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Michael Bildhauer

On the free boundary of surfaces with bounded mean curvature: the non-perpendicular case

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Abstract. Hölder continuity up to the free boundary is proved for minimizing solutions if they meet the supporting surface in an angle which is bounded away from zero. The problem is localized by proving the continuity of the distance function, a result which is also true for stationary points.

1. Introduction

Given a vector field $Q \in C^1(\mathbb{R}^3, \mathbb{R}^3)$, we consider twodimensional weak solutions $X : B_1(0) \subset \mathbb{R}^2 \to \mathbb{R}^3$ of the following variational problem $\mathscr{P}$, which arises for example in the investigation of partitioning problems (see [1], [12]): find minimizers or stationary points of the functional

$$\begin{aligned} \mathcal{F}[Y] &= \frac{1}{2} \iint_{B_1(0)} |\nabla Y|^2 \, du\, dv + \iint_{B_1(0)} Q(Y) \cdot (Y_u \wedge Y_v) \, du\, dv \\ &= D[Y] + V^Q[Y] \end{aligned}$$

in a suitable class $\mathcal{C}$, which defines *partially free* or *free boundary values* on a *supporting surface* S. The *Dirichlet integral* is denoted by $D[Y]$ with $|\nabla Y|^2 := |(Y_u, Y_v)|^2 = |Y_u|^2 + |Y_v|^2$. The functional V^Q will be called the *volume functional*, although capillary forces are involved if the *normal component with respect to the supporting surface* S of the vector field Q is not vanishing. In fact, this situation is studied here. A smooth solution of $\mathscr{P}$ is known to be a surface of mean curvature $H = \mathrm{div} Q/2$ satisfying the free boundary condition

$$|Q \cdot N| = \cos \alpha,$$

where α denotes the angle in which X meets the supporting surface S at the free boundary, and N is the outward normal unit vector of S (see [1], [12]). We do not consider existence problems (see [4] or [12] for references) and

M. Bildhauer: Fachbereich Mathematik, Universität des Saarlandes,
D-66041 Saarbrücken, Germany. e-mail: bibi@math.uni-sb.de

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assume in what is following that X is a $H^{1,2}$–solution of the problem $\mathcal{P}$, where we have to distinguish minimizing and stationary points. A detailed description of the known results on regularity theory is given in [4] and we can restrict ourself to the most important references.

Since div$Q$ is always assumed to be bounded, interior regularity is completely proved in [8] and [9]. Regularity for Plateau boundary values is also known (see [4]), thus only the behavior of the solution at the free boundary has to be studied. The general case of stationary points is treated in [11] (minimal surfaces, that is $Q \equiv 0$) and in [12] (surfaces of bounded mean curvature). To prove smoothness up to the free boundary, the supporting surface S is assumed to be of class $C^{m,\beta}$, $m \geq 3$, and the solution has to meet S perpendicular, that is $Q \cdot N \equiv 0$ on S. It is still an open question if this condition can be dropped. While Grüter, Hildebrandt and Nitsche extended the argumentation of [8] and [9] which is based upon methods from geometric measure theory, Dziuk ([5]) studied minimal surfaces almost simultaneously by using Jägers reflection principle ([14]) and then refering to the interior regularity results of Grüter. However, this approach requires as an additional assumption the *continuity of the distance function*, i.e.

$$\text{dist}(X(w), S) \to 0 \quad \text{as} \quad w \to w_0 \in \partial B_1(0),$$

where $w_0 \in \partial B_1(0)$ is a point corresponding to the free boundary.

A direct method to prove Hölder continuity up to the free boundary is applicable in the case of minimizing solutions. Here S is assumed to fulfil a *chord–arc–condition*, which was recognized (also almost simultaneously) by Nitsche ([16]) and Goldhorn–Hildebrandt ([7]) to be sufficient. Due to an example of Courant and Cheung (see [4], pp. 43–44), without this condition we cannot expect smooth minimizers. The weakest requirement on the vector field Q was given by Hildebrandt ([13]), but he still had to impose a *global smallness condition*, namely $|Q| < q < 1$ for some real number q.

In Section 4 and Section 5 of this paper we study this smallness condition for Q in the case of minimizing solutions: on one hand, regularity at the free boundary should follow from a *local condition* in a neighbourhood of the supporting surface S. This is proved in Section 4.

To do this, we first prove a result of general interest, namely the continuity of the distance function. It should be emphasized, that this observation is true not only for minimizers but also for stationary points and that no conditions for S are required. Especially and in agreement with the example of Courant and Cheung, even a chord–arc–condition is not necessary. Our result shows that the assumption of Dziuk ([5]), which can be traced back to Jäger ([14]), Lewy ([15]) and Courant ([2]), in fact is a conclusion from stationarity. As a corollary we obtain bounded solutions in the case of bounded supporting surfaces. Recently Grüter ([10]) also proved the continuity of the distance function for weakly harmonic maps.

On the other hand, in Section 5 we reduce the technical smallness condition of Hildebrandt to a *geometrical* one. The contact angle is the geometrical property which controls the boundary behavior of a solution. If this angle converges to zero, then we cannot exclude unbounded solutions of bounded mean curvature and of bounded area. However, if this angle is bounded away from zero, that is if we impose a smallness condition on the *normal component* of Q, then Hölder continuity of minimizing solutions up to the free boundary is proved in Theorem 5.3. For some technical reasons and in order to speak of a normal component, the supporting surface is again assumed to be smooth in some sense. But let us first fix the notation and make the assumptions precise.

2. Notation

The Euclidean space $\mathbb{R}^2$ is identified with the complex plane $\mathbb{C}$, so $w = (u, v) \in \mathbb{R}^2$ is the equivalent counterpart to $w = u + iv \in \mathbb{C}$. We always consider a supporting surface S and a rectifiable arc Γ with end points $P_1 \neq P_2$ in S. The general assumption on the supporting surface (which is in fact not needed in Section 3) is a chord–arc–condition.

Definition 2.1. *A set S in $\mathbb{R}^3$ is said to fulfil a chord–arc–condition with constants M and δ, $M \geq 1$ and $\delta > 0$, if it is closed and if any two points P_1 and P_2 of S whose distance $|P_1 - P_2|$ is less than or equal to δ can be connected in S by a rectifiable arc Γ^* with length $L(\Gamma^*) \leq M\,|P_1 - P_2|$.*

The suitable variational class $\mathcal{C}$ of *admissible surfaces* is slightly different from the natural setting in the context of Dirichlet's integral (see [3], pp. 255–256) since $\mathcal{F}$ is invariant only with respect to *orientation preserving* conformal mappings. Partially free boundary values are considered without loss of generality (see [12]).

