the dependence of the energy of one-phase equilibria on the parameter h.

3. AUXILIARY RESULTS

In this section we prove (under the hypotheses stated in Section 1) a series of auxiliary results which are needed in Section 4 to show Theorem 1.1. We start with two lemmata estimating the X-norm of equilibrium states.

Lemma 3.1. Consider an equilibrium state $(\hat{u}, \hat{\chi})$ of $I[\cdot,\cdot,h,\sigma]$. Then

$$\sigma\|\hat{u}\|_H^p + (v - \mu)\|\hat{u}\|_X^2 \leqslant 2|A^-\xi^- - A^+\xi^+|\sqrt{|\Omega|}\,\|\hat{u}\|_X \tag{12}$$

holds true, in particular, there is a constant R, not depending on h, σ, such that

$$\|\hat{u}\|_X = \|\varepsilon(\hat{u})\|_{L^2(\Omega;\mathbb{S}^d)} \leqslant R \tag{13}$$

Proof. The minimizing property yields $I[\hat{u}, \hat{\chi}, h, \sigma] \leqslant I[0, \hat{\chi}, h, \sigma]$, i.e.

$$\sigma\|\hat{u}\|_H^p + \int_\Omega \hat{\chi}\langle (A^+ - A^-)\varepsilon(\hat{u}), \varepsilon(\hat{u})\rangle\,\mathrm{d}x + \int_\Omega \langle A^-\varepsilon(\hat{u}), \varepsilon(\hat{u})\rangle\,\mathrm{d}x$$
$$+2\int_\Omega \hat{\chi}\langle \varepsilon(\hat{u}), A^-\xi^- - A^+\xi^+\rangle\,\mathrm{d}x \leqslant 0$$

thus the assertions follow from (1) to (3). □

Lemma 3.2. There is a real number $\delta > 0$ such that we have for any equilibrium state $(\hat{u}, \hat{\chi})$ of $I[\cdot,\cdot,h,\sigma]$, $\hat{u} \not\equiv 0$,

$$\|\hat{u}\|_X^{1-p} \geqslant \delta\sigma \tag{14}$$

Proof. From the Calderon–Zygmund regularity results (compare, for example Reference [12], Theorems 9.14 and 9.15), we deduce the existence of a positive number $\kappa = \kappa(\Omega, d)$ such that

$$\|\hat{u}\|_X = \|\varepsilon(\hat{u})\|_{L^2(\Omega;\mathbb{S}^d)} \leqslant \|\hat{u}\|_{W_2^2(\Omega;\mathbb{R}^d)} \leqslant \kappa\|\Delta\hat{u}\|_{L^2(\Omega;\mathbb{R}^d)} = \kappa\|\hat{u}\|_H$$

(12) gives

$$\sigma\|\hat{u}\|_H^p \leqslant 2|A^+\xi^+ - A^-\xi^-|\sqrt{|\Omega|}\,\|\hat{u}\|_X \leqslant 2|A^+\xi^+ - A^-\xi^-|\sqrt{|\Omega|}\,\|\hat{u}\|_X^{1-p}\kappa^p\|\hat{u}\|_H^p$$

implying Lemma 3.2 since

$$\|\hat{u}\|_X^{1-p} \geqslant \sigma \frac{1}{2|A^+\xi^+ - A^-\xi^-|\sqrt{|\Omega|}\kappa^p} =: \sigma\delta \qquad \square$$

In the next lemma we investigate the relation between one-phase equilibrium states and the vanishing of the associated deformation field.

Lemma 3.3. Consider an equilibrium state $(\hat{u}, \hat{\chi})$ of $I[\cdot, \cdot, h, \sigma]$. Then
(a) if $(\hat{u}, \hat{\chi})$ is one-phase, i.e. $\hat{\chi} \equiv 0$ or $\hat{\chi} \equiv 1$, then $\hat{u} \equiv 0$;
(b) if $h \neq \hat{h}$ and if $\hat{u} \equiv 0$, then $(\hat{u}, \hat{\chi})$ is a one-phase equilibrium;
(c) if $h = \hat{h}$ and if $\hat{u} \equiv 0$, then any $\chi \in M$ provides an equilibrium state $(0, \chi)$.

Proof. Assume that $\hat{\chi} \equiv 1$ ($\hat{\chi} \equiv 0$), thus $I[\cdot, \hat{\chi}, h, \sigma] = I^+[\cdot, h, \sigma]$ $(= I^-[\cdot, \sigma])$, hence by Lemma 2.4 $\hat{u} \equiv 0$ and (a) is verified. Next observe that for any $\chi \in M$

$$\begin{aligned} I[0, \chi, h, \sigma] &= [\langle A^+\xi^+, \xi^+\rangle - \langle A^-\xi^-, \xi^-\rangle + h] \int_\Omega \chi \, \mathrm{d}x + |\Omega| \langle A^-\xi^-, \xi^-\rangle \\ &= (h - \hat{h}) \int_\Omega \chi \, \mathrm{d}x + |\Omega| \langle A^-\xi^-, \xi^-\rangle \end{aligned}$$

In the case $h > \hat{h}$, it is seen that

$$I[0, \chi, h, \sigma] \geqslant |\Omega| \langle A^-\xi^-, \xi^-\rangle$$

and equality is true if and only if $\chi \equiv 0$. This proves part (b) for $h > \hat{h}$, the case $h < \hat{h}$ is treated in the same manner. Finally $h = \hat{h}$ implies $I[0, \chi, h, \sigma] = |\Omega| \langle A^-\xi^-, \xi^-\rangle$ for any $\chi \in M$, thus we have (c). $\square$

As a next step, we ensure that the existence of one-phase (two-phase) equilibria depends continuously on h and σ.

