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Obstacle Problems with Linear Growth: Hölder Regularity for the Dual Solution

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Abstract. For a strictly convex integrand $f : \mathbb{R}^n \to \mathbb{R}$ with linear growth we discuss the variational problem

$$J(u) \;=\; \int_\Omega f(\nabla u)\,dx \;\longrightarrow\; \min$$

among mappings $u : \mathbb{R}^n \supset \Omega \to \mathbb{R}$ of Sobolev class W^1_1 with zero trace satisfying in addition $u \geq \psi$ for a given function ψ such that $\psi|_{\partial\Omega} < 0$. We introduce a natural dual problem which admits a unique maximizer σ. In further sections the smoothness of σ is investigated using a special J–minimizing sequence with limit $u^* \in C^{1,\alpha}(\Omega)$ for which the duality relation $\sigma = \frac{\partial f}{\partial P}(\nabla u^*)$ holds.

1. Introduction

In a previous paper [BF] we investigated the dual variational problem to the minimization problem

$$(1.1) \qquad J(u) \;=\; \int_\Omega f(\nabla u)\,dx \;\longrightarrow\; \min \quad \text{on} \quad u_0 + \overset{\circ}{W}{}^1_1\big(\Omega, \mathbb{R}^N\big)$$

with strictly convex integrand $f : \mathbb{R}^{nN} \to \mathbb{R}$ of linear growth. Here Ω denotes a bounded domain in $\mathbb{R}^n$. The study of the dual problem to (1.1) is motivated by the fact that (1.1) is in general not solvable whereas the dual problem admits a unique solution σ which has a clear physical or geometrical meaning (see [FS] for a detailed list of references). Besides other things we proved in [BF] that σ is partially Hölder continuous on Ω which means that there exists an open subset Ω_0 of Ω with full measure such that $\sigma \in C^{0,\alpha}\big(\Omega_0, \mathbb{R}^{nN}\big)$ for any $0 < \alpha < 1$.

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In the present paper we concentrate on the scalar case $N = 1$ and replace (1.1) by the obstacle problem

$$(1.2)\quad J(u) \longrightarrow \min \quad \text{on} \quad \overset{\circ}{W}{}^1_{1,\psi}(\Omega) := \Big\{u \in \overset{\circ}{W}{}^1_1(\Omega) : u \geq \psi \ \text{ almost everywhere}\Big\}$$

where $\psi : \overline{\Omega} \to \mathbb{R}$ is a given smooth function such that $\psi\big|_{\partial\Omega} < 0$ and $\psi(x) > 0$ at some interior point $x \in \Omega$. Problem (1.2) admits a natural dual formulation with unique solution σ and such that the inf–sup relation holds. We prove that σ has weak derivatives in $L^2_{loc}(\Omega, \mathbb{R}^n)$, moreover, we show that σ is Hölder continuous on Ω. The proofs of these facts are based on the construction of a special minimizing sequence for (1.2) whose weak limit u^* is of class $C^{1,\alpha}(\Omega)$ and in addition satisfies the duality relation $\sigma = \frac{\partial f}{\partial q}(\nabla u^*)$.

2. Notation and results

On a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ consider a function $\psi \in W^2_\infty(\Omega)$ such that $\psi\big|_{\partial\Omega} < 0$ and $\psi(x) > 0$ at some interior point $x \in \Omega$. We define the classes

$$\overset{\circ}{W}{}^1_{p,\psi}(\Omega) := \Big\{u \in \overset{\circ}{W}{}^1_p(\Omega) : u \geq \psi \ \text{ almost everywhere}\Big\}, \quad 1 \leq p \leq \infty,$$

consisting of all Sobolev functions with zero trace respecting the obstacle ψ. On $\overset{\circ}{W}{}^1_{1,\psi}(\Omega)$ we let $J(u) = \int_\Omega f(\nabla u)\, dx$ with density f satisfying

Assumption 2.1. The function $f : \mathbb{R}^n \to \mathbb{R}$ is smooth, convex and of linear growth in the following sense:

(i) $f \in C^2(\mathbb{R}^n, \mathbb{R})$.

(ii) There are numbers μ_1, μ_2, ν_1 and ν_2 in $\mathbb{R}^+_0$ such that for all P, $Q \in \mathbb{R}^n$

$$\begin{aligned} \frac{\mu_1}{\sqrt{1+|P|^2}}\left(|Q|^2 - \frac{(Q\cdot P)^2}{1+|P|^2}\right) &\leq D^2 f(P)(Q,Q) \\ &\leq \frac{\mu_2}{\sqrt{1+|P|^2}}\left(|Q|^2 - \frac{(Q\cdot P)^2}{1+|P|^2}\right), \\ \nu_1\sqrt{1+|P|^2} - \nu_2 &\leq \nabla f(P)\cdot P\,. \end{aligned}$$

(iii) There is a real number A such that $|\nabla f(P)| \leq A$ for all $P \in \mathbb{R}^n$.

Clearly (ii) implies with suitable real numbers $a, b \in \mathbb{R}$, $a > 0$, the growth estimate

$$f(P) \geq a\,|P| + b \quad \text{for all} \quad P \in \mathbb{R}^n\,.$$

We may assume $f \geq 0$ and in addition $\nabla f(0) = 0$ since we have for $u \in \overset{\circ}{W}{}^1_1(\Omega)$

$$J(u) = \int_\Omega \big(f(\nabla u) - \nabla f(0)\cdot\nabla u\big)\, dx\,.$$

Let f^* denote the conjugate function of f (compare [ET] or [Ro]). Then

$$J(u) \;=\; \sup_{\tau \in L^\infty(\Omega, \mathbb{R}^n)} l(u,\tau)$$

for any $u \in \overset{\circ}{W}{}^1_{1,\psi}(\Omega)$, where

$$l(u,\tau) \;=\; \int_\Omega \big(\tau\cdot\nabla u - f^*(\tau)\big)\,dx\,, \quad u \in \overset{\circ}{W}{}^1_{1,\psi}(\Omega)\,, \quad \tau \in L^\infty(\Omega,\mathbb{R}^n)\,,$$

is the Lagrangian of the problem. Finally, we introduce the dual functional

$$R \;:\; L^\infty(\Omega,\mathbb{R}^n) \ni \tau \;\longmapsto\; \inf_{u \in \overset{\circ}{W}{}^1_{1,\psi}(\Omega)} l(u,\tau) \in \overline{\mathbb{R}}$$

and the corresponding dual problem

$$R \;\longrightarrow\; \max \quad \text{on} \quad L^\infty(\Omega,\mathbb{R}^n)\,. \tag{2.1}$$

We now formulate our main results:

Theorem 2.2. *Problem* (2.1) *admits a unique solution* σ*,*

$$R(\sigma) \;=\; \sup_{\tau \in L^\infty(\Omega,\mathbb{R}^n)} R(\tau) \quad \left(= \inf_{u \in \overset{\circ}{W}{}^1_{1,\psi}(\Omega)} J(u)\right)\,,$$

with the following properties:
(i) *The maximizer* σ *is of class* $W^1_{2,loc}(\Omega,\mathbb{R}^n)$*.*
(ii) σ *is Hölder continuous in the interior of* Ω *with any exponent* $\alpha \in (0,1)$*.*

Remark 2.3. In [BF] partial regularity of the maximizer σ has been established under much weaker assumptions as stated above. First of all, it is an easy exercise to show that (i) of Theorem 2.2 holds under the hypothesis of the paper [BF] but unfortunately we could not prove (ii) in this modified setting. The technical reason is that our proof of Theorem 2.2 (ii) is based on a regularity result for local minimizers of obstacle problems in BV which we only obtained under the assumption that the minimal surface type ellipticity condition for f — taken from [GMS] — holds true. We further conjecture that Theorem 2.2 (ii) remains valid if we replace the lower bound in Assumption 2.1 (ii) by the weaker condition

$$D^2 f(P)(Q,Q) \;\geq\; \frac{\mu_1}{\sqrt{1+|P|^2}^{\,3}}\,|Q|^2\,,$$

but we did not discuss this question seriously.[1)]

Remark 2.4. (i) These results are also valid in the case of non vanishing finite boundary values.