Definition 2.2. *For $B = B_1(0) \subset \mathbb{R}^2$ the class $\mathcal{C}(\Gamma, S)$ is the set of all Sobolev functions $Y \in H^{1,2}\left(B, \mathbb{R}^3\right)$ with the following properties: there is an arc $C = \{e^{i\theta} : 0 \leq \theta_1 \leq \theta \leq \theta_2 < 2\pi\}$ such that the L^2–traces of Y satisfy:*

- *(i)* Free boundary values: $Y(w) \in S$ *for $\mathcal{H}^1$–almost all* $w \in \partial B \sim C$;
- *(ii)* Plateau boundary values: $Y_{|C} : C \to \Gamma$ *is a continuous, weakly monotonic mapping onto Γ with $Y(e^{i\theta_1}) = P_{i_1}$ and $Y(e^{i\theta_2}) = P_{i_2}$ for* $\{i_1, i_2\} = \{1, 2\}$.

Remark 2.3. The permutation in (ii) is needed to preserve orientation.

A family of surfaces $Y_\varepsilon \in \mathcal{C}(\Gamma, S)$, $|\varepsilon| < \varepsilon_0$ for some number $\varepsilon_0 > 0$, is said to be an *admissible variation of a surface* $Y \in \mathcal{C}(\Gamma, S)$, if $\{Y_\varepsilon\}_{|\varepsilon|<\varepsilon_0}$ is of one of the following types:

Type 1 (inner variations): $Y_\varepsilon(w) = Y(\tau_\varepsilon(w))$ where $\{\tau_\varepsilon\}_{|\varepsilon|<\varepsilon_0}$ is a family of diffeomorphisms $\overline{B} \longrightarrow \overline{B}$ such that τ_0 is the identity and that $\tau(w,\varepsilon) := \tau_\varepsilon(w) \in C^1\left(\overline{B} \times (-\varepsilon_0, \varepsilon_0), \overline{B}\right)$.
Type 2 (outer variations): $Y_\varepsilon(w) = Y(w) + \varepsilon\Psi(w,\varepsilon)$ where the Dirichlet integrals $D[\Psi(\cdot,\varepsilon)]$ are uniformly bounded and there exists $\Phi \in H^{1,2}\left(B, \mathbb{R}^3\right) \cap L^\infty(B, \mathbb{R}^3)$ with

$$\Psi(w,\varepsilon) \to \Phi(w) \quad \text{for almost all} \quad w \in B \quad \text{as} \quad \varepsilon \to 0.$$

Remark 2.4. In contrast to [3], p. 330, no deformation of B is admitted for inner variations according to the definition of admissible surfaces in the unit ball.

Finally $X \in \mathcal{C}(\Gamma, S)$ is a *stationary point of* $\mathcal{F}$ in this class, if

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \{\mathcal{F}[X_\varepsilon] - \mathcal{F}[X]\} = 0$$

for all admissible variations. This condition is especially fulfilled for minimizers. If X is a stationary point, then it is parametrized conformally, i.e.

$$|X_u|^2 = |X_v|^2, \quad X_u \cdot X_v = 0 \quad \text{almost everywhere in } B.$$

Since this is proved by considering inner variations, the well known arguments of [3], pp. 242, cannot be cited (see Remark 2.4). So we refer to [17], Theorem 3.1, p. 9, observing that for ε sufficiently small the volume functional is invariant with respect to inner variations. Once the proof of the conformality relations is done, inner variations are no longer needed in this paper and now B, C and $\partial B \sim C$ can be replaced by the standard notation:

$$B := \{w \in \mathbb{R}^2 : |w| < 1, v > 0\},$$
$$C := \{w \in \mathbb{R}^2 : |w| = 1, v > 0\}, I := \partial B \sim C.$$

3. Continuity of the distance function

The above mentioned methods from geometric measure theory are used in this section to prove the continuity of the distance function for stationary solutions. Since the conclusion of the following theorem is drawn only from inner estimates, the chord–arc–condition as a general assumption in fact is not necessary.

Theorem 3.1. *Consider a boundary configuration* $\langle\Gamma, S\rangle$ *as above and define*

$$U_\tau := \{z \in \mathbb{R}^3 : \operatorname{dist}(z, S) < \tau\} \quad \textit{for any given} \quad \tau > 0.$$

A vector field $Q(z) \in C^1\left(\mathbb{R}^3, \mathbb{R}^3\right)$ *is assumed to satisfy*

$$|\operatorname{div} Q(z)| \le H_0 < \infty \quad \textit{for a constant} \quad H_0 > 0 \quad \textit{and for all} \quad z \in \mathbb{R}^3.$$

If X is a stationary point of the functional $\mathcal{F}$ in the class $\mathcal{C}(\Gamma, S)$, then for all $\tau_0 > 0$ and for all $\hat{w} \in I$ there exists a real number $0 < R = R(\hat{w}, \tau_0)$ such that

$$X\left(S_R(\hat{w})\right) \subset U_{\tau_0}.$$

Here and below we set for $w_0 \in I$ and $r > 0$: $S_r(w_0) := \{w \in \mathbb{R}^2 : |w - w_0| < r,\ v > 0\} \cap B$ and $C_r(w_0) := \{w \in \mathbb{R}^2 : |w - w_0| = r,\ v > 0\} \cap B$.

Proof of Theorem 3.1. Fix $\tau_0 > 0$ and define $h_0 := H_0/2$. For a given $\hat{w} \in I$ choose $0 < \tilde{R} < 1 - |\hat{w}|$ such that

$$\iint_{S_{\tilde{R}}(\hat{w})} |\nabla X|^2\, du\, dv < c_0(\tau_0, H_0) := \frac{\pi \tau_0^2}{2} e^{-h_0 \tau_0}. \tag{3.1}$$