Lemma 3.4. Given two sequences $\{h_n\}$, $\{\sigma_n\}$ assume that $h_n \to h_0$ and $\sigma_n \to \sigma_0 > 0$ as $n \to \infty$. As usual denote by $(\hat{u}_n, \hat{\chi}_n)$, $(\hat{u}_0, \hat{\chi}_0)$ equilibrium states of $I[\cdot, \cdot, h_n, \sigma_n]$ and $I[\cdot, \cdot, h_0, \sigma_0]$, respectively.
(a) If $\hat{u}_n \equiv 0$ ($\hat{u}_n \not\equiv 0$) at least for a subsequence, then there exists an equilibrium state $(\hat{u}_0, \hat{\chi}_0)$ satisfying $\hat{u}_0 \equiv 0$ ($\hat{u}_0 \not\equiv 0$).
(b) If $\hat{\chi}_n \equiv 0$ ($\hat{\chi}_n \equiv 1$) for a subsequence, then $I[\cdot, \cdot, h_0, \sigma_0]$ admits an equilibrium state satisfying $\hat{u}_0 \equiv 0$, $\hat{\chi}_0 \equiv 0$ ($\hat{\chi}_0 \equiv 1$).
(c) If $h_0 \neq \hat{h}$ and if $0 \not\equiv \hat{\chi}_n \not\equiv 1$, again at least for a subsequence, then there is a solution with $0 \not\equiv \hat{\chi}_0 \not\equiv 1$.

Proof. From Lemma 2.1 we deduce

$$\begin{aligned} \frac{\nu}{2} \|\hat{u}_n\|_X^2 + \sigma_n \|\hat{u}_n\|_H^p &\leqslant I[\hat{u}_n, \hat{\chi}_n, h_n, \sigma_n] + h_n |\Omega| + \frac{4 + \nu^2}{\nu^3} (|\xi^+|^2 + |\xi^-|^2) \\ &\leqslant I[0, 0, h_n, \sigma_n] + h_n |\Omega| + \frac{4 + \nu^2}{\nu^3} (|\xi^+|^2 + |\xi^-|^2) \end{aligned}$$

hence (recall that $\sigma_0 > 0$) there is a real number $c > 0$ such that $\|\hat{u}_n\|_H \leqslant c < +\infty$. Passing to a subsequence (not relabelled) we may assume that

$$\hat{u}_n \rightharpoonup: \hat{u}_0 \text{ in } H \quad \text{as } n \to \infty$$

Sobolev's embedding theorem then gives the existence of a real number $r > 1$ such that

$$\hat{u}_n \to \hat{u}_0 \text{ in } W^1_{2r}(\Omega; \mathbb{R}^d) \quad \text{as } n \to \infty$$

Moreover, we may assume (again passing to a subsequence if necessary) that

$$\hat{\chi}_n \overset{n\to\infty}{\rightharpoonup:} \tilde{\chi}_0 \text{ in } L^2(\Omega), \quad 0 \leqslant \tilde{\chi}_0 \leqslant 1 \quad \text{a.e.}$$

and applying Lemma 2.2 we see for all $(u, \chi) \in H \times M$

$$I[\hat{u}_0, \tilde{\chi}_0, h_0, \sigma_0] \leqslant \liminf_{n\to\infty} I[\hat{u}_n, \hat{\chi}_n, h_n, \sigma_n] \leqslant \liminf_{n\to\infty} I[u, \chi, h_n, \sigma_n] = I[u, \chi, h_0, \sigma_0]$$

As done in the proof of Theorem 2.3 (compare also Remark 1.4 and Remark 1.5), we may replace $\tilde{\chi}_0$ by a characteristic function $\hat{\chi}_0 \in M$, which provides an admissible minimizer $(\hat{u}_0, \hat{\chi}_0)$ of $I[\cdot, \cdot, h_0, \sigma_0]$.

ad (a) If $\hat{u}_n = 0$ for a subsequence, then by the above arguments we clearly may take $\hat{u}_0 \equiv 0$. If $\hat{u}_n \not\equiv 0$ for a subsequence, Lemma 3.2 gives $\|\hat{u}_n\|_X^{1-p} \geqslant \delta\sigma_n$, hence strong convergence in $W^1_{2r}(\Omega; \mathbb{R}^d)$ proves $\|\hat{u}_0\|_X^{1-p} \geqslant \delta\sigma_0$, i.e. $\hat{u}_0 \not\equiv 0$.

ad (b) The case $\hat{\chi}_n \equiv 0$ for a subsequence shows (with the above notation) $\tilde{\chi}_0 \equiv 0$ and $(\hat{u}_0, 0)$ is seen to be minimizing. The first assertion of Lemma 3.3 ensures the statement $\hat{u}_0 \equiv 0$. The case $\hat{\chi}_n \equiv 1$ is covered by the same arguments.

ad (c) We may assume that $h_n \neq \hat{h}$ for all n sufficiently large. Moreover, by Lemma 3.3 (b) we then observe that $\hat{u}_n \not\equiv 0$, in conclusion Lemma 3.2 gives $\|\hat{u}_n\|_X^{1-p} \geqslant \delta\sigma_n$ and therefore the limit $\hat{u}_0$ does not vanish. The claim now follows from Lemma 3.3a). □

The volume of the phases depends in a monotonic manner on the parameter h, more precisely

Lemma 3.5. Denote by $(\hat{u}_i, \hat{\chi}_i)$ equilibrium states of $I[\cdot, \cdot, h_i, \sigma]$, $i = 1, 2$. Then we have

$$(h_1 - h_2)(\|\hat{\chi}_1\|_{L^1(\Omega)} - \|\hat{\chi}_2\|_{L^1(\Omega)}) \leqslant 0$$