(ii) Of course, the following arguments also cover the case without an obstacle and Remark 6.2 (ii) of [BF] is proved.

[1)] Very recently the first author succeeded in proving Theorem 2.2(ii) assuming that $D^2 f(P)(Q,Q) \geq \mu_1(1+|P|^2)^{-\mu/2}|Q|^2$ holds for some exponent $\mu < 3$, we refer to [BI2].

3. Proof of $W^1_{2,\mathrm{loc}}$–regularity via a suitable regularization

From now on we assume that all the hypotheses stated in and before Theorem 2.2 are valid. Then the existence of a solution σ to problem (2.1) can be deduced along the lines of [ET] or just by considering a maximizing sequence, the uniqueness of σ is a consequence of the results established in [BI1]. Clearly we have

$$R(\sigma) \;=\; \sup_{\tau\in L^\infty(\Omega,\mathbb{R}^n)} R(\tau) \;\le\; \inf_{u\in \overset{\circ}{W}{}^1_{1,\psi}(\Omega)} J(u)\,, \tag{3.1}$$

the reverse inequality again follows from [ET] but is also a byproduct of the following considerations: in order to construct a special maximizing sequence converging to σ we let $(0<\delta\le 1)$

$$\begin{aligned} f_\delta(P) \;&=\; \frac{\delta}{2}\,|P|^2 + f(P)\,, \qquad P\in\mathbb{R}^n\,,\\ J_\delta(u) \;&=\; \int_\Omega f_\delta(\nabla u)\,dx\,, \qquad u\in \overset{\circ}{W}{}^1_{2,\psi}(\Omega)\,. \end{aligned}$$

For functions u from this space we introduce

$$J_\delta(u) \;=\; \sup_{\tau\in L^2(\Omega,\mathbb{R}^n)} \int_\Omega \big(\tau\cdot\nabla u - f^*_\delta(\tau)\big)\,dx \;=:\; \sup_{\tau\in L^2(\Omega,\mathbb{R}^n)} l_\delta(u,\tau)\,.$$

Define

$$R_\delta(\tau) \;=\; \inf_{v\in \overset{\circ}{W}{}^1_{2,\psi}(\Omega)} l_\delta(v,\tau)\,, \qquad \tau\in L^2(\Omega,\mathbb{R}^n)\,,$$

and let u_δ denote the unique solution of

$$J_\delta \;\longrightarrow\; \min \quad \text{on} \quad \overset{\circ}{W}{}^1_{2,\psi}(\Omega)\,.$$

Lemma 3.1. (i) $\sigma_\delta := \frac{\partial f_\delta}{\partial P}(\nabla u_\delta)$ *is the unique solution of* $R_\delta\to\max$ *on* $L^2(\Omega,\mathbb{R}^n)$.
(ii) *The δ–version of the inf–sup relation holds, i. e.* $J_\delta(u_\delta)=R_\delta(\sigma_\delta)$.

The proof is standard (see [ET], p. 85), for the reader's convenience we give a short and selfcontained proof using the next result which is of great importance for the rest of the paper.

Lemma 3.2. *For any* $\delta>0$ *we have*
(i) $g_\delta \;:=\; \chi_{[u_\delta=\psi]}\big(-\operatorname{div}\{Df_\delta(\nabla\psi)\}\big) \;\ge\; 0$ *almost everywhere,*
(ii) $\int_\Omega \sigma_\delta\cdot\nabla\varphi\,dx = \int_\Omega g_\delta\,\varphi\,dx$ *for all* $\varphi\in C^1_0(\Omega)$.

Proof of Lemma 3.2. Following [Fu1], [Fu2] and [FL] we define for any $\delta>0$

$$w^\varepsilon_t \;:=\; u_\delta + t\,\eta\,h_\varepsilon\circ(u_\delta-\psi)$$

where $0 \leq \eta \in C^1_0(\Omega)$, $\varepsilon, t > 0$ and $h_\varepsilon \in C^1(\mathbb{R})$ is satisfying $0 \leq h_\varepsilon \leq 1$, $h_\varepsilon = 1$ on $(0,\varepsilon)$ and $h_\varepsilon = 0$ on $(2\varepsilon, \infty)$. Since w^ε_t is of class $\overset{\circ}{W}{}^1_{2,\psi}$,

$$\frac{1}{t}\left[\int_\Omega f_\delta\big(\nabla w^\varepsilon_t\big)\, dx - \int_\Omega f_\delta(\nabla u_\delta)\, dx\right] \;\geq\; 0$$

holds true. Passing to the limit $t \to 0$ we see

$$\int_\Omega Df_\delta(\nabla u_\delta) \cdot \nabla\big(\eta\, h_\varepsilon \circ (u_\delta - \psi)\big)\, dx \;\geq\; 0\,,$$

i.e. there exists a Radon measure $\lambda = \lambda_\delta$ such that for all $\varphi \in C^1_0(\Omega)$

$$\int_\Omega Df_\delta(\nabla u_\delta) \cdot \nabla\big(\varphi\, h_\varepsilon \circ (u_\delta - \psi)\big)\, dx \;=\; \int_\Omega \varphi\, d\lambda\,.$$

Notice that λ does not depend on ε. This can be proved by using $\widetilde{w} = u_\delta + \eta\, t\big\{h_\varepsilon \circ (u_\delta - \psi) - h_{\varepsilon'} \circ (u_\delta - \psi)\big\}$, $\varepsilon < \varepsilon'$, as testfunction provided t is small enough. By standard arguments u_δ is seen to be of class $W^2_{2,loc}$ (see [HW] and also Lemma 3.3, where the testfunction is explicitly given) and we obtain by a partial integration

$$-\int_\Omega \operatorname{div}\big\{Df_\delta(\nabla u_\delta)\big\} h_\varepsilon \circ (u_\delta - \psi) \varphi\, dx \;=\; \int_\Omega \varphi\, d\lambda\,. \tag{3.2}$$

Now the right–hand side of (3.2) is independent of ε and we may pass to the limit $\varepsilon \to 0$. Since $\nabla u_\delta = \nabla \psi$ almost everywhere on $[u_\delta = \psi]$ (see [GT], Lemma 7.7, p. 145), we get

$$\int_\Omega g_\delta\, \varphi\, dx \;=\; \int_\Omega \varphi\, d\lambda \quad \text{for all} \quad \varphi \in C^1_0(\Omega)\,.$$

Thus the first conclusion is proved because λ is non–negative, the second one since obviously

$$\int_\Omega Df_\delta(\nabla u_\delta) \cdot \nabla\Big(\varphi\big(1 - h_\varepsilon \circ (u_\delta - \psi)\big)\Big)\, dx \;=\; 0\,. \qquad \square$$

P r o o f of Lemma 3.1. Again $\sup R_\delta \leq \inf J_\delta$ is evident and we claim

$$R_\delta(\sigma_\delta) \;\geq\; J_\delta(u_\delta) \;=\; \inf_{v \in \overset{\circ}{W}{}^1_{2,\psi}(\Omega)} J_\delta(v)\,, \tag{3.3}$$

implying Lemma 3.1. To verify (3.3) we recall the duality relation

$$f^*_\delta(\sigma_\delta) + f_\delta(\nabla u_\delta) \;=\; \sigma_\delta \cdot \nabla u_\delta\,.$$

Thus Lemma 3.2 gives

$$\begin{aligned} R_\delta(\sigma_\delta) \;&=\; \inf_{v \in \overset{\circ}{W}{}^1_{2,\psi}(\Omega)} \int_\Omega \big[\nabla v \cdot \sigma_\delta - f^*_\delta(\sigma_\delta)\big]\, dx \\ &=\; \inf_{v \in \overset{\circ}{W}{}^1_{2,\psi}(\Omega)} \int_\Omega \sigma_\delta \cdot (\nabla v - \nabla u_\delta)\, dx + \int_\Omega f_\delta(\nabla u_\delta)\, dx \;=\; \end{aligned}$$

$$= \inf_{v \in \overset{\circ}{W}{}^1_{2,\psi}(\Omega)} \int_\Omega (v - u_\delta) g_\delta \, dx + J_\delta(u_\delta) \ \geq \ J_\delta(u_\delta)$$

since for all $v \in \overset{\circ}{W}{}^1_{2,\psi}$ we have $\int_\Omega (v - u_\delta) g_\delta \, dx = \int_{[u_\delta = \psi]} (v - \psi) g_\delta \, dx \geq 0$. □

Now we want to show that $\{\sigma_\delta\}$ is a maximizing sequence which converges to σ.