The modified Courant–Lebesgue Lemma (see [11], Lemma 2, p. 393) with $r \in [\tilde{R}/n, \tilde{R}]$, $n \in \mathbb{N}$ sufficiently large, ensures the existence of a real number R, $0 < R < \tilde{R}$, satisfying

$$\operatorname{osc}_{C_R(\hat{w})} X < \tau_0/2. \tag{3.2}$$

Since the proof of the Courant–Lebesgue Lemma does not depend on measure zero sets of radii and since $X \in H^{1,2}\left(B, \mathbb{R}^3\right)$, we can assume the limit $O(R)$ to exist in S:

$$O(R) := \lim_{\theta \to \pi - 0} X(\hat{w} + Re^{i\theta}) \in S. \tag{3.3}$$

Following [12], the proof is completed by an indirect argument: assume that there exists $w^* \in S_R(\hat{w})$ such that $X(w^*) \notin U_{\tau_0}$, that is

$$\operatorname{dist}(X(w^*), S) \geq \tau_0. \tag{3.4}$$

The relations (3.2)–(3.4) imply

$$\inf_{w \in C_R(\hat{w})} |X(w) - X(w^*)| > \tau_0/2. \tag{3.5}$$

On the other hand, X is a stationary point of the functional $\mathcal{F}$ in the class $\mathcal{C}(\Gamma, S)$, especially X is a conformally parametrized solution of

$$\begin{gathered}\iint_B \{\nabla X \cdot \nabla \eta + \operatorname{div} Q(X)\, \eta \cdot (X_u \wedge X_v)\}\, du\, dv = 0 \\ \text{for all } \eta \in H_0^{1,2}(B, \mathbb{R}^3) \cap L^\infty(B, \mathbb{R}^3).\end{gathered} \tag{3.6}$$

Now consider $\lambda(s) \in C^1(\mathbb{R}, \mathbb{R})$ with $\lambda'(s) \geq 0$ and $\lambda(s) = 0$ for $s \leq 0$ and define

$$\Psi(\rho) = \frac{1}{2} \iint_{S_R(\hat{w})} \lambda\left(\rho - |X(w) - X(w^*)|\right) |\nabla X|^2\, du\, dv \quad \forall\, 0 < \rho < \tau_0/2,$$

as well as

$$\eta(w) := \begin{cases} \lambda(\rho - |X(w) - X(w^*)|)\,(X(w) - X(w^*)) & : w \in S_R(\hat{w}) \\ 0 & : w \in B \sim S_R(\hat{w}) \end{cases} .$$

By (3.5), $|X(w) - X(w^*)| > \tau_0/2$ is true for $w \in C_R(\hat{w})$, and by assumption for almost every $w \in I$ we have $|X(w) - X(w^*)| \geq \tau_0$. Thus η is seen to be admissible in (3.6) and the argumentation of [12], pp. 130–131, gives a monotonicity formula for $\Psi(\rho)/\rho^2$ and with [12], Lemma 1, p. 129, a contradiction to (3.1), i.e. the theorem is proved. □

Of course the continuity of a solution up to the free boundary is not implied. Only a special kind of singularities is excluded and we obtain the following corollary.

Corollary 3.2. *If S is bounded, then a stationary solution is also bounded.*

From now on the context of stationary solutions is left and the continuity of minimizers is studied in the next sections.

4. A local problem

In order to formulate a theorem concerning the local character of regularity results for minimizers, that is to require only a local smallness condition in a neighbourhood of S, we have to introduce some further notation:

$Z_d := \{w \in B : |w| < 1 - d\}$ for $0 < d < 1$ and $S_r(w_0) := B \cap B_r(w_0)$ for all $w_0 \in \overline{B}$. Given $\hat{w} \in I$ and $0 < R < 1 - |\hat{w}|$ define

$$Z_d(R, \hat{w}) = \{w \in B : |w - \hat{w}| < R - d\} \ \text{ for } \ 0 < d < R,$$

$$S_r^{R,\hat{w}}(w_0) = S_R(\hat{w}) \cap B_r(w_0) \ \text{ for all } \ w_0 \in \overline{S_R(\hat{w})}.$$

Here the condition $0 < R < 1 - |\hat{w}|$ implies $S_R(\hat{w}) = \{w : |w - \hat{w}| < R, v > 0\} \subset B$ and $C_R(\hat{w}) = \{w : |w - \hat{w}| = R, v > 0\} \subset B$. With this notation, the continuity of the distance function is the main tool to prove the following theorem.

Theorem 4.1. *Consider a boundary configuration $\langle \Gamma, S \rangle$ as above, especially S is assumed to fulfil a chord–arc–condition with constants M und δ. Consider a vector field Q which satisfies besides the assumptions of Theorem 3.1*

$$\|Q\|_{C^0(U_{\tau_0}, \mathbb{R}^3)} < 1 \ \textit{for} \ \tau_0 > 0. \tag{4.1}$$

According to Theorem 3.1 choose for all $\hat{w} \in I$ a real number $R_0 = R_0(\hat{w}, \tau_0/2)$. If X is a minimizer of the functional $\mathcal{F}$ in the class $\mathcal{C}(\Gamma, S)$, then

$$\Phi^{R,\hat{w}}(r, w_0) := \iint_{S_r^{R,\hat{w}}(w_0)} |\nabla X|^2 \, du \, dv \le \left(\frac{2r}{d}\right)^{2\kappa} \iint_{S_R(\hat{w})} |\nabla X|^2 \, du \, dv$$

is true for all $\hat{w} \in I$, for all $0 < R < R_0$ satisfying

$$\iint_{S_R(\hat{w})} |\nabla X|^2 \, du \, dv \le c_1(\tau_0, M, \delta, \kappa) := \min\left\{\frac{\tau_0^2}{4M^2\kappa\pi}, \frac{\tau_0^2}{8\kappa\pi}, \frac{\delta^2}{\kappa\pi}\right\}, \tag{4.2}$$

for all $d \in (0, R)$, for all $w_0 \in \overline{Z_d(R, \hat{w})}$ and for all $r > 0$, where κ is given by

$$\kappa := \frac{1 - \|Q\|_{C^0(U_{\tau_0}, \mathbb{R}^3)}}{1 + \|Q\|_{C^0(U_{\tau_0}, \mathbb{R}^3)}} (1 + M^2)^{-1}.$$

So X is of class $C^{0,\kappa}\left(\overline{Z_d(R, \hat{w})}, \mathbb{R}^3\right)$ and there is a constant $c_2(\kappa) > 0$ such that

$$[X]_{\kappa, \overline{Z_d(R,\hat{w})}} \le c_2(\kappa)\, d^{-\kappa} \sqrt{\min\{c_0(\tau_0/2, H_0), c_1(\tau_0, M, \delta, \kappa)\}}\,.$$

Thus, for all $d \in (0, 1)$ the minimizer X is a Hölder continuous function on $\overline{Z_d}$.