Proof. The proof is an immediate consequence of

$$I[\hat{u}_1, \hat{\chi}_1, h_1, \sigma] \leqslant I[\hat{u}_2, \hat{\chi}_2, h_1, \sigma]$$

$$I[\hat{u}_2, \hat{\chi}_2, h_2, \sigma] \leqslant I[\hat{u}_1, \hat{\chi}_1, h_2, \sigma]$$

□

Remark 3.6. If there exists an equilibrium state $(\hat{u}_0, \hat{\chi}_0)$ of $I[\cdot, \cdot, h_0, \sigma]$ satisfying $\hat{\chi}_0 \equiv 0$ ($\hat{\chi}_0 \equiv 1$), then by Lemma 3.5 for $h > h_0$ ($h < h_0$) any equilibrium state $(\hat{u}, \hat{\chi})$ of $I[\cdot, \cdot, h, \sigma]$ is one-phase, i.e. $\hat{\chi} \equiv 0$ ($\hat{\chi} \equiv 1$).

If we want two-phase equilibria to exist, then we have to restrict the admissible values for the parameters h and σ. A precise formulation is given in the next two lemmata.

Lemma 3.7. There is a real number $h_0 > 0$ with the following property: for any $h > h_0$ ($h < -h_0$), for all $\sigma > 0$ and for any equilibrium state $(\hat{u}, \hat{\chi})$ of $I[\cdot, \cdot, h, \sigma]$ we have $\hat{u} \equiv 0$ and $\hat{\chi} \equiv 0$ ($\hat{\chi} \equiv 1$).

Proof. The idea is to find a real number $h_0 > 0$ such that for any $\sigma > 0$ and for any $(u, \chi) \in H \times M$

$$I[u, \chi, h_0, \sigma] \geqslant I[0, 0, h_0, \sigma] \tag{15}$$

Once (15) is established, $(0, 0)$ is seen to be an equilibrium state of $I[\cdot, \cdot, h_0, \sigma]$ and the first assertion follows from Remark 3.6. The case $h < -h_0$ is treated in the same manner, where we have to increase h_0 if necessary. Thus, it remains to show (15) which is equivalent to

$$\int_\Omega \chi[\langle (A^+ - A^-)\varepsilon(u), \varepsilon(u)\rangle - 2\langle A^+\xi^+ - A^-\xi^-, \varepsilon(u)\rangle + \langle A^+\xi^+, \xi^+\rangle$$
$$-\langle A^-\xi^-, \xi^-\rangle + h_0]\,\mathrm{d}x + \int_\Omega \langle A^-\varepsilon(u), \varepsilon(u)\rangle\,\mathrm{d}x + \sigma\|u\|_H^p \geqslant 0 \tag{16}$$

We may estimate ($0 < \lambda < 1$)

$$\langle A^-\varepsilon(u), \varepsilon(u)\rangle + \chi\langle (A^+ - A^-)\varepsilon(u), \varepsilon(u)\rangle$$
$$\geqslant \langle A^-\varepsilon(u), \varepsilon(u)\rangle - |\langle (A^+ - A^-)\varepsilon(u), \varepsilon(u)\rangle|, 2|\langle A^\pm\xi^\pm, \varepsilon(u)\rangle|$$
$$\leqslant \lambda\langle A^\pm\varepsilon(u), \varepsilon(u)\rangle + \frac{1}{\lambda}\langle A^\pm\xi^\pm, \xi^\pm\rangle$$

thus (16) is implied by

$$\int_\Omega [\langle A^-\varepsilon(u), \varepsilon(u)\rangle - |\langle (A^+ - A^-)\varepsilon(u), \varepsilon(u)\rangle| - \lambda\langle (A^+ + A^-)\varepsilon(u), \varepsilon(u)\rangle]\,\mathrm{d}x$$
$$+ \int_\Omega \chi\left[h_0 + \left(1 - \frac{1}{\lambda}\right)\langle A^+\xi^+, \xi^+\rangle - \left(\frac{1}{\lambda} + 1\right)\langle A^-\xi^-, \xi^-\rangle\right]\,\mathrm{d}x \geqslant 0 \tag{17}$$

By (1) and (2) the first integral on the left-hand side of (17) is greater than or equal to

$$(v - \mu - 2\lambda v^{-1})\|u\|_X^2$$

hence positive if we choose λ sufficiently small. Decreasing λ, if necessary, we finally let

$$h_0 := \left(1 + \frac{1}{\lambda}\right)\langle A^-\xi^-, \xi^-\rangle - \left(1 - \frac{1}{\lambda}\right)\langle A^+\xi^+, \xi^+\rangle > 0$$

With this choice (17), hence (16), holds and in conclusion the lemma is valid. □

Except for $h \neq \hat{h}$ the existence of two-phase equilibria requires also the boundedness of σ:

Lemma 3.8. There exists a real number $\sigma_0 > 0$ with the following property: for any $\sigma > \sigma_0$ and for any $h \in \mathbb{R}$ the functional $I[\cdot, \cdot, h, \sigma]$ admits only equilibria $(\hat{u}, \hat{\chi})$ satisfying $\hat{u} \equiv 0$.