We let $\tau_\delta = \frac{\partial f}{\partial P}(\nabla u_\delta)$, i. e. $\sigma_\delta = \delta \nabla u_\delta + \tau_\delta$, and observe the following a priori bounds

$$\|\tau_\delta\|_{L^\infty} \ \leq \ c_1 \,, \quad \|\sigma_\delta\|_{L^2} \ \leq \ c_2 \,, \quad \delta \int_\Omega |\nabla u_\delta|^2 \, dx \ \leq \ c_3 \,.$$

After passing to subsequences we may assume

$$\tau_\delta \ \overset{*}{\rightharpoondown}: \ \tau' \quad \text{in} \quad L^\infty(\Omega, \mathbb{R}^n) \,, \quad \sigma_\delta \ \rightharpoondown: \ \sigma' \quad \text{in} \quad L^2(\Omega, \mathbb{R}^n) \,,$$

and $\delta \nabla u_\delta \to 0$ in $L^2(\Omega, \mathbb{R}^n)$, in particular $\sigma' = \tau'$, and after passing to a further subsequence $\delta \nabla u_\delta \to 0$ a. e. on Ω. Next we claim

Lemma 3.3. *The functions σ_δ are uniformly bounded* (*with respect to δ*) *in the space $W^1_{2,loc}(\Omega, \mathbb{R}^n)$.*

Assuming this for the moment we may select another subsequence such that

$$\sigma_\delta(x) \ \longrightarrow \ \sigma'(x) \quad \text{for almost all} \quad x \in \Omega \,, \tag{3.4}$$

and this enables us to prove

Lemma 3.4. *σ' is R–maximizing and the whole sequence $\{\sigma_\delta\}$ converges to $\sigma' = \sigma$.*

Proof of Lemma 3.4. Suppose by contradiction that for some $\varepsilon > 0$

$$R(\sigma') \ \leq \ \sup_{\tau \in L^\infty(\Omega, \mathbb{R}^n)} R(\tau) - \varepsilon \ \leq \ \inf_{u \in \overset{\circ}{W}{}^1_{1,\psi}(\Omega)} J(u) - \varepsilon \,.$$

Then by the definition of J_δ

$$R(\sigma') \ \leq \ J_\delta(u_\delta) - \varepsilon \ = \ R_\delta(\sigma_\delta) - \varepsilon \,. \tag{3.5}$$

To study the right–hand side of (3.5) observe that for any fixed $w \in \overset{\circ}{W}{}^1_{2,\psi}(\Omega)$

$$\limsup_{\delta \downarrow 0} R_\delta(\sigma_\delta) \ \leq \ \int_\Omega \sigma' \cdot \nabla w \, dx + \limsup_{\delta \downarrow 0} \int_\Omega \big(-f^*_\delta(\sigma_\delta)\big) \, dx \,. \tag{3.6}$$

By definition we have for all $Z \in \mathbb{R}^n$

$$f^*_\delta(Z) \ = \ \sup_{Y \in \mathbb{R}^n} \left[Y \cdot Z - \frac{\delta}{2} |Y|^2 - f(Y) \right] \ \geq \ -f(0)$$

and Fatou's Lemma proves

$$\limsup_{\delta \downarrow 0} \int_\Omega -f^*_\delta(\sigma_\delta) \, dx \ = \ -\liminf_{\delta \downarrow 0} \int_\Omega f^*_\delta(\sigma_\delta) \, dx \ \leq \ \int_\Omega -\liminf_{\delta \downarrow 0} f^*_\delta(\sigma_\delta) \, dx \,.$$

Now (3.4) and

$$f_\delta^*\big(\sigma_\delta(x)\big) \;\geq\; Y\cdot\sigma_\delta(x) - \frac{\delta}{2}|Y|^2 - f(Y) \quad \text{for all} \quad Y\in \mathbb{R}^n$$

imply almost everywhere

$$\liminf_{\delta\downarrow 0} f_\delta^*(\sigma_\delta(x)) \;\geq\; Y\cdot\sigma'(x) - f(Y) \quad \text{for all} \quad Y\in \mathbb{R}^n$$

and the definition of f^* finally gives

$$\liminf_{\delta\downarrow 0} f_\delta^*(\sigma_\delta(x)) \;\geq\; f^*(\sigma'(x))\,.$$

Thus (3.6) proves for all $w\in \overset{\circ}{W}{}^1_{2,\psi}(\Omega)$

$$\limsup_{\delta\downarrow 0} R_\delta(\sigma_\delta) \;\leq\; \int_\Omega \big[\sigma'\cdot\nabla w - f^*(\sigma')\big]\,dx\,.$$

Given $w\in \overset{\circ}{W}{}^1_{1,\psi}(\Omega)$ we now approximate $W^1_1(\Omega)\ni v := w-\psi\geq 0$ with respect to the W^1_1–norm by a sequence $\{v_k\}\subset W^1_2(\Omega)$, $v_k\geq 0$. Setting $w_k := v_k+\psi\in \overset{\circ}{W}{}^1_{2,\psi}(\Omega)$ and observing $\sigma'\in L^\infty(\Omega,\mathbb{R}^n)$ we arrive at

$$\limsup_{\delta\downarrow 0} R_\delta(\sigma_\delta) \;\leq\; R(\sigma')\,.$$

So with (3.5) one obtains the contradiction

$$R(\sigma') \;\leq\; R(\sigma') - \varepsilon\,.$$

The uniqueness of σ' as maximizer of R yields the convergence of the whole sequence. □

Remark 3.5. The well–known inf–sup relation for R and J is a byproduct of the above considerations since the assumption $\sup R < \inf J$ also implies (3.5).

It remains to give the

Proof of Lemma 3.3. Fix $e\in\mathbb{R}^n$, $|e|=1$, and define for $\mathbb{R}\ni h\neq 0$

$$\Delta_h g(x) \;=\; \frac{1}{h}\big\{g(x+he)-g(x)\big\}\,.$$

By a direct calculation it is easy to check that for $\eta\in C_0^\infty(\Omega)$, $0\leq\eta\leq 1$,

$$v \;:=\; u_\delta + \varepsilon\Delta_{-h}\big(\eta^2\Delta_h[u_\delta-\psi]\big)\in \overset{\circ}{W}{}^1_{2,\psi}(\Omega)$$

provided ε is small enough (see also [HW]). This gives the variational inequality

$$\int_\Omega \sigma_\delta\cdot\nabla\big\{\Delta_{-h}\left(\eta^2\Delta_h[u_\delta-\psi]\right)\big\}\,dx \;\geq\; 0$$

and by a "partial integration"

$$\int_\Omega \Delta_h\sigma_\delta\cdot\nabla\bigl\{\eta^2\Delta_h[u_\delta-\psi]\bigr\}\,dx \;\le\; 0\,. \tag{3.7}$$