Proof of Theorem 4.1. The idea of constructing a harmonic function is given in [13], here we refer to the detailed proof given in [4], Chapter 7.5. The additional ideas to extend this proof to our situation are the following:
fix $\hat{w} \in I$ and consider a real number $R < R_0$, where R_0 is chosen according to Theorem 3.1. In our context assertion (6) of [4], p. 50, reads as follows:

Assertion 1. For all $d \in (0, R)$, for all $w_0 \in I$ satisfying $|w_0 - \hat{w}| \le R - d$ and for all $r \in (0, d]$ we have

$$\Phi^{R,\hat{w}}(r, w_0) \le \left(\frac{r}{d}\right)^{2\kappa} \Phi^{R,\hat{w}}(d, w_0).$$

Proof of Assertion 1. Fix d and w_0 as above. Because of $S_r(w_0) = S_r^{R,\hat{w}}(w_0)$, the indices R und $\hat{w}$ can be omitted and the argumentation of [4] is carried over until we arrive at (12), p. 51. Notice that the set $\mathcal{N}$ satisfies

$$\begin{aligned} &(i) \;\; \Phi'(r, w_0) \;\text{ exists with }\; \Phi'(r, w_0) = 2\, r^{-1} \textstyle\int_0^\pi |X_\theta(r, \theta)|^2 \, d\theta, \\ &(ii) \;\; O_1(r) := \lim_{\theta \to \pi - 0} X(r, \theta) \;\text{ exisits in }\; S \end{aligned} \tag{4.3}$$

for all $r \in (0, d) \sim \mathcal{N}$. The counterpart of [4], condition (12), is:
Case 1. Consider any $r \in (0, d) \sim \mathcal{N}$ for which

$$\int_0^{\pi} |X_\theta(r, \theta)|^2 \, d\theta \leq \min \left\{ \frac{\delta^2}{\pi}, \frac{\tau_0^2}{4M^2\pi} \right\} \tag{4.4}$$

holds true. Then a vector valued harmonic function, defined on $B_r(w_0)$, with appropriate admissible boundary values (i.e. boundary values of X on $\partial B_r(w_0) \cap \{v > 0\}$ and a *chord–arc–curve* ζ on S elsewhere) satisfies (19) of [4], p. 52:

$$\iint_{B_r(w_0)} |\nabla H|^2 \, du \, dv \leq \frac{1}{2}(1 + M^2) \, r \, \Phi'(r, w_0). \tag{4.5}$$

Now, define on $B \cup B_r(w_0)$

$$Y(w) := \begin{cases} H(w) & : \quad w \in B_r(w_0) \\ X(w) & : \quad w \in B \sim B_r(w_0) \end{cases},$$

by construction a function of class $H^{1,2}(B \cup B_r(w_0), \mathbb{R}^3)$, and consider the *orientation preserving* homeomorphism, which maps $\overline{B}$ conformally onto $\overline{B \cup B_r(w_0)}$, keeping the set $\{-1, 1\}$ as well as the point i fixed. This homeomorphism yields the desired comparison function $Z := Y \circ \tau \in \mathcal{C}(\Gamma, S)$. By the minimality of X and by conformal, orientation preserving invariance of $\mathcal{F}$

$$\begin{aligned} \mathcal{F}_r^{w_0}[X] &:= \frac{1}{2} \iint_{S_r(w_0)} |\nabla X|^2 \, du \, dv + \iint_{S_r(w_0)} Q(X) \cdot (X_u \wedge X_v) \, du \, dv \\ &\leq \frac{1}{2} \iint_{B_r(w_0)} |\nabla H|^2 \, du \, dv + \iint_{B_r(w_0)} Q(H) \cdot (H_u \wedge H_v) \, du \, dv \end{aligned} \tag{4.6}$$

holds true. Because of $R < R_0$ and by Theorem 3.1, the assumption (4.1) gives an estimate for $Q \circ X_{|S_r(w_0)}$: setting $K := \|Q\|_{C^0(U_{\tau_0}, \mathbb{R}^3)} < 1$ we obtain

$$\frac{1}{2} \iint_{S_r(w_0)} |\nabla X|^2 \, du \, dv - \frac{K}{2} \iint_{S_r(w_0)} |\nabla X|^2 \, du \, dv \leq \mathcal{F}_r^{w_0}[X]. \tag{4.7}$$

In order to use the assumption on the right hand side of (4.6), the following lemma has to be proved. For bounded vector fields this lemma will give a better Hölder exponent than estimates using the global bound. Furthermore, notice that the theorem is also true for unbounded vector fields.

Lemma 4.2. *Using the above notation and assumptions, the condition (4.4) implies for all $r \in (0, d) \sim \mathcal{N}$:*

$$H\,(B_r(w_0)) \subset U_{\tau_0}.$$

Proof of Lemma 4.2. The set $\mathcal{N}$ of measure zero was chosen to fulfil

$$H_0 := H(w_0 + re^{i\pi}) = \lim_{\theta \to \pi - 0} X(w_0 + re^{i\theta}) \in S.$$

Since H is harmonic, the function $|H(w) - H_0|^2$ is subharmonic. The maximum principle proves for all $w \in B_r(w_0)$

$$|H(w) - H_0| \le \sup_{w \in \partial B_r(w_0)} |H(w) - H_0|.$$

With the help of (4.4)

$$|X(r, \theta_1) - X(r, \theta_2)| \le \int_{\theta_1}^{\theta_2} |X_\theta(r, \theta)|\, d\theta \le \frac{\tau_0}{2M}$$

is verified for $0 < \theta_1 < \theta_2 < \pi$. For $w \in C_r(w_0)$ we have $H(w) = X(w)$ and

$$\sup_{w \in C_r(w_0)} |H(w) - H_0| \le \frac{\tau_0}{2M} \le \frac{\tau_0}{2}.$$

Given $w \in \partial B_r(w_0) \sim C_r(w_0)$, then $H(r, \theta)$ is defined via the chord–arc–condition and the length of the corresponding curve ist estimated in [4], p. 51, by $l^* \le \tau_0/2$, where our assumption has to be observed. This gives

$$\sup_{w \in \partial B_r(w_0) \sim C_r(w_0)} |H(w) - H_0| \le \frac{\tau_0}{2}$$

and the lemma is proved. □

Now (4.5), (4.6) and (4.7) are seen to imply for all $r \in (0, d) \sim \mathcal{N}$ satisfying (4.4)

$$\Phi(r, w_0) \le \frac{1 + K}{1 - K} \iint_{B_r(w_0)} |\nabla H|^2\, du\, dv \le \frac{1}{2\kappa}\, r\, \Phi'(r, w_0). \tag{4.8}$$