Proof. Recalling (12) and (13) one gets

$$\sigma \|\hat{u}\|_H^p \leqslant 2|A^+\xi^+ - A^-\xi^-\|\Omega|^{1/2} R$$

$$\text{i.e.} \quad \sigma \|\hat{u}\|_X^p \leqslant 2|A^+\xi^+ - A^-\xi^-\|\Omega|^{1/2} R\kappa^p$$

hence we may estimate

$$\sigma^{(1-p)/p} \|\hat{u}\|_X^{1-p} \leqslant R' := (2|A^+\xi^+ - A^-\xi^-\|\Omega|^{1/2} R\kappa^p)^{(1-p)/p}$$

If $\hat{u} \not\equiv 0$ is supposed, then (14) gives

$$\sigma^{(1-p)/p} \delta\sigma \leqslant R' \Leftrightarrow \sigma \leqslant (R'/\delta)^p$$

thus the lemma is proved by letting $\sigma_0 := (R'/\delta)^p$. □

As a last auxiliary result on the distribution of phases, a sufficient condition for the existence of two phase equilibria is given.

Lemma 3.9. If $\sigma > 0$ is sufficiently small, then $I[\cdot, \cdot, \hat{h}, \sigma]$ admits only equilibria $(\hat{u}, \hat{\chi})$ satisfying $\hat{u} \not\equiv 0$.

Proof. Suppose by contradiction that there is a sequence $\{\sigma_n\}$ of positive real numbers, $\sigma_n \downarrow 0$ as $n \to \infty$, such that $I[\cdot, \cdot, \hat{h}, \sigma_n]$ admits a one-phase equilibrium state, i.e., $\hat{\chi}_n \equiv 0$ or $\hat{\chi}_n \equiv 1$ and, by Lemma 3.3, $\hat{u}_n \equiv 0$. Minimality implies for any $(u, \chi) \in H \times M$

$$I[u, \chi, \hat{h}, \sigma_n] \geqslant I[0, \hat{\chi}_n, \hat{h}, \sigma_n] = |\Omega| \langle A^-\xi^-, \xi^- \rangle$$

Using the definition of $\hat{h}$ this can be rewritten as

$$\int_\Omega \chi[\langle (A^+ - A^-)\varepsilon(u), \varepsilon(u) \rangle - 2\langle \varepsilon(u), A^+\xi^+ - A^-\xi^- \rangle]\,dx + \int_\Omega \langle A^-\varepsilon(u), \varepsilon(u) \rangle\,dx$$
$$+ \sigma_n \|u\|_H^p \geqslant 0 \quad \text{for any } (u, \chi) \in H \times M$$

If we replace u by $\sigma_n u$, divide through σ_n and pass to the limit $n \to \infty$, we get

$$-\int_\Omega \chi \langle \varepsilon(u), A^+\xi^+ - A^-\xi^- \rangle\,dx \geqslant 0 \quad \text{for any } (u, \chi) \in H \times M$$

In fact, equality is true since we may consider $-u$ instead of u. Let $\gamma = A^-\xi^- - A^+\xi^+$, fix $x_0 \in \Omega$ and consider $\rho > 0$ such that $B_{2\rho}(x_0) \Subset \Omega$. Finally we choose $\chi = \mathbf{1}_{B_\rho(x_0)}$, $\varphi \in C_0^\infty(\Omega)$, $\varphi \equiv 1$ on $B_{2\rho}(x_0)$ and let $v_k(x) = e\varphi(x)x_k$ with $1 \leqslant k \leqslant d, e \in \mathbb{R}^d$. This choice implies on $B_{2\rho}(x_0)$

$$\varepsilon(v_k) = \frac{1}{2}(e^i\delta_{jk} + e^j\delta_{ik})_{1 \leqslant i,j \leqslant d}$$

hence we get

$$0=\int_{\Omega}\chi\,\mathrm{d}x\frac{1}{2}(\gamma_{ij}e^{i}\delta_{jk}+\gamma_{i,j}e^{j}\delta_{ik})=|B_{\rho}(x_0)|(\gamma e)_k$$

This gives the contradiction $\gamma=0$ and the lemma is proved. □

We finish this section with the following.

Lemma 3.10. For any $h\in\mathbb{R}$ and for any real number $\sigma>0$ we let

$$I_1(\sigma,h):=\inf_{(u,\chi)\in H\times M} I[u,\chi,h,\sigma]$$

Then $I_1(\sigma,h)$ is a concave function, in particular, $I_1(\sigma,h)$ is continuous.

Proof. Note that for h and σ as above $I_1(\sigma,h)$ is well defined. Moreover, for any fixed $(u,\chi)\in H\times M$ the mapping $(h,\sigma)\mapsto I[u,\chi,h,\sigma]$ is a linear function in h and σ, hence concave. Since the infimum of a family of concave functions again in concave, the lemma is seen to be valid. □

4. PROOF OF THEOREM 1.1, (i)–(iii)

Step 1: (Definition of the set B). Note that by construction we have

$$I_1(\sigma,h)\leqslant I_0(h)\quad\text{for any } h\in\mathbb{R},\ \sigma>0 \tag{18}$$

Inequality (18) leads to the definition

$$B:=\{(\sigma,h)\in\mathbb{R}^{+}\times\mathbb{R}: I_1(\sigma,h)<I_0(h)\}$$

and we observe that

$$(\sigma_0,h_0)\in B\Leftrightarrow I[\cdot,\cdot,h_0,\sigma_0]\quad\text{admits only two-phase equilibria } (\hat{u},\hat{\chi})$$