Since u_δ is of class $W^2_{2,loc}$,

$$\sigma_\delta \;=\; \delta\nabla u_\delta+\frac{\partial f}{\partial Q}(\nabla u_\delta)\in W^1_{2,loc}$$

and we may pass to the limit $h\to 0$ in (3.7). Setting $e=e_\gamma$, $\gamma=1,\dots,n$, and summing up from 1 to n with respect to γ this yields

$$\int_\Omega \partial_\gamma\sigma_\delta\cdot\nabla\bigl\{\eta^2\partial_\gamma[u_\delta-\psi]\bigr\}\,dx \;\le\; 0\,,$$

implying (observe the definition of σ_δ)

$$\begin{aligned}
&\int_\Omega \eta^2D^2f_\delta(\nabla u_\delta)\bigl(\nabla\partial_\gamma u_\delta,\nabla\partial_\gamma u_\delta\bigr)\,dx\\
\le\;&\int_\Omega \eta^2D^2f_\delta(\nabla u_\delta)\bigl(\nabla\partial_\gamma u_\delta,\nabla\partial_\gamma\psi\bigr)\,dx\\
&+\int_\Omega D^2f_\delta(\nabla u_\delta)\bigl(\nabla\partial_\gamma u_\delta,2\eta\nabla\eta[\partial_\gamma\psi-\partial_\gamma u_\delta]\bigr)\,dx\\
\le\;&\int_\Omega\bigl\{D^2f_\delta(\nabla u_\delta)\bigl(\eta\nabla\partial_\gamma u_\delta,\eta\nabla\partial_\gamma u_\delta\bigr)\bigr\}^{\frac12}\bigl\{D^2f_\delta(\nabla u_\delta)\bigl(\eta\nabla\partial_\gamma\psi,\eta\nabla\partial_\gamma\psi\bigr)\bigr\}^{\frac12}\,dx\\
&+\int_\Omega\bigl\{D^2f_\delta(\nabla u_\delta)\bigl(\eta\nabla\partial_\gamma u_\delta,\eta\nabla\partial_\gamma u_\delta\bigr)\bigr\}^{1/2}\\
&\quad\times\bigl\{D^2f_\delta(\nabla u_\delta)\bigl(2\nabla\eta[\partial_\gamma\psi-\partial_\gamma u_\delta],2\nabla\eta[\partial_\gamma\psi-\partial_\gamma u_\delta]\bigr)\bigr\}^{1/2}\,dx\,.
\end{aligned}$$

Using Young's inequality we see

$$\begin{aligned}
&\int_\Omega \eta^2D^2f_\delta(\nabla u_\delta)\left(\nabla\partial_\gamma u_\delta,\nabla\partial_\gamma u_\delta\right)\,dx\\
\le\;& c(\eta)\int_\Omega \bigl\|D^2f_\delta(\nabla u_\delta)\bigr\|\bigl[|\nabla^2\psi|^2+|\nabla\psi|^2+|\nabla u_\delta|^2\bigr]\,dx
\end{aligned}$$

and our Assumption 2.1 (ii) proves

$$\begin{aligned}
&\int_\Omega \eta^2D^2f_\delta(\nabla u_\delta)(\nabla\partial_\gamma u_\delta,\nabla\partial_\gamma u_\delta)\,dx\\
\le\;& c_1(\eta)\int_\Omega\left[\delta+\frac{\mu_2}{\sqrt{1+|\nabla u_\delta|^2}}\right]\bigl[|\nabla^2\psi|^2+|\nabla\psi|^2+|\nabla u_\delta|^2\bigr]\,dx\\
\le\;& c_2(\eta,\psi)+c_3(\eta)\int_\Omega\left[\delta\,|\nabla u_\delta|^2+\frac{|\nabla u_\delta|^2}{\sqrt{1+|\nabla u_\delta|^2}}\right]\,dx\,,
\end{aligned}$$

where the right–hand side is uniformly bounded with respect to δ. Summarizing the results,

$$\int_{\Omega'} \partial_\gamma \sigma_\delta \cdot \nabla \partial_\gamma u_\delta \, dx \;=\; \int_\Omega D^2 f_\delta(\nabla u_\delta)(\nabla \partial_\gamma u_\delta, \nabla \partial_\gamma u_\delta)\, dx \;\le\; c(\Omega') \;<\; \infty \quad \text{for all} \quad \Omega' \subset\subset \Omega$$

is proved and the lemma follows immediately on account of

$$|\nabla \sigma_\delta|^2 \;\le\; c\, \partial_\gamma \sigma_\delta \cdot \nabla \partial_\gamma u_\delta$$

which again is a consequence of Young's inequality. □

4. Proof of part (ii) of Theorem 2.2

Let $\mathcal{M}$ denote the set of all L^1–limits of J–minimizing sequences from $\overset{\circ}{W}{}^1_{1,\psi}(\Omega)$. First of all we note that any J–minimizing sequence $\{u_m\} \subset \overset{\circ}{W}{}^1_{1,\psi}(\Omega)$ is bounded in $BV(\Omega)$ so that at least for a subsequence we have $u_m \to: u$ in $L^1(\Omega)$, i.e. $u \in \mathcal{M}$. From now on we fix some u^* in $\mathcal{M}$ and a J–minimizing sequence $\{u_m\}$ generating u^*. Let

$$\hat{J}(w, \hat{\Omega}) \;:=\; \inf \left\{ \liminf_{k\to\infty} J(w_k) :\; w_k \in C^1(\hat{\Omega}),\, w_k \to w \;\text{ in }\; L^1_{loc}(\hat{\Omega}) \right\} .$$

We state the properties of $\hat{J}$ which are needed in the following:

Lemma 4.1. (i) *$\hat{J}$ is lower semicontinuous with respect to $L^1_{loc}(\hat{\Omega})$–convergence.*
(ii) *On $W^1_1(\hat{\Omega})$, $\hat{J}$ and $J\big|_{\hat{\Omega}}$ coincide.*
(iii) $\hat{J}(u^*, \Omega) \le \inf\Big\{ J(u) :\; u \in u_0 + \overset{\circ}{W}{}^1_{1,\psi}(\Omega, \mathbb{R}^N) \Big\}$.

This lemma is easily proved (see [BF], Proposition 5.3) if we observe the lower semicontinuity of J on $W^1_1(\hat{\Omega})$ with respect to the L^1_{loc}–topology (see [AD]). A deeper result on $\hat{J}$ is the following representation formula of Goffman and Serrin (see [GS]):

Lemma 4.2. *The representation formula*

$$\hat{J}(u, \hat{\Omega}) \;=\; \int_{\hat{\Omega}} f(\nabla^a u)\, dx + \int_{\hat{\Omega}} f_\infty\left(\frac{\nabla^s u}{|\nabla^s u|} \right) d\, |\nabla^s u|$$

is true for all $u \in BV(\hat{\Omega})$, where f_∞ is the recession function of f defined by

$$f_\infty(X) \;=\; \limsup_{t\to+\infty} \frac{f(tX)}{t} \, .$$

The absolutely continuous part of Du with respect to the Lebesgue measure is here denoted by $\nabla^a u$, the singular part by $\nabla^s u$ and $\nabla^s u / |\nabla^s u|$ is the Radon–Nikodym derivative.

The third lemma follows from [AG], Theorem 2.1 respectively Proposition 2.2, p. 247 (see also [GMS] and [Re]).

Lemma 4.3. *Consider* $u \in BV(\hat\Omega)$ *and a sequence* $\{u_m\} \subset W^1_1(\hat\Omega)$ *such that:*

(i) $u_m \to u$ *in* $L^1(\hat\Omega)$ *as* $m \to \infty$.

(ii) $\int_{\hat\Omega} \sqrt{1+|\nabla u_m|^2}\,dx \to \int_{\hat\Omega} \sqrt{1+|\nabla u|^2}$ *as* $m \to \infty$.

(iii) $\nabla u_m \rightharpoondown \nabla u$ *in the sense of measures as* $m \to \infty$.

Then $\hat J$ *is continuous with respect to this kind of convergence, i. e.*

$$\hat J(u_m, \hat\Omega) \;\longrightarrow\; \hat J(u, \hat\Omega) \quad as \quad m \to \infty\,.$$

Next we show (as an alternative one can study a relaxed problem in the space $BV_{u_0,\psi}(\Omega)$, see [GMS]):

Lemma 4.4. *The function* u^* *is a local* $\hat J$*–minimizer on any ball* $B_R(x) \subset\subset \Omega$ *with respect to all* BV*–functions* v *satisfying* $v \geq \psi$ *almost everywhere on* $B_R(x)$.