Case 2. Consider any $r \in (0, d) \sim \mathcal{N}$ for which (4.4) is not true. By the choice of R (see (4.2)), by (4.3 (i)) and by assumption, in this case we also get

$$\begin{aligned} \Phi(r, w_0) &\le \left(\iint_{S_R(\hat{w})} |\nabla X|^2\, du\, dv \right) \frac{r}{2}\, \Phi'(r, w_0) \left(\min\left\{ \frac{\delta^2}{\pi}, \frac{\tau_0^2}{4M^2\pi} \right\} \right)^{-1} \\ &\le \frac{1}{2\kappa}\, r\, \Phi'(r, w_0). \end{aligned} \tag{4.9}$$

Now (4.8) and (4.9) prove by integration Assertion 1. □

Assertion 2. *For all* $w_0 \in S_R(\hat{w})$ *satisfying* $|w_0 - \hat{w}| \le R - \rho$ *and* $Im(w_0) \ge \rho$ *for some* $\rho \in (0, R)$ *we have*

$$\Phi(r, w_0) \le \left(\frac{r}{\rho}\right)^{2\kappa} \Phi(\rho, w_0) \ \textit{for all} \ r \in [0, \rho].$$

Proof of Assertion 2. Distinguishing the cases $\int_0^{2\pi} |X_\theta(r, \theta)|^2 \, d\theta \le (>) \tau_0^2/(8\pi)$ and observing that $X(w) \in U_{\tau_0/2}$ for $w^* \in \partial B_r(w_0)$, we prove Assertion 2 as above. □

Now distinguishing three cases exactly as in [4], p. 54, and refering to *Dirichlet's Growth Theorem* the proof of Theorem 4.1 is completed. □

5. The normal component

In this section a geometrical theorem will be proved, namely a theorem on the Hölder continuity of minimizers if the contact angle is not allowed to tend to zero. As mentioned in the introduction, this assumption on the contact angle seems to be the best possible. Consider again a boundary configuration $\langle \Gamma, S \rangle$ as above and a vector field $Q \in C^{1,\beta}(\mathbb{R}^3, \mathbb{R}^3)$, $0 < \beta < 1$, satisfying

$$|\mathrm{div} Q(z)| \le H_0 < \infty \qquad \text{for a constant } H_0 > 0 \text{ and for all } z \in \mathbb{R}^3,$$

$$\|Q\|_{C^0(\mathbb{R}^3, \mathbb{R}^3)} \le Q_0 < \infty \ \text{for a constant } Q_0 > 0.$$

Since we will have to bend the supporting surface locally to a plane, since we will have to control the behavior of S at infinity and since we have to define a normal of S, we impose the following smoothness condition, which includes the bound on the normal component of Q. We will distinguish twodimensional balls B and threedimensional balls $\mathcal{B}$.

Assumption 5.1. *There is a neighbourhood* U_{τ_0} *of* S *such that:*

(i) *There are positive, real constants* $\tilde{\rho}$, $\bar{\rho}$ *and a countable number of points* $z_i \in S$ *such that*

$$U_{\tau_0} \subset \bigcup_i \mathcal{B}_{\rho_i}(z_i) \ \textit{for} \ \tilde{\rho} < \rho_i < \bar{\rho}.$$

(ii) *For any* z_i, *a* C^2*–diffeomorphism* $h_i : h_i^{-1}(\mathcal{B}_{15\rho_i}(z_i)) \to \mathcal{B}_{15\rho_i}(z_i)$ *exists with*

$$\left(h_i^{-1}(z)\right)^3 = 0 \ \textit{for all} \ z \in \mathcal{B}_{15\rho_i}(z_i) \cap S.$$

(iii) *Consider* $h(y)(= h_i(y))$ *and define* $g_{mn}(y) := \sum_{l=1}^{3} h^l_{y^m} h^l_{y^n}$, $1 \le m, n \le 3$. *Then* $G(y) = (g_{mn}(y))_{mn} \in \mathbb{R}^{3,3}$ *is positive definite, and there is a constant* K *such that for all* i *and for all* $y \in h^{-1}(\mathcal{B}_{15\rho_i}(z_i))$ *the norms of* G, DG, G^{-1} *are bounded by* K *as well as for all* $z \in \mathcal{B}_{15\rho_i}(z_i)$ *the norm of* Dh^{-1}.

(iv) *There is a constant* q, $0 < q < 1$, *such that for all* i

$$\|Dh_i^{-1}(z)\|^2 \left| Q(z) \cdot \left(\frac{\partial h_i}{\partial y^1} \wedge \frac{\partial h_i}{\partial y^2} \right) \circ (h_i^{-1}(z)) \right| < q$$
$$\text{for all } \; z \in \mathcal{B}_{6\rho_i}(z_i),$$

where $\| \, \|$ *is the operator norm of the linear mapping* Dh_i^{-1}.

Remark 5.2. For example each compact C^2–surface – such that the absolute value of the normal component of the vector field is bounded by $\tilde{q} < 1$ – is easily seen to fulfil Assumption 5.1.

Now we can state our main theorem. The essential idea is given in Lemma 5.5, which is due to the fact that the volume functional associates the controled normal components of Q and X with the tangential components of X and Q. It is remarkable that – distinguishing different cases as in the last section – we have precisely the room to move which is needed in proving the smallness of terms invoking the tangential component of Q.

Theorem 5.3. *There are positive real numbers* $\kappa \in (0, 1)$ *and* $\tau < \tau_0$, *depending only on* $\tilde{\rho}$, $\bar{\rho}$, K, M, δ, H_0, Q_0 *and* q, *such that: if* X *is a minimizer of the functional* $\mathcal{F}$ *in the class* $\mathcal{C}(\Gamma, S)$, *then* $X(B) \subset U_\tau$ *implies for all* $d \in (0, 1)$, *for all* $w_0 \in \overline{Z}_d$ *and for all* $r > 0$

$$\Phi(r, w_0) := \iint_{S_r(w_0)} |\nabla X|^2 \, du \, dv \le \left(\frac{2r}{d} \right)^{2\kappa} \iint_B |\nabla X|^2 \, du \, dv.$$

According to Theorem 3.1 and to the arguments of Theorem 4.1 we immediately see:

Corollary 5.4. *For all* $d \in (0, 1)$ *there is a real number* $\kappa \in (0, 1)$ *such that a minimizer* X *is of class* $C^{0,\kappa}(\overline{Z}_d, \mathbb{R}^3)$.

Proving Theorem 5.3 we need the following lemma to see that only the normal component of Q is of geometric significance. The lemma will be proved at the end of this section.