By Lemma 3.9, B is known to be non-empty, moreover, B is seen to be open on account of $B=(I_0-I_1)^{-1}(0,\infty)$ and the continuity of I_0, I_1. Finally, Lemma 3.7 and Lemma 3.8 prove B to be bounded. Given $\sigma_0>0$ let

$$L(\sigma_0):=\{h\in\mathbb{R}:\ (\sigma_0,h)\in B\}$$

Lemma 4.1. Either we have $L(\sigma_0)=\emptyset$ or there exist two uniquely defined real numbers $h^{\pm}(\sigma_0), h^{-}(\sigma_0)<\hat{h}<h^{+}(\sigma_0)$, such that

$$L(\sigma_0)=(h^{-}(\sigma_0),h^{+}(\sigma_0))$$

Proof. Suppose that $L(\sigma_0)\neq\emptyset$, i.e. there exists a real number $h\in\mathbb{R}$ such that $(\sigma_0,h)\in B$. Since B is open $L(\sigma_0)$ is also open, thus

$$L(\sigma_0)=\bigcup_{n=1}^{N} I_n,\quad N\in\mathbb{N}\cup\{\infty\}$$

where $I_n \neq \emptyset$ denote some open, bounded, mutually disjoint intervals. If we fix one of these intervals $I_n = (\alpha, \beta)$, then α, β do not belong to $L(\sigma_0)$, hence $(\sigma_0, \alpha), (\sigma_0, \beta) \notin B$. This proves

$$I_1(\sigma_0, \alpha) = I_0(\alpha), \quad I_1(\sigma_0, \beta) = I_0(\beta), \quad I_1(\sigma_0, h) < I_0(h) \tag{19}$$

for any $h \in (\alpha, \beta)$. Now we claim that $\alpha < \hat{h} < \beta$, which clearly gives the lemma. Suppose by contradiction that $\alpha \geqslant \hat{h}$. From $I_1(\sigma_0, \alpha) = I_0(\alpha)$ we see the existence of at least one one-phase equilibrium at (σ_0, α). The assumption $\alpha \geqslant \hat{h}$ gives

$$I_0(\alpha) = |\Omega| \langle A^- \xi^-, \xi^- \rangle = I[0, 0, \alpha, \sigma_0]$$

hence the one-phase equilibrium with $\hat{u} \equiv 0$, $\hat{\chi} \equiv 0$ exists for (σ_0, α). One the other hand, Remark 3.6 then proves that for $h > \alpha$, only one-phase equilibria with $\hat{\chi} \equiv 0$ exist which contradicts (19) and the lemma is proved since analogous arguments show the second inequality $\hat{h} < \beta$. □

Step 2: (Definition of the functions $h^{\pm}(\sigma)$). Following Lemma 4.1 we define for any $\sigma > 0$ satisfying $L(\sigma) \neq \emptyset$

$$h^+(\sigma) := \sup L(\sigma), \quad h^-(\sigma) := \inf L(\sigma)$$

If $L(\sigma) = \emptyset$ then we let

$$h^+(\sigma) := h^-(\sigma) := \hat{h}$$

Step 3: (Definition of the sets A and C). The sets A and C are defined via

$$A := \{(\sigma, h): \ \sigma > 0, \ h > h^+(\sigma)\}$$
$$C := \{(\sigma, h): \ \sigma > 0, \ h < h^-(\sigma)\}$$

and we claim that for $(\sigma, h) \in A$ ($(\sigma, h) \in C$) the functional $I[\cdot, \cdot, h, \sigma]$ admits only one-phase equilibria $(\hat{u}, \hat{\chi})$ with $\hat{u} \equiv 0$ and $\hat{\chi} \equiv 0$ ($\hat{\chi} \equiv 1$). To verify our claim we assume $(\sigma, h) \in A$, hence $h > h^+(\sigma) \geqslant \hat{h}$. Recalling (19) we have $I_1(\sigma, h^+(\sigma)) = I_0(h^+(\sigma))$ and by Remark 3.6 $I[\cdot, \cdot, h, \sigma]$ admits only a one-phase equilibrium which on account of $h > \hat{h}$ is of type $\hat{\chi} \equiv 0$. The case $(\sigma, h) \in C$ is treated in the same way, and the claim is proved. Now let

$$A' := \{(\sigma, h): \ \sigma > 0, \ h \geqslant \hat{h}, \ I_1(\sigma, h) = I_0(h) = \langle A^- \xi^-, \xi^- \rangle |\Omega|\}$$

It is easily seen that

$$A' = A \cup \operatorname{graph} h^+$$

In fact, if $(\sigma, h) \in A'$, then we either have $h > h^+(\sigma)$ or $h = h^+(\sigma)$ since $h < h^+(\sigma)$ would imply two-phase equilibria which are excluded by the definition of A'. Thus the inclusion '$\subset$' is proved. The other inclusion follows from Lemma 3.4(b). In a similar way we define

$$C' = \{(\sigma, h): \ \sigma > 0, \ h \leqslant \hat{h}, \ I_1(\sigma, h) = I_0(h) = (\langle A^+ \xi^+, \xi^+ \rangle + h)|\Omega|\}$$
$$C' = C \cup \operatorname{graph} h^-$$

Lemma 4.2. A' and C' are convex sets.

Proof. Fix two points $(\sigma_i, h_i) \in A'$, $i = 1, 2$, a real number $0 \leqslant \tau \leqslant 1$, and let $\sigma_\tau := \tau\sigma_1 + (1 - \tau)\sigma_2$, $h_\tau := \tau h_1 + (1-\tau)h_2$. Since $\sigma_1, \sigma_2 > 0$ and since $h_1, h_2 \geqslant \hat{h}$ the assertions $\sigma_\tau > 0$ and $h_\tau \geqslant \hat{h}$ are trivial, it remains to show

$$I_1(\sigma_\tau, h_\tau) = I_0(h_\tau) = |\Omega| \langle A^- \xi^-, \xi^- \rangle$$

However, these equalities are known to be true for σ_i, h_i and since in addition I_1 is concave (see Lemma 3.10), we obtain

$$\begin{aligned} I_1(\sigma_\tau, h_\tau) &\geqslant \tau I_1(\sigma_1, h_1) + (1-\tau) I_1(\sigma_2, h_2) \\ &= \tau I_0(h_1) + (1-\tau) I_0(h_2) = |\Omega| \langle A^- \xi^-, \xi^- \rangle \end{aligned}$$

On the other hand, $I_1(\sigma, h) \leqslant I_0(h)$ holds for any $h \in \mathbb{R}$, $\sigma > 0$. This together with $h_\tau \geqslant \hat{h}$ gives

$$I_1(\sigma_\tau, h_\tau) \leqslant I_0(h_\tau) = |\Omega| \langle A^- \xi^-, \xi^- \rangle$$

This proves that the convexity of A', C' is handled with analogous arguments. □

Step 4: (Properties of the functions $h^\pm(\sigma)$).