Proof. The proof follows the one given in [BF] (see Section 5 and Section 7), so we only sketch the main arguments. Given the minimizing sequence $\{u_m\}$, we claim that one can fix traces in the following way: for $x_0 \in \Omega$ choose $B_R(x_0)$, $B_{2R}(x_0) \subset\subset \Omega$, and a sequence $\{w_m\}_{m\in\mathbb{N}} \subset \overset{\circ}{W}{}^1_{1,\psi}(\Omega)$ such that:

(i) $w_m \to u^*$ in $L^1(\Omega)$ as $m \to \infty$,

(ii) $\lim_{m\to\infty} J(w_m) \leq \lim_{m\to\infty} J(u_m)$,

(iii) $w_m\big|_{\partial B_R(x_0)} = u^*\big|_{\partial B_R(x_0)}$, where the traces are well defined functions of class $L^1\big(\partial B_R(x_0)\big)$.

To prove this claim we follow exactly Section 7 of [BF], where we have to take care of the obstacle: here, the main new feature is to extend the standard approximation procedure for BV functions in such a way that the obstacle is respected. This approximation argument is outlined in Appendix A and the above claim is seen to be true. Now define

$$\text{(a)} \qquad I \;:\; W^1_1\big(B_R(x_0)\big) \;\longrightarrow\; \mathbb{R}\,, \quad I(w) \;:=\; \int_{B_R(x_0)} f(\nabla w)\,dx\,,$$

$$\text{(b)} \qquad \mathbf{K}_\psi \;:=\; \Big\{ w \in W^1_{1,\psi}\big(B_R(x_0)\big) : w\big|_{\partial B_R(x_0)} = u^*\big|_{\partial B_R(x_0)} \Big\}\,,$$

where $W^1_{1,\psi}\big(B_R(x_0)\big) := \big\{ w \in W^1_1\big(B_R(x_0)\big) \,:\, w \geq \psi \text{ almost everywhere}\big\}$, and let $v_m = w_m\big|_{B_R(x_0)}$. Then

$$\inf_{\mathbf{K}_\psi} I \;=\; \liminf_{m\to\infty} I(v_m) \tag{4.1}$$

follows immediately as in [BF].

Consider now $\varphi \in BV\big(B_R(x_0)\big)$, $\varphi \geq \psi$ almost everywhere, such that $\operatorname{spt}(\varphi - u^*) \subset\subset B_R(x_0)$. Then we choose (with a slight modification of Appendix A) a sequence $\varphi_m \in \mathbf{K}_\psi$ satisfying $\varphi_m \to \varphi$ in $L^1\big(B_R(x_0)\big)$ and $\int_{B_R(x_0)} \sqrt{1+|\nabla\varphi_m|^2}\,dx \to \int_{B_R(x_0)} \sqrt{1+|\nabla\varphi|^2}$, hence by Lemma 4.1 and Lemma 4.3

$$\hat{J}\big(\varphi, B_R(x_0)\big) \;=\; \lim_{m\to\infty} \hat{J}\big(\varphi_m, B_R(x_0)\big) \;=\; \lim_{m\to\infty} I(\varphi_m) \;\geq\; \inf_{\mathbf{K}_\psi} I \;\overset{(4.1)}{=}\; \liminf_{m\to\infty} I(v_m)$$
$$\geq\; \hat{J}\big(u^*, B_R(x_0)\big)\,,$$

where the last inequality follows from $v_m \to u^*$ in $L^1(B_R)$ (see (i)) and the definition of $\hat{J}(\,\cdot\,, B_R)$. Thus the Lemma is proved. □

Let us now consider the sequence $\{u_\delta\}$ introduced at the beginning of Section 3. By construction $\{u_\delta\}$ is a J–minimizing sequence, and after passing to a subsequence we have convergence to a function $u^* \in \mathcal{M}$. Let us fix a ball $B_R(x_0)$ compactly contained in Ω. Then we may apply Lemma B.1 to see that u^* is of class $C^{1,\alpha}\big(B_R(x_0)\big)$, hence the singular part of ∇u^* vanishes. Recalling Lemma 4.4 and using the representation formula from Lemma 4.2, we can derive the Euler–Lagrange equation exactly along the lines of Lemma 3.2 with the result

$$\int_{B_R(x_0)} \frac{\partial f}{\partial P}(\nabla u^*)\cdot\nabla\varphi\,dx \;=\; \int_{B_R(x_0)} g\,\varphi\,dx \quad \text{for all} \quad \varphi \in C_0^\infty\big(B_R(x_0)\big)\,,$$

where $g \in L^\infty\big(B_R(x_0)\big)$ is defined as in Lemma 3.2 (i) with u^* and f in place of u_δ and f_δ. Recalling

$$\int_{B_R(x_0)} \sigma_\delta\cdot\nabla\varphi\,dx \;=\; \int_{B_R(x_0)} g_\delta\,\varphi\,dx \quad \text{for all} \quad \varphi \in C_0^\infty\big(B_R(x_0)\big)$$

with $\|g_\delta\|_{L^\infty}$ bounded independent of δ, we deduce

$$\begin{aligned} &\int_{B_R(x_0)} \left(\sigma_\delta - \frac{\partial f}{\partial P}(\nabla u^*)\right)\cdot\nabla\left(\eta^2[u_\delta-u^*]\right)\,dx \\ &= \int_{B_R(x_0)} (g_\delta-g)\eta^2(u_\delta-u^*)\,dx\,, \end{aligned} \tag{4.2}$$

where $\eta \in C_0^\infty\big(B_R(x_0)\big)$, $0\leq\eta\leq 1$, is fixed. As shown in [BF], proof of Theorem 6.1, equation (4.2) yields $\nabla u_\delta \to \nabla u^*$ almost everywhere provided

$$\int_{B_R(x_0)} (g_\delta-g)\eta^2(u_\delta-u^*)\,dx \;\longrightarrow\; 0 \tag{4.3}$$

as $\delta\downarrow 0$. But (4.3) follows from $u_\delta\to u^*$ in $L^1(\Omega)$ combined with the L^∞–bounds for g_δ and g. The pointwise convergence $\nabla u_\delta \to \nabla u^*$ finally implies $\sigma = \frac{\partial f}{\partial Q}(\nabla u^*)$ on $B_R(x_0)$ (compare again [BF], proof of Theorem 6.1) and since $B_R(x_0)$ was arbitrary $\sigma \in C^{0,\alpha}(\Omega,\mathbb{R}^n)$ is established. □

Appendix A. Approximation of BV–functions in the presence of an obstacle

We now prove that the standard approximation of a function with bounded variation can be modified such that the obstacle is respected.

Lemma A.1. *Consider a ball B_R and a function $u \in BV(B_R)$ satisfying $u \geq \psi$ almost everywhere. Then there exists a sequence $\{u_j\} \subset C^\infty(B_R)$ with the following properties:*

(i) $\lim_{j\to\infty} \int_{B_R} |u - u_j|\,dx \;=\; 0\,,$

(ii) $\lim_{j\to\infty} \int_{B_R} |Du_j|\,dx \;=\; \int_{B_R} |Du|\,,$

(iii) $u_j\big|_{\partial B_R} \;= u\big|_{\partial B_R}\,,$

(iv) $u_j(x) - \psi(x) \;>\; 0$ *on* B_R.

Proof. Consider $u \in BV(B_R)$ such that $u \geq \psi$ almost everywhere. Fix a function $g \in C^\infty(B_R)$ satisfying $g > 0$ on B_R and $g\big|_{\partial B_R} \equiv 0$, for example we may take a smooth function approximating $\tilde g(x) = R - |x|$. Then for any fixed $\varepsilon_0 > 0$ and for $\delta \in \mathbb{R}^+$ small enough, $w_\delta := u + \delta g$ satisfies

$$\text{(A.1)} \qquad \int_{B_R} |u - w_\delta|\,dx \;<\; \varepsilon_0 \quad \text{and} \quad \left| \int_{B_R} |Dw_\delta| - \int_{B_R} |Du| \right| \;<\; \varepsilon_0\,.$$

Furthermore we have by construction

$$\text{(A.2)} \qquad w_\delta \;>\; \psi \;\text{ on }\; B_R \quad \text{and} \quad w_\delta\big|_{\partial B_R} \;=\; u\big|_{\partial B_R}\,.$$