Lemma 5.5. *There are positive real numbers $\tau < \tau_0$ and $\varepsilon_0 > 0$, depending only on $\tilde{\rho}$, $\bar{\rho}$, K, M, δ, H_0, Q_0 and q, such that:*
suppose X is a minimizer of $\mathcal{F}$ in the class $\mathcal{C}(\Gamma, S)$ satisfying $X(B) \subset U_\tau$. Then:

$$\left| \iint_{S_r(w_0)} Q(X) \cdot (X_u \wedge X_v)\, du\, dv \right| \le \frac{1}{2} \frac{1+q}{2} \iint_{S_r(w_0)} |\nabla X|^2\, du\, dv$$

(i) for all $w_0 \in I$ and for all $r \in (0, 1 - |w_0|)$ satisfying

$$\int_0^\pi |X_\theta(r,\theta)|^2\, d\theta \le \Phi(r, w_0)\, \varepsilon_0 \ \ \textit{and}$$
$$O(r) := \lim_{\theta \to \pi - 0} X(r, \theta) \ \ \textit{exisits in} \ \ S,$$

(ii) for all $S_r(w_0) = B_r(w_0) \subset B$ satisfying

$$\int_0^{2\pi} |X_\theta(r,\theta)|^2\, d\theta \le \Phi(r, w_0)\, \varepsilon_0.$$

Proof of Theorem 5.3. Choose τ and ε_0 according to Lemma 5.5. Then there exists a constant κ, depending only on the above quantities, such that:

Assertion 1. The condition $X(B) \subset U_\tau$ implies for all $d \in (0, 1)$, for all $w_0 \in I$ satisfying $|w_0| \le 1 - d$ and for all $r \in (0, d)$

$$\Phi(r, w_0) \le \left(\frac{r}{d}\right)^{2\kappa} \Phi(d, w_0).$$

Proof of Assertion 1. Fix d and w_0 as above. Then the arguments of Assertion 1 of the last section are completely the same until we arrive at Case 1. Observe that $\mathcal{N}$ is chosen to satisfy (4.3(i)). Now Case 1 reads as follows.
Case 1: Consider any $r \in (0, d) \sim \mathcal{N}$ for which

$$\int_0^\pi |X_\theta(r,\theta)|^2\, d\theta \le \Phi(r, w_0)\, \varepsilon_0 \tag{5.1}$$

holds true, where we may assume without loss of generality $\Phi(r, w_0)\, \varepsilon_0 \le \delta^2/\pi$. Constructing a harmonic function as above and using Lemma 5.5 (i) we see

$$\Phi(r, w_0) \le \frac{1 + Q_0}{1 - q}(1 + M^2)\, r\, \Phi'(r, w_0).$$

Case 2: Consider any $r \in (0, d) \sim \mathcal{N}$ for which (5.1) is not true. Then we get

$$\Phi(r, w_0) \leq \Phi(r, w_0) \left\{ \left(\int_0^\pi |X_\theta(r, \theta)|^2 \, d\theta \right) (\Phi(r, w_0)\, \varepsilon_0)^{-1} \right\}$$
$$\leq \frac{1}{2} r\, \Phi'(r, w_0)\, \varepsilon_0^{-1}$$

and the assertion. □

In the same manner Assertion 2 is proved and again the theorem follows:

Assertion 2. *The condition $X(B) \subset U_\tau$ implies for all w_0 satisfying $|w_0| \leq 1 - \rho$ and $Im(w_0) \geq \rho$ for some $\rho \in (0, 1)$*

$$\Phi(r, w_0) \leq \left(\frac{r}{\rho} \right)^{2\kappa} \Phi(\rho, w_0) \ \text{ for all } \ r \in (0, \rho).$$

□

We finish this paper with the proof of Lemma 5.5. Notice that there is no gap between the assumptions of Case 2 and of Lemma 5.5, both of them being precisely needed.

Proof of Lemma 5.5. Consider the situation (i) for fixed w_0 and r. According to Assumption 5.1 (i) and to the covering theorem of Vitali (see [6], pp. 26) we obtain a *disjoint* subcollection, again denoted by $\{\mathcal{B}_{\rho_i}(z_i)\}_{i \in \mathcal{I} \subset \mathbb{N}}$, such that

$$U_{\tau_0} \subset \bigcup_{i \in \mathcal{I}} \tilde{V}_i \ \text{ for } \ V_i := \mathcal{B}_{6\rho_i}(z_i) \ \text{ and } \ \tilde{V}_1 := V_1,$$
$$\tilde{V}_i := V_i \sim \bigcup_{k=1}^{i-1} V_k \ \text{ for all } \ i > 1.$$

Given $A \subset \mathbb{R}^3$, set $A^{-1} := \{w \in B : X(w) \in A\} \subset B \subset \mathbb{R}^2$. By construction, the collection $\{\tilde{V}_i^{-1}\}_{i \in \mathcal{I}}$ is a disjoint covering of B since $\tau < \tau_0$ and since X is assumed to be a mapping into U_{τ_0}. If h (omitting the index i) is the diffeomorphism corresponding to V_i, if we set $Y_i = Y := h^{-1} \circ X$ and if $\tilde{Q}(y) \in \mathbb{R}^3$ denotes

$$\left(Q(h(y)) \frac{\partial h(y)}{\partial y^2} \wedge \frac{\partial h(y)}{\partial y^3}, \right.$$
$$\left. Q(h(y)) \frac{\partial h(y)}{\partial y^3} \wedge \frac{\partial h(y)}{\partial y^1}, \, Q(h(y)) \frac{\partial h(y)}{\partial y^1} \wedge \frac{\partial h(y)}{\partial y^2} \right),$$

then we are led to

$$
\begin{aligned}
&\left|\iint_{S_r(w_0)} Q(X)\cdot(X_u\wedge X_v)\,du\,dv\right|\\
&=\left|\sum_{i\in\mathcal{I}}\iint_{S_r(w_0)\cap\tilde V_i^{-1}} Q(X)\cdot(X_u\wedge X_v)\,du\,dv\right|\\
&=\left|\sum_{i\in\mathcal{I}}\iint_{S_r(w_0)\cap\tilde V_i^{-1}} \tilde Q(Y_i)\cdot(Y_{i\,u}\wedge Y_{i\,v})\,du\,dv\right| \qquad (5.2)\\
&\le\sum_{i\in\mathcal{I}}\iint_{S_r(w_0)\cap\tilde V_i^{-1}}\left|\tilde Q^3(Y_i)(Y_{i\,u}\wedge Y_{i\,v})^3+\sum_{k=1}^{2}\tilde Q^k(Y_i)(Y_{i\,u}\wedge Y_{i\,v})^k\right|du\,dv\\
&\le\sum_{i\in\mathcal{I}}\frac12\iint_{S_r(w_0)\cap\tilde V_i^{-1}}|\tilde Q^3(Y_i)|\,|\nabla Y_i|^2\,du\,dv+c(Q_0,K)J,
\end{aligned}
$$

where J is given by (set $Y_i^{(1,2)}=(y_i^1,y_i^2,0)$)

$$
J=\sum_{i\in\mathcal{I}}\left(\iint_{S_r(w_0)\cap\tilde V_i^{-1}}|\nabla Y_i^{(1,2)}|^2\,du\,dv\right)^{\frac12}\left(\iint_{S_r(w_0)\cap\tilde V_i^{-1}}|\nabla y_i^3|^2\,du\,dv\right)^{\frac12}.
$$