Lemma 4.3. The functions $h^\pm$ are bounded and depend continuously on $\sigma > 0$. Moreover, $h^+(\sigma)$ is convex on $(0, \infty)$, whereas $h^-(\sigma)$ is concave on $(0, \infty)$.

Proof. In Step 1, it was shown that B is bounded, hence with Lemma 4.1 the functions $h^\pm$ are seen to be uniformly bounded on $(0, \infty)$. Thus we only have to prove that $h^+(h^-)$ is convex (concave) which will imply continuity. Now fix $\sigma_1, \sigma_2 > 0$, $0 \leqslant \tau \leqslant 1$, and observe that $(\sigma_i, h^+(\sigma_i)) \in A'$, $i = 1, 2$. In fact, $h^+(\sigma_i) \geqslant \hat{h}$ is proved in Lemma 4.1, and the existence of a one-phase equilibrium of type $\hat{\chi} \equiv 0$ follows from Lemma 3.4(b). Convexity of A' then yields

$$(\underbrace{\tau\sigma_1 + (1-\tau)\sigma_2}_{\tilde{\sigma}}, \underbrace{\tau h^+(\sigma_1) + (1-\tau) h^+(\sigma_2)}_{=:\,\tilde{h}}) \in A'$$

Since $(\tilde{\sigma}, \tilde{h}) \in A'$ immediately gives (compare Step 3.) $\tilde{h} \geqslant h^+(\tilde{\sigma})$, we have proved the convexity of h^+:

$$\tau h^+(\sigma_1) + (1-\tau) h^+(\sigma_2) = \tilde{h} \geqslant h^+(\tilde{\sigma}) = h^+(\tau\sigma_1 + (1-\tau)\sigma_2)$$

Using the same arguments h^- is seen to be concave and the lemma is verified. □

Lemma 4.4. There is a real number $\sigma^* > 0$ such that h^+ is strictly decreasing on $(0, \sigma^*)$, whereas h^- is strictly increasing on this interval. On (σ^*, ∞) both h^+ and h^- are equal to $\hat{h}$.

Proof. By Lemma 3.9 we know that $h^-(\sigma) < \hat{h} < h^+(\sigma)$ if $\sigma \ll 1$ is sufficiently small. On the other hand, $\sigma \gg 1$ implies according to Lemma 3.8 $h^-(\sigma) = \hat{h} = h^+(\sigma)$. Hence, we may define

$$\sigma_+^* := \inf\{\sigma > 0 \colon\ h^+ = \hat{h} \text{ on } (\sigma, \infty)\}$$

Now assume by contradiction that h^+ is not strictly decreasing on $(0,\sigma_+^*)$, i.e. for some positive numbers $0<\sigma_1<\sigma_2<\sigma_+^*$ we have $h^+(\sigma_1)\leqslant h^+(\sigma_2)$. Together with this assumption, convexity of h^+ gives for any $\sigma>\sigma_2$.

$$\frac{h^+(\sigma)-h^+(\sigma_2)}{\sigma-\sigma_2}\geqslant\frac{h^+(\sigma_2)-h^+(\sigma_1)}{\sigma_2-\sigma_1}\geqslant 0$$

Since $\sigma_2<\sigma_+^*$ implies $h^+(\sigma_2)>\hat{h}$, we obtain the contradiction $h^+(\sigma)\geqslant h^+(\sigma_2)>\hat{h}$ for any $\sigma>\sigma_2$. Up to now, it is proved that h^+ is strictly decreasing on $(0,\sigma_+^*)$. Analogous considerations prove the existence of a real number $\sigma_-^*\in(0,\infty)$ such that $h^-\equiv\hat{h}$ for $\sigma\geqslant\sigma_-^*$ and such that h^- is strictly increasing on $(0,\sigma_-^*)$. It remains to verify $\sigma_+^*=\sigma_-^*$: to this purpose observe that by Lemma 4.1 $h^-(\sigma)\neq h^+(\sigma)$ implies $\hat{h}\in(h^-(\sigma),h^+(\sigma))$. If we assume that $\sigma_-^*<\sigma_+^*$, then we may find $\sigma\in(\sigma_-^*,\sigma_+^*)$ such that $(h^-(\sigma),h^+(\sigma))\neq\emptyset$ and such that $h^-(\sigma)=\hat{h}$. This gives the contradiction $\hat{h}\notin(h^-(\sigma),h^+(\sigma))$. Again the case $\sigma_-^*>\sigma_+^*$ is excluded with the same arguments, and the proof of Lemma 4.4 is complete. □

5. EQUILIBRIUM STATES OF $I[\cdot,\cdot,h,\sigma]$ FOR POINTS (σ,h) ON THE GRAPHS OF $h^\pm$

In this section we prove (iv)–(vii) of Theorem 1.1.