Fixing some δ as above we write $w := w_\delta$ and now approximate this function following for example [Gi], Theorem 1.17, p. 14: for any fixed $0 < \varepsilon < \varepsilon_0$ there exists a number $m \in \mathbb{N}$ such that

$$\int_{B_R \sim B^0} |Dw| \;<\; \varepsilon\,.$$

Here we have abbreviated

$$B^k \;:=\; \left\{ x \in B_R : \operatorname{dist}(x, \partial B_R) > \frac{1}{m+k} \right\} \quad \text{for} \quad k \;=\; 0\,,1\,,2\,,\ldots$$

and $B^{-j} := \emptyset$ for all $j \in \mathbb{N}$. The sets A_i, $i \in \mathbb{N}$, are defined by induction:

$$A_1 \;:=\; B^2\,, \quad A_i \;:=\; B^{i+1} \sim \overline{B^{i-1}}\,, \quad i \;=\; 2\,,3\,,\ldots\,.$$

As usual consider a partition of the unity $\{\varphi_i\}$ subordinate to the covering $\{A_i\}$, i.e.

$$\varphi_i \in C_0^\infty(A_i)\,, \quad 0 \;\leq\; \varphi_i \leq 1\,, \quad \sum_{i=1}^\infty \varphi_i \;=\; 1 \;\text{ on }\; B_R\,.$$

Let η denote a positive symmetric mollifier and define $\varepsilon_i > 0$ according to

(a) $\operatorname{spt} \eta_{\varepsilon_i} * (w\varphi_i) \subset B^{\,i+2} \sim \overline{B}^{\,i-2}$,

(b) $\int_{B_R} |\eta_{\varepsilon_i} * (w\varphi_i) - w\varphi_i| \, dx < \varepsilon 2^{-i}$,

(c) $\int_{B_R} |\eta_{\varepsilon_i} * (wD\varphi_i) - wD\varphi_i| \, dx < \varepsilon 2^{-i}$.

Now introduce the function

$$v_\varepsilon \; := \; \sum_{i=1}^{\infty} \eta_{\varepsilon_i} * (w\varphi_i) \, .$$

Then we have to prove that v_ε respects the obstacle. To do this, observe that

$$w\big|_{B^{i+4} - \overline{B^{i-4}}} \; > \; \psi + \rho_i$$

for a real number $\rho_i > 0$. Setting $\varphi_0 \equiv \varphi_{-1} \equiv 0$, for all $i_0 \in \mathbb{N}$ and for all $x \in A_{i_0}$

$$\text{(A.3)} \qquad v_\varepsilon(x) \; = \; \sum_{i=i_0-2}^{i_0+2} \eta_{\varepsilon_i} * (w\varphi_i)(x) \; > \; \sum_{i=i_0-2}^{i_0+2} \eta_{\varepsilon_i} * ((\psi + \rho_{i_0})\varphi_i)(x)$$

follows. In addition to (a) – (c) now choose ε_i small enough such that for $i_0 - 2 \le i \le i_0 + 2$ and for all $x \in A_{i_0}$

$$|\eta_{\varepsilon_i} * (\psi\varphi_i) - \psi\varphi_i| \; \ll \; \rho_{i_0} \quad \text{and} \quad |\eta_{\varepsilon_i} * \varphi_i - \varphi_i| \; \ll \; \rho_{i_0} \, .$$

Then (A.3) implies for $x \in A_{i_0}$

$$v_\varepsilon(x) \; > \; \sum_{i=i_0-2}^{i_0+2} \Big[\psi(x) + \frac{\rho_{i_0}}{2}\Big]\varphi_i(x) \; = \; \psi(x) + \frac{\rho_{i_0}}{2}$$

and since this is true for all $i_0 \in \mathbb{N}$, we have proved

$$\text{(A.4)} \qquad v_\varepsilon(x) \; > \; \psi(x) \quad \text{for all} \quad x \in B_R \, .$$

If $\varepsilon \to 0$, then convergence in L^1 and also

$$\text{(A.5)} \qquad \int_{B_R} |v_\varepsilon - w| \, dx \; \le \; \sum_{i=1}^{\infty} \int_{B_R} |\eta_{\varepsilon_i} * (w\varphi_i) - w\varphi_i| \, dx \; < \; \varepsilon_0$$

follows from (b) and $w = \sum_{i=1}^{\infty} w\varphi_i$. Semicontinuity, that is

$$\int_{B_R} |Dw| \; \le \; \liminf_{\varepsilon \to 0} \int_{B_R} |Dv_\varepsilon| \, dx \, ,$$

(see [Gi], Theorem 1.9, p. 7) proves at least for some small enough $\varepsilon > 0$

$$\text{(A.6)} \qquad \int_{B_R} |Dw| \; \le \; \int_{B_R} |Dv_\varepsilon| \, dx + \varepsilon_0 \, .$$

To show the reverse inequality we fix a function $\tau \in C^1_0(B_R)$, $|\tau| \le 1$ and prove exactly as in [Gi], p. 15,

$$\text{(A.7)} \qquad \int_{B_R} v_\varepsilon \, \mathrm{div}\tau \, dx \; \le \; \int_{B_R} |Dw| + \varepsilon_0 \, .$$

Observe that τ is compactly supported and only finite sums are to be considered, i.e. we can interchange summation and integration. If we choose δ and ε such that (A.1) and (A.5) – (A.7) are fulfilled for $\varepsilon_0 = 1/j$, $j \in \mathbb{N}$, then the sequence $\{u_j\}$, $u_j = v_\varepsilon$, is by (A.2) and (A.4) seen to satisfy all the conclusions of the lemma. $\square$

Appendix B. A regularity result for BV–obstacle problems

Let $\{u_\delta\}$ denote the sequence introduced in Section 3 and consider a weak cluster point u^* of $\{u_\delta\}$, i.e. the weak limit for a suitable sequence $\delta \downarrow 0$. We claim

Lemma B.1. *u^* is of class $C^{1,\alpha}(\Omega)$ for any $0 < \alpha < 1$.*

This regularity result is a consequence of

Lemma B.2. *The first weak derivatives of u^* are locally bounded functions, i.e. u^* belongs to the space $W^1_{\infty,loc}(\Omega)$.*

Conjecture B.3. *This statement is true for any generalized minimizer $v \in \mathcal{M}$.*[2)]

Given Lemma B.2, the proof of Lemma B.1 in fact is standard: since the singular part of ∇u^* vanishes, Lemma 4.2 and Lemma 4.4 together imply that u^* locally minimizes

$$w \;\longmapsto\; \int_{B_R} f(\nabla w)\,dx$$

on any ball $B_R \subset\subset \Omega$ subject to the constraint $w \geq \psi$ almost everywhere on B_R. We fix a ball B_R and insert the admissible comparison function

$$v \;=\; u^* + \varepsilon\Delta_{-h}\big(\eta^2\Delta_h[u^* - \psi]\big)$$

for $\eta \in C_0^\infty(B_R)$, $0 \leq \eta \leq 1$, where $\Delta_{\pm h}$ denotes the difference quotient in a given coordinate direction e and ε is sufficiently small. Then

$$\int_{B_R}\left\{\int_0^1 D^2 f\big(\nabla u^*(x) + th\Delta_h\nabla u^*(x+he)\big)\,dt\right\} \\ \times \big(\Delta_h\nabla u^*(x), \nabla\big[\eta^2(x)\Delta_h(u^* - \psi)(x)\big]\big)\,dx \;\leq\; 0\,,$$

and due to Lemma B.2 the bilinear form $\{\ldots\}$ is uniformly elliptic. Thus the above inequality implies $u^* \in W^2_{2,loc}(B_R)$, i.e. $u^* \in W^2_{2,loc}(\Omega)$. On the other hand, we may repeat the proof of Lemma 3.2 to get

$$\int_\Omega Df(\nabla u^*)\cdot\nabla\varphi\,dx \;=\; \int_\Omega g\varphi\,dx$$

valid for all $\varphi \in C_0^1(\Omega)$ with g defined according to Lemma 3.2 (i) where of course now the index δ has to be dropped. Let $v = \partial_s u^*$ denote any weak derivative. Then

$$\int_\Omega D^2 f(\nabla u^*)\big(\nabla v, \nabla\varphi\big)\,dx \;=\; -\int_\Omega g\partial_s\varphi\,dx$$

for all $\varphi \in C_0^1(\Omega)$. The coefficients $\frac{\partial^2 f}{\partial p_\alpha \partial p_\beta}(\nabla u^*)$ of this equation are uniformly elliptic on any subdomain, Hölder continuity of v can be deduced from [GT], Theorem 8.22. □

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2) In the mean time we obtained a positive answer to conjecture B3 since it can be shown that generalized minimizers are unique up to a constant.