By Assumption 5.1 (iv) and since the sets $\tilde V_i^{-1}$ are mutually disjoint, the first term on the right hand side of (5.2) is estimated from above:

$$
\sum_{i\in\mathcal{I}}\frac12\iint_{S_r(w_0)\cap\tilde V_i^{-1}}|\tilde Q^3(Y_i)|\,|\nabla Y_i|^2\,du\,dv\le\frac{q}{2}\iint_{S_r(w_0)}|\nabla X|^2\,du\,dv. \qquad (5.3)
$$

To get an estimate for J, define non–negative, real valued, smooth functions

$$
\lambda_i(s)=\begin{cases}1 & : \quad s\le 6\rho_i\\ 0 & : \quad s\ge 7\rho_i\end{cases},\quad \lambda_i'(s)\le 2/\rho_i\le 2/\tilde\rho.
$$

Setting $W_i=\mathcal{B}_{7\rho_i}(z_i)$ we observe: it is possible to arrange the balls W_i in $c_1(n,\tilde\rho,\bar\rho)$ mutually disjoint subcollections $\{W_i\}_{i\in\mathcal{I}_k\subset\mathcal{I}}$, where n is the dimension of the surrounding Euclidean space, $n=3$. Indeed, following the idea of Besicovitchs covering theorem (see [6], pp. 30), the balls are distributed to several “rows” by induction. The first element in the first row is W_1. Assume the balls $W_1,\dots,W_j$ are mutually disjoint arranged in m rows. Then the ball W_{j+1} becomes an element of the first of these m rows, where the intersection with all other elements is empty. If no such row exists, then W_{j+1} is the first element of the row with number $m+1$. By construction, the balls $\mathcal{B}_{\rho_i}$ are mutually disjoint and the ratios of the radii are uniformly bounded.

So the intersection of a given ball W_i with another ball W_j is nonempty only for a finite number $c_1(n, \tilde{\rho}, \bar{\rho})$ and the observation is proved. With this observation we will obtain an estimate for J using the definition of λ_i:

$$J \le \sum_{k=1}^{c_1} \sum_{i\in\mathcal{I}_k} \left(\iint_{S_r(w_0)} \lambda_i(|X - z_i|)\, |\nabla Y_i^{(1,2)}|^2\, du\, dv \right)^{\frac{1}{2}} \times \left(\iint_{S_r(w_0)} \lambda_i(|X - z_i|)\, |\nabla y_i^3|^2\, du\, dv \right)^{\frac{1}{2}}. \tag{5.4}$$

Here we have to give an estimate for the last integral in (5.4): the smoothness of Q implies $X \in C^2(B, \mathbb{R}^3)$ (see [9]) and a partial integration proves for all $i \in \mathcal{I}$

$$\begin{aligned} &\iint_{S_r(w_0)} \lambda_i(|X - z_i|)\, |\nabla y_i^3|^2\, du\, dv \\ &= - \iint_{S_r(w_0)} \lambda_i'(|X - z_i|)\, \frac{\langle \nabla X, (X - z_i)\rangle \cdot \nabla y_i^3}{|X - z_i|}\, y_i^3\, du\, dv \\ &\quad - \iint_{S_r(w_0)} \lambda_i(|X - z_i|)\, \Delta y_i^3\, y_i^3\, du\, dv \\ &\quad + \int_{\partial S_r(w_0)} \lambda_i(|X - z_i|)\, y_i^3\, \nabla y_i^3 \cdot \nu\, d\mathcal{H}^1, \end{aligned} \tag{5.5}$$

where ν is the outer unit normal to $\partial S_r(w_0)$. Now, assume $X(w) \in U_\tau$ for all $w \in S_r(w_0)$, where $\tau < \tau_0$ is chosen below in an appropriate way. Since z_i is an element of the supporting surface, for all $w \in S_r(w_0) \cap W_i^{-1}$ there is a point $f \in \mathcal{B}_{15\rho_i}$ such that $f \in S$ and that $|X(w) - f| < 2\tau$. The third component of $h_i^{-1}(f)$ vanishes by construction and the mean value theorem yields for all $w \in S_r(w_0) \cap W_i^{-1}$ a point $\xi \in \mathcal{B}_{15\rho_i}$ satisfying

$$|y_i^3(w)| \le |D(h^{-1})^3|(\xi)\, |X(w) - f| \le c_2(K)\, \tau. \tag{5.6}$$

With the assumption on the diffeomorphism h (again omitting the fixed index i) and by virtue of the variational equation (see (3.6) and [4], pp. 64)

$$\begin{aligned} &g_{ij}(Y)\Delta y^i + \frac{\partial g_{ij}(Y)}{\partial y^k} \nabla y^i \nabla y^k \\ &= \frac{1}{2} \frac{\partial g_{ik}(Y)}{\partial y^j} \nabla y^i \nabla y^k + 2H(h(Y))\sqrt{g(Y)}\, (Y_u \wedge Y_v)^j \end{aligned}$$

we obtain the estimates

$$\begin{aligned} |\nabla Y_i(w)|^2 &\le c_3(K)\, |\nabla X(w)|^2 \ \text{ for all }\ w \in S_r(w_0) \cap W_i^{-1} \\ \text{and } |\Delta Y_i(w)| &\le c_4(K, H_0)\, |\nabla Y_i|^2 \text{ for all }\ w \in S_r(w_0) \cap W_i^{-1}. \end{aligned} \tag{5.7}$$

By (5.5), (5.6) and (5.7) there is a constant $c_5(K, H_0, \tilde{\rho})$, independent of i, such that

$$\iint_{S_r(w_0)} \lambda_i(|X - z_i|)\, |\nabla y_i^3|^2 \, du\, dv$$
$$\leq \tau c_5 \iint_{S_r(w_0) \cap W_i^{-1}} |\nabla X|^2 \, du\, dv + \int_{\partial S_r(w_0)} \lambda_i(|X - z_i|)\, y_i^3 \, \nabla y_i^3 \cdot \nu \, d\mathcal{H}^1.$$