ad (iv). Consider the case $0<\sigma<\sigma^*$ and $h=h^+(\sigma)$. Letting $\sigma_n\equiv\sigma$ and by considering a sequence $\{h_n\}$ satisfying $h_n\uparrow h$ as $n\to\infty$ we may assume $(\sigma_n,h_n)\in B$ for n sufficiently large, hence there exists a sequence of two-phase equilibria $(\hat{u}_n,\hat{\chi}_n)$ of $I[\cdot,\cdot,h_n,\sigma_n]$. Since $\lim_{n\to\infty} h_n=h=h^+(\sigma)>\hat{h}$, Lemma 3.4(b) is applicable and $I[\cdot,\cdot,h^+(\sigma),\sigma]$ is seen to admit a two-phase equilibrium. On the other hand, now letting $\sigma_n\equiv\sigma$ and considering a sequence $\{h_n\}$, $h_n\downarrow h$ as $n\to\infty$, we have $(\sigma_n,h_n)\in A$ and the same reasoning proves the existence of a one-phase equilibrium, which on account of Remark 3.6 can only be of type $\hat{\chi}\equiv 0$.

ad (v). We can apply the same arguments as used for (iv) with obvious modifications.

ad (vi). For $h=\hat{h}$ and $\sigma>\sigma^*$ we again apply Lemma 3.4 to find $(\hat{u},\hat{\chi})$, $\hat{u}\equiv 0$, as an equilibrium state of $I[\cdot,\cdot,\hat{h},\sigma]$. Here, Lemma 3.3(c) shows any characteristic function $\hat{\chi}$ to be admissible. Equilibrium states satisfying $\hat{u}\not\equiv 0$ are not possible: if we assume the existence of an equilibrium state $(\hat{u}_0,\hat{\chi}_0)$ of $I[\cdot,\cdot,\hat{h},\sigma_0]$, $\sigma_0>\sigma^*$, $\hat{u}_0\not\equiv 0$, then we obtain for any $\sigma\in(\sigma^*,\sigma_0)$

$$I_0(\hat{h})=I_1(\sigma,\hat{h})\leqslant I[\hat{u}_0,\hat{\chi}_0,\hat{h},\sigma]<I[\hat{u}_0,\hat{\chi}_0,\hat{h},\sigma_0]=I_1(\sigma_0,\hat{h})=I_0(\hat{h})$$

where we used the existence of equilibria of type $\hat{u}\equiv 0$ for the parameters $\sigma=\sigma_0$, $h=\hat{h}$.

ad (vii). Finally, the case $h=\hat{h}$ and $\sigma=\sigma^*$ has to be discussed. As in (vi) equilibrium states of type $\hat{u}\equiv 0$, $\hat{\chi}\equiv$ arbitrary characteristic function, are found. The existence of a two-phase equilibrium state satisfying $\hat{u}\not\equiv 0$ is proved by considering a sequence $\{\sigma_n\}$, $\sigma_n\uparrow\sigma^*$ as $n\to\infty$, $h_n\equiv\hat{h}$, i.e. $(\sigma_n,\hat{h})\in B$. By the definition of B we have $I_1(\sigma_n,\hat{h})<I_0(\hat{h})$ and, as a consequence (compare Lemma 3.3(c)), $\hat{u}_n\not\equiv 0$ if $(\hat{u}_n,\hat{\chi}_n)$ denotes a corresponding equilibrium state of $I[\cdot,\cdot,\hat{h},\sigma_n]$. With Lemma 3.4(a) assertion (vii) holds and the whole theorem is proved. □

6. PROOF OF THEOREM 1.3

W.l.o.g. assume that $\hat{u} \not\equiv 0$. Then we have $\int_\Omega |\Delta\hat{u}|^2 \, dx > 0$ and letting $u_t := \hat{u} + t\varphi$, $t \in \mathbb{R}$, $\varphi \in C_0^\infty(\Omega; \mathbb{R}^d)$, minimality of $(\hat{u}, \hat{\chi})$ implies

$$\begin{aligned} 0 &= \frac{d}{dt}_{|t=0} I[u_t, \hat{\chi}, h, \sigma] \\ &= 2\int_\Omega \left\langle \hat{\chi} A^+(\varepsilon(\hat{u}) - \xi^+) + (1-\hat{\chi})A^-(\varepsilon(\hat{u}) - \xi^-), \varepsilon(\varphi) \right\rangle dx \\ &\quad + p\sigma \left(\int_\Omega |\Delta\hat{u}|^2 \right)^{p/2-1} \int_\Omega \Delta\hat{u} : \Delta\varphi \, dx \end{aligned}$$

hence, letting $T = c(\hat{\chi} A^+(\varepsilon(\hat{u}) - \xi^+) + (1-\hat{\chi})A^-(\varepsilon(\hat{u}) - \xi^-))$ for a suitable real number $c > 0$, we obtain

$$\int_\Omega \Delta\hat{u} : \Delta\varphi \, dx = \int_\Omega \nabla\varphi : T \, dx \quad \text{for all } \varphi \in C_0^\infty(\Omega; \mathbb{R}^d) \tag{20}$$

Now we abbreviate $U := \Delta\hat{u} \in L^2(\Omega; \mathbb{R}^d)$ and denote by U^ρ, T^ρ the standard mollifications of U and T, respectively, where $\rho > 0$ is chosen sufficiently small. Then (20) is valid for U^ρ, T^ρ in the following sense:

$$\int_\Omega \nabla U^\rho : \nabla\varphi \, dx = -\int_\Omega \nabla\varphi : T^\rho \, dx, \quad \varphi \in C_0^\infty(\Omega; \mathbb{R}^d), \quad \operatorname{dist}(\operatorname{spt}\varphi, \partial\Omega) > \rho \tag{21}$$