Now it remains to estimate the boundary integrals. Assumption 5.1 (ii) shows

$$\int_{\partial S_r(w_0)} \lambda_i(|X - z_i|)\, |y_i^3|\, |\nabla y_i^3 \cdot \nu| \, d\mathcal{H}^1 \tag{5.8}$$
$$\leq c_6(K) \sup_{w \in C_r(w_0) \cap W_i^{-1}} |y_i^3(w)| \int_{C_r(w_0) \cap W_i^{-1}} |\nabla X| \, d\mathcal{H}^1,$$

and so only the balls W_i satisfying $X\,(C_r(w_0)) \cap W_i \neq \emptyset$ are to be considered. Fix one of these balls and $a_i \in X\,(C_r(w_0)) \cap W_i$. For all $w \in C_r(w_0)$ we have by assumption

$$|X(w) - a_i| \leq \sqrt{\pi} \left(\int_0^{\pi} |X_\theta(r, \theta)|^2 \, d\theta \right)^{\frac{1}{2}} \leq \sqrt{\Phi(r, w_0)\varepsilon_0}\, \sqrt{\pi}\,.$$

Thus for ε_0 sufficiently small, $\varepsilon_0 < \tilde{c}_7(D[X], \tilde{\rho}) \leq c_7(\tilde{\rho})$, the set $X(C_r(w_0))$ and the point $O(r)$ defined in (i) are seen to stay inside the ball $\mathcal{B}_{15\rho_i}$. This gives for all $w \in C_r(w_0)$

$$|y_i^3(w)| = |h_i^{-1}(X(w))^3 - h_i^{-1}(O(r))^3|$$
$$\leq c_8(K) \left(\int_0^{\pi} |X_\theta(r, \theta)|^2 \, d\theta \right)^{\frac{1}{2}} \leq c_8(K) \sqrt{\Phi(r, w_0)\varepsilon_0}.$$

Finally the conformality relations imply

$$\int_{C_r(w_0)} |\nabla X| \, d\mathcal{H}^1 \leq \sqrt{2} \int_{C_r(w_0)} \left(\frac{1}{r^2} |X_\theta(r, \theta)|^2 \right)^{\frac{1}{2}} d\mathcal{H}^1$$
$$\leq \frac{\sqrt{2\pi}}{\sqrt{r}} \left(\int_{C_r(w_0)} |X_\theta(r, \theta)|^2 \, d\mathcal{H}^1 \right)^{\frac{1}{2}} \leq \sqrt{2\pi\, \Phi(r, w_0)\varepsilon_0},$$

and the above computations prove for all i satisfying $X\,(C_r(w_0)) \cap W_i \neq \emptyset$

$$\int_{\partial S_r(w_0)} \lambda_i(|X - z_i|)\, |y_i^3|\, |\nabla y_i^3 \cdot \nu| \, d\mathcal{H}^1 \leq c_9(K, \delta)\, \Phi(r, w_0)\, \varepsilon_0.$$

By the choice of ε_0, the intersection of $X(C_r(w_0))$ with W_i is nonempty only for $c_{10}(n,\tilde{\rho},\bar{\rho})$ balls, while the other balls can be omitted in estimating (5.8). Furthermore, the subcollections $\{W_i\}_{i\in\mathcal{I}_k}$ are mutually disjoint and summarizing the results we have found an upper bound for J:

$$\sum_{k=1}^{c_1}\sum_{i\in\mathcal{I}_k}\left(\iint_{S_r(w_0)\cap W_i^{-1}}|\nabla Y_i|^2\,du\,dv\right)^{\frac{1}{2}}$$
$$\times\left(\iint_{S_r(w_0)}\lambda_i(|X-z_i|)|\nabla y_i^3|^2\,du\,dv\right)^{\frac{1}{2}}$$
$$\le c_{11}(K)\sum_{k=1}^{c_1}\sum_{i\in\mathcal{I}_k}\left(\iint_{S_r(w_0)\cap W_i^{-1}}|\nabla X|^2\,du\,dv\right)^{\frac{1}{2}}\cdot$$
$$\left(\tau c_5\iint_{S_r(w_0)\cap W_i^{-1}}|\nabla X|^2\,du\,dv+\int_{\partial S_r(w_0)}\lambda_i(|X-z_i|)\,|y_i^3|\,|\nabla y_i^3\cdot\nu|\,d\mathcal{H}^1\right)^{\frac{1}{2}}$$
$$\le c_{12}(n,\tilde{\rho},\bar{\rho},K,h_0,\delta)\left(\sqrt{\tau}+\sqrt{\varepsilon_0}\right)\iint_{S_r(w_0)}|\nabla X|^2\,du\,dv.$$

If we choose τ and ε_0 to be smaller than a constant $C(n,\tilde{\rho},\bar{\rho},K,H_0,\delta,Q_0,q)$ and if we recall (5.3), then we obtain the first conclusion of Lemma 5.5.
To prove the lemma in situation (ii), observe that there are at most $c_{10}(n,\tilde{\rho},\bar{\rho})$ balls W_i such that $X(\partial B_r(w_0))\cap W_i\neq\emptyset$. The other balls are treated as above since the boundary integrals are vanishing. If there is a point $a_i\in X(\partial B_r(w_0))\cap W_i$, then define $P_i:=h_i^{-1}(a_i)$. As above we see

$$\iint_{S_r(w_0)}\lambda_i(|X-z_i|)\,|\nabla y_i^3|^2\,du\,dv$$
$$=-\iint_{S_r(w_0)}\lambda_i'(|X-z_i|)\,\frac{\langle\nabla X,(X-z_i)\rangle\cdot\nabla y_i^3}{|X-z_i|}\,(y_i^3-P_i^3)\,du\,dv$$
$$-\iint_{S_r(w_0)}\lambda_i(|X-z_i|)\,\Delta y_i^3\,(y_i^3-P_i^3)\,du\,dv$$
$$+\int_{\partial S_r(w_0)}\lambda_i(|X-z_i|)\,(y_i^3-P_i^3)\,\nabla y_i^3\cdot\nu\,d\mathcal{H}^1.$$

Now consider a_i and argue as in (5.6) to prove for $w\in S_r(w_0)\cap W_i^{-1}$

$$|y_i^3(w)-P_i^3|\le|y_3^i(w)|+|P_i^3|\le 2\,c_2(K)\,\tau.$$

The boundary estimates are the same if we substitute y_i^3 by $y_i^3-P_i^3$ and $O(r)$ by a_i and the lemma is proved. □

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