Since $\eta^2 U^\rho$, $\eta \in C_0^\infty(\Omega)$, $0 \leqslant \eta \leqslant 1$, is admissible in (21) for ρ sufficiently small, this implies

$$\begin{aligned} &\int_\Omega \eta^2 |\nabla U^\rho|^2 \, dx + 2\int_\Omega \eta \nabla\eta \otimes U^\rho : \nabla U^\rho \, dx \\ &\quad = -\int_\Omega \eta^2 \nabla U^\rho : T^\rho \, dx - 2\int_\Omega \eta\nabla\eta \otimes U^\rho : T^\rho \, dx \end{aligned}$$

hence, with the help of Young's inequality

$$\int_\Omega \eta^2 |\nabla U^\rho|^2 \, dx \leqslant \tilde{c}(\eta) \left(\int_{\operatorname{spt}\eta} |U^\rho|^2 \, dx + \int_{\operatorname{spt}\eta} |T^\rho|^2 \, dx \right)$$

This proves $\{U^\rho\}$ to be uniformly bounded in $W^1_{2,loc}(\Omega; \mathbb{R}^d)$ which, together with $U^\rho \to U$ in $L^2_{loc}(\Omega; \mathbb{R}^d)$ as $\rho \to 0$, gives $U \in W^1_{2,loc}(\Omega; \mathbb{R}^d)$. As a result, we have the equation

$$\int_\Omega \nabla U : \nabla\varphi \, dx = -\int_\Omega T : \nabla\varphi \, dx \quad \text{for all } \varphi \in C_0^\infty(\Omega; \mathbb{R}^d) \tag{22}$$

Now we apply the standard L^p-theory for weak solutions of "$\Delta v = \nabla T$" as well as the Calderon–Zygmund regularity results. To be precise let us first consider the case $d = 2$. Here $\varepsilon(u) \in W_2^1(\Omega; \mathbb{R}^{d\times d})$ implies $T \in L^p(\Omega; \mathbb{R}^{d\times d})$ for any $p < \infty$. L^p-theory gives $\nabla U \in L^p_{loc}$

$(\Omega;\mathbb{R}^{d\times d})$ (compare Reference [13], Section 4.3, in particular p. 73), hence $\Delta u\in W^1_{p,loc}(\Omega;\mathbb{R}^d)$ for any $p<\infty$ and we obtain $\Delta u\in C^{0,\alpha}_{loc}(\Omega;\mathbb{R}^d)$ for any $\alpha\in(0,1)$. Finally, the assertion follows from the interior Schauder estimates (see Reference [13], Theorem 3.6). Next we assume that $d\geqslant 3$ and let $s_l:=2d/(d-2l)$. Then it is easy to see that

$$\begin{aligned}
&\hat{u}\in W^2_2(\Omega;\mathbb{R}^d) && \Rightarrow\ \varepsilon(\hat{u})\in L^{s_1}(\Omega;\mathbb{R}^{d\times d})\ \Rightarrow\ T\in L^{s_1}(\Omega;\mathbb{R}^{d\times d})\\
\Rightarrow\ &\nabla(\Delta\hat{u})\in L^{s_1}_{loc}(\Omega;\mathbb{R}^{d\times d}) && \Rightarrow\ \Delta\hat{u}\in W^1_{s_1,loc}(\Omega;\mathbb{R}^d)\\
\Rightarrow\ &\Delta\hat{u}\in L^{s_2}_{loc}(\Omega;\mathbb{R}^d) && \Rightarrow\ \hat{u}\in W^2_{s_2,loc}(\Omega;\mathbb{R}^{d\times d})\ \Rightarrow\ T\in L^{s_3}(\Omega;\mathbb{R}^{d\times d})\\
&\dots
\end{aligned} \tag{23}$$

This procedure stops if $d\leqslant 2l$. Thus, denote by l^* the maximum of all $l\in\mathbb{N}$ such that $d-2l>0$. Then s_{l^*} is well defined and satisfies $s_{l^*}\geqslant d$. In fact, the latter inequality is equivalent to $2\geqslant d-2l^*$ which is true on account of the maximality of l^*. Now assume that l^* is an even number. Then (23) implies for any $p<\infty$

$$\begin{aligned}
&\hat{u}\in W^2_{s_{l^*},loc}(\Omega\mathbb{R}^d)\ \Rightarrow\ \varepsilon(\hat{u})\in W^1_{d,loc}(\Omega;\mathbb{R}^{d\times d})\ \Rightarrow\ \varepsilon(\hat{u})\in L^p_{loc}(\Omega;\mathbb{R}^{d\times d})\\
&\Rightarrow\ T\in L^p_{loc}(\Omega;\mathbb{R}^{d\times d})
\end{aligned}$$

thus $\Delta\hat{u}\in W^1_{p,loc}(\Omega;\mathbb{R}^d)$ for any $p<\infty$ (again compare Reference [13], Section 4.3) and as a consequence $\Delta\hat{u}\in C^{0,\alpha}_{loc}(\Omega,\mathbb{R}^d)$ for all $0<\alpha<1$. Again the interior Schauder estimates (see Reference [13], Theorem 3.6) prove the result. In the case that l^* is an odd number, we conclude

$$\begin{aligned}
&\Delta\hat{u}\in W^1_{s_{l^*},loc}(\Omega;\mathbb{R}^d)\ \Rightarrow\ \Delta\hat{u}\in W^1_{d,loc}(\Omega;\mathbb{R}^d)\ \Rightarrow\ \Delta\hat{u}\in L^p_{loc}(\Omega\mathbb{R}^d)\\
&\Rightarrow\ \hat{u}\in W^2_{p,loc}(\Omega;\mathbb{R}^d)
\end{aligned}$$

which again is valid for any $p<\infty$, hence $\varepsilon(\hat{u})\in L^p_{loc}(\Omega;\mathbb{R}^{d\times d})$ for any $p<\infty$ and we proceed as before, i.e. Theorem 1.3 is proved. □

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