It remains to prove Lemma B.2. Actually Lemma B.2 corresponds to Proposition 3.6 of [GMS] but since the proof given by Giaquinta – Modica – Souček is quite condensed and since due to the presence of the obstacle our setting is slightly different, we prefer to present the necessary steps in somewhat more detail. We also would like to point out that similar gradient bounds occur in the papers of Gerhardt [GE] and Ladyzhenskaya – Ural'tseva [LU]. Following [GMS] and also [LU] we introduce the following notation:

$$\begin{aligned} \omega_\delta &:= \ln\big(1+|\nabla u_\delta|^2\big)\,, & \nu_\delta(x) &:= \frac{1}{\sqrt{1+|\nabla u_\delta|^2}}\big(-\nabla u_\delta(x),1\big)\,,\\ A_{\delta,k} &:= \big\{x\in\Omega : \omega_\delta(x)\geq k\big\}\,, & S_{\delta,k} &:= \big\{\big(x,u_\delta(x)\big) : x\in A_{\delta,k}\big\}\,. \end{aligned}$$

For $\xi\in\mathbb{R}^n$ we let $\hat{\xi}:=(\xi,0)\in\mathbb{R}^{n+1}$ and

$$\xi' \;:=\; \hat{\xi}-\big(\hat{\xi}\cdot\nu_\delta\big)\nu_\delta$$

$\big($more precisely we should write $\xi'=\xi'_\delta\big)$, then our ellipticity condition implies

$$\frac{\mu_1}{\sqrt{1+|\nabla u_\delta|^2}}\,|\xi'|^2 \;\leq\; D^2 f(\nabla u_\delta)\big(\xi,\xi\big) \;\leq\; \frac{\mu_2}{\sqrt{1+|\nabla u_\delta|^2}}\,|\xi'|^2\,.$$

If $w:\Omega\to\mathbb{R}$ denotes a function for which the next expression makes sense, we let

$$\mathcal{D}w \;:=\; \nabla\hat{w}-(\nabla\hat{w}\cdot\nu_\delta)\nu_\delta\,.$$

Proposition B.4. (Compare [GMS], inequality (3.12).) *There exist positive constants c, C and $k_0=k_0(\psi)$ independent of δ such that for all $\eta\in C^1_0(\Omega)$, $0\leq\eta\leq 1$, we have*

$$\begin{aligned} &c\int_{S_{\delta,k}}|\mathcal{D}\omega_\delta|^2\eta^2\,d\mathcal{H}^n+\delta\int_{A_{\delta,k}}\big(1+|\nabla u_\delta|^2\big)|\nabla\omega_\delta|^2\eta^2\,dx\\ \leq\;& C\int_{S_{\delta,k}}(\omega_\delta-k)^2\,|\mathcal{D}\eta|^2\,d\mathcal{H}^n+\delta\int_{A_{\delta,k}}\big(1+|\nabla u_\delta|^2\big)(\omega_\delta-k)^2\,|\nabla\eta|^2\,dx \end{aligned}$$

being valid for all $k\geq k_0(\psi)$.

Remark B.5. For functions $w:\Omega\to\mathbb{R}$ we have by definition

$$\begin{aligned} \int_{S_{\delta,k}} w\,d\mathcal{H}^n &= \int_{S_{\delta,k}} w(x_1,\dots,x_n)\,d\mathcal{H}^n(x_1,\dots,x_{n+1})\\ &= \int_{A_{\delta,k}} w\sqrt{1+|\nabla u_\delta|^2}\,dx\,. \end{aligned}$$

Proof of Proposition B.4. We recall the Euler equation from Lemma 3.2

$$\int_\Omega Df_\delta(\nabla u_\delta)\cdot\nabla\varphi\,dx \;=\; \int_\Omega g_\delta\varphi\,dx$$

with right–hand side g_δ supported on the coincidence set $\big[u_\delta=\psi\big]$. Fix a coordinate direction $s=1,\dots,n$ and let $\Delta_{\pm h}$ denote the corresponding difference quotients. For

$k > 0$ and η as above we let $\varphi := \Delta_{-h}\big(\Delta_h u_\delta \max\{\omega_\delta - k, 0\}\eta^2\big)$ and "perform an integration by parts" in order to get after passing to the limit $h \to 0$ (from now on summation with respect to s)

$$\begin{aligned}
&\int_{A_{\delta,k}} D^2 f_\delta(\nabla u_\delta)\big(\partial_s \nabla u_\delta, \nabla\big(\partial_s u_\delta(\omega_\delta - k)\eta^2\big)\big)\, dx \\
&= -\int_{A_{\delta,k}} g_\delta \partial_s\big(\partial_s u_\delta \eta^2 \max\{\omega_\delta - k, 0\}\big)\, dx\,.
\end{aligned} \tag{B.1}$$

We have (compare [GT], Lemma 7.7)

$$g_\delta \max\{\omega_\delta - k, 0\} \;=\; g_\delta \max\big\{\ln\big(1 + |\nabla\psi|^2\big) - k, 0\big\}\,,$$

hence the right–hand side of (B.1) vanishes for $k \geq k_0(\psi)$. On the left–hand side we observe

$$\int_{A_{\delta,k}} D^2 f_\delta\big(\partial_s \nabla u_\delta, \partial_s \nabla u_\delta\big)(\omega_\delta - k)\eta^2\, dx \;\geq\; 0\,,$$

and according to $f_\delta = f + \frac{\delta}{2}|\cdot|^2$ we get four additional terms which are handled as follows:

$$\begin{aligned}
&\int_{A_{\delta,k}} D^2 f(\nabla u_\delta)\big(\partial_s \nabla u_\delta, \partial_s u_\delta \nabla\omega_\delta \eta^2\big)\, dx \\
=\;& \frac{1}{2}\int_{A_{\delta,k}} D^2 f(\nabla u_\delta)\big(\nabla|u_\delta|^2, \nabla\omega_\delta\eta^2\big)\, dx \\
=\;& \frac{1}{2}\int_{A_{\delta,k}} D^2 f(\nabla u_\delta)\big(\nabla\omega_\delta, \nabla\omega_\delta\big)(1 + |\nabla u_\delta|^2)\eta^2\, dx \\
\geq\;& \frac{\mu_1}{2}\int_{A_{\delta,k}} |\mathcal{D}\omega_\delta|^2 \sqrt{1 + |\nabla u_\delta|^2}\,\eta^2\, dx \\
=\;& \frac{\mu_1}{2}\int_{S_{\delta,k}} |\mathcal{D}\omega_\delta|^2\,\eta^2\, d\mathcal{H}^n\,,
\end{aligned}$$

$$\delta\int_{A_{\delta,k}} \partial_s \nabla u_\delta \cdot \partial_s u_\delta \nabla\omega_\delta\eta^2\, dx \;=\; \frac{\delta}{2}\int_{A_{\delta,k}} |\nabla\omega_\delta|^2\big(1 + |\nabla u_\delta|^2\big)\eta^2\, dx\,,$$

the third term is estimated from above:

$$\begin{aligned}
&\left|\int_{A_{\delta,k}} D^2 f(\nabla u_\delta)\big(\partial_s \nabla u_\delta, \partial_s u_\delta(\omega_\delta - k)\nabla\eta^2\big)\, dx\right| \\
\leq\;& \int_{A_{\delta,k}} \big(D^2 f(\nabla u_\delta)\big(\nabla\,|\nabla u_\delta|^2, \nabla\,|\nabla u_\delta|^2\big)\big)^{\frac{1}{2}} \big(D^2 f(\nabla u_\delta)\big(\nabla\eta, \nabla\eta\big)\big)^{\frac{1}{2}}\,\eta(\omega_\delta - k)\, dx \\
\leq\;& \int_{A_{\delta,k}} \left(\frac{\mu_2}{\sqrt{1 + |\nabla u_\delta|^2}}\,\big|\mathcal{D}\,|\nabla u_\delta|^2\big|^2\right)^{\frac{1}{2}} \left(\frac{\mu_2}{\sqrt{1 + |\nabla u_\delta|^2}}\,|\mathcal{D}\eta|^2\right)^{\frac{1}{2}} \eta(\omega_\delta - k)\, dx \\
=\;& \int_{A_{\delta,k}} \mu_2\,|\mathcal{D}\omega_\delta|\,|\mathcal{D}\eta|\,\eta(\omega_\delta - k)\sqrt{1 + |\nabla u_\delta|^2}\, dx \\
=\;& \int_{S_{\delta,k}} \mu_2\,|\mathcal{D}\omega_\delta|\,|\mathcal{D}\eta|\,\eta(\omega_\delta - k)\, d\mathcal{H}^n\,,
\end{aligned}$$

and the last one satisfies

$$\left|\delta\int_{A_{\delta,k}}\partial_s\nabla u_\delta\cdot\nabla\eta^2\partial_s u_\delta(\omega_\delta-k)\,dx\right| \;\le\; \delta\int_{A_{\delta,k}}\big(1+|\nabla u_\delta|^2\big)\,|\nabla\omega_\delta|\,|\nabla\eta|\,\eta(\omega_\delta-k)\,dx\,.$$

Now, if we let $k > k_0(\psi)$ and if we apply Young's inequality, the claim of Proposition B.4 follows from Equation (B.1) together with the above estimates. □

Proposition B.6. (Compare [GMS], inequality (3.13).) *There are constants C and $k_0 = k_0(\psi)$ not depending on δ and k such that for all $k > k_0$ and $\eta \in C_0^1(\Omega)$, $0 \le \eta \le 1$, we have*

$$\begin{aligned}
&\delta\int_{A_{\delta,k}}(\omega_\delta-k)^2\left|\nabla^2 u_\delta\right|^2\eta^2\,dx\\
&\le\; C\left\{\int_{S_{\delta,k}}(\omega_\delta-k)^2\,|\mathcal{D}\eta|^2\,d\mathcal{H}^n+\delta\int_{A_{\delta,k}}\big(1+|\nabla u_\delta|^2\big)\,|\nabla\eta|^2(\omega_\delta-k)^2\,dx\right\}.
\end{aligned}$$

Proof. Replacing the derivative ∂_s by the difference quotients Δ_{-h} and Δ_h we let $\varphi = \partial_s\big(\partial_s u_\delta \max\{\omega_\delta-k,0\}^2\eta^2\big)$ and deduce from the Euler equation for u_δ at least for $k \ge k_0(\psi)$

$$\begin{aligned}
0 \;&=\; \int_{A_{\delta,k}} D^2 f_\delta(\nabla u_\delta)\Big(\partial_s\nabla u_\delta,\nabla\big[\partial_s u_\delta(\omega_\delta-k)^2\eta^2\big]\Big)\,dx\\
&=\; \int_{A_{\delta,k}} D^2 f_\delta(\nabla u_\delta)\big(\partial_s u_\delta\partial_s\nabla u_\delta,2(\omega_\delta-k)\nabla\omega_\delta\eta^2\big)\,dx\\
&\quad+\int_{A_{\delta,k}} D^2 f_\delta(\nabla u_\delta)\big(\partial_s\nabla u_\delta,\partial_s\nabla u_\delta\big)(\omega_\delta-k)^2\eta^2\,dx\\
&\quad+\int_{A_{\delta,k}} D^2 f_\delta(\nabla u_\delta)\big(\partial_s u_\delta\partial_s\nabla u_\delta,\nabla\eta^2\big)(\omega_\delta-k)^2\,dx \;\;=:\;\; T_0+T_1+T_2\,.
\end{aligned}$$

Since $\partial_s u_\delta\partial_s\nabla u_\delta = \frac{1}{2}\nabla|\nabla u_\delta|^2$ and $\nabla\omega_\delta = \big(1+|\nabla u_\delta|^2\big)^{-1}\nabla|\nabla u_\delta|^2$, we get $T_0 \ge 0$. The ellipticity of $D^2 f$ implies

$$T_1 \;\ge\; \mu_1\int_{A_{\delta,k}}\frac{1}{\sqrt{1+|\nabla u_\delta|^2}}\,|\mathcal{D}\partial_s u_\delta|^2(\omega_\delta-k)^2\eta^2\,dx+\delta\int_{A_{\delta,k}}\left|\nabla^2 u_\delta\right|^2(\omega_\delta-k)^2\eta^2\,dx\,.$$

For T_2 we use the upper bound imposed on $D^2 f$ together with the Cauchy–Schwarz inequality with the result

$$\begin{aligned}
|T_2| &\le \mu_2\int_{A_{\delta,k}}\frac{|\mathcal{D}\eta|\,\eta(\omega_\delta-k)^2}{\sqrt{1+|\nabla u_\delta|^2}}\left|\mathcal{D}\,|\nabla u_\delta|^2\right|\,dx+\delta\int_{A_{\delta,k}}\left|\nabla\,|\nabla u_\delta|^2\right|\,|\nabla\eta|\,\eta(\omega_\delta-k)^2\,dx\\
&\le \varepsilon\mu_2\int_{A_{\delta,k}}\sqrt{1+|\nabla u_\delta|^2}\,|\mathcal{D}\omega_\delta|^2\eta^2(\omega_\delta-k)^2\,dx+\frac{\mu_2}{\varepsilon}\int_{S_{\delta,k}}|\mathcal{D}\eta|^2(\omega_\delta-k)^2\,d\mathcal{H}^n\\
&\quad+\delta\int_{A_{\delta,k}}|\nabla\omega_\delta|\big(1+|\nabla u_\delta|^2\big)\,|\nabla\eta|\,\eta(\omega_\delta-k)^2\,dx
\end{aligned}$$

being valid for any $\varepsilon > 0$. It is easy to check that

$$|\mathcal{D}\omega_\delta| \;\le\; \frac{2}{1+|\nabla u_\delta|^2}\,|\nabla u_\delta|\,\sqrt{\mathcal{D}(\partial_s u_\delta)\cdot\mathcal{D}(\partial_s u_\delta)}$$

holds, hence we get after appropriate choice of ε

$$\begin{aligned}&\delta\int_{A_{\delta,k}}\big|\nabla^2 u_\delta\big|^2\big(\omega_\delta-k\big)^2\eta^2\,dx\\ \le\;& C\left\{\int_{S_{\delta,k}}|\mathcal{D}\eta|^2(\omega_\delta-k)^2\,d\mathcal{H}^n+\delta\int_{A_{\delta,k}}|\nabla\omega_\delta|\big(1+|\nabla u_\delta|^2\big)\,|\nabla\eta|\,\eta(\omega_\delta-k)^2\,dx\right\}.\end{aligned}$$

In a last step we observe

$$\begin{aligned}&\int_{A_{\delta,k}}|\nabla\omega_\delta|\big(1+|\nabla u_\delta|^2\big)\,|\nabla\eta|\,\eta(\omega_\delta-k)^2\,dx\\ \le\;& 2\varepsilon\int_{A_{\delta,k}}\eta^2\,\big|\nabla^2u_\delta\big|^2\,(\omega_\delta-k)^2\,dx+\frac{1}{\varepsilon}\int_{A_{\delta,k}}|\nabla\eta|^2\,\big(1+|\nabla u_\delta|^2\big)(\omega_\delta-k)^2\,dx\,,\end{aligned}$$

and the claim of Proposition B.6 follows. □

For balls B_r compactly contained in Ω and positive numbers h we let

$$\begin{aligned}A(h,r)\;&:=\;A_\delta(h,r)\;:=\;A_{\delta,h}\cap B_r\,,\\ S(h,r)\;&:=\;S_\delta(h,r)\;:=\;(A(h,r)\times\mathbb{R})\cap S_\delta\;=\;\big\{\big(x,u_\delta(x)\big):\,x\in A(h,r)\big\}\,,\\ a(h,r)\;&:=\;a_\delta(h,r)\;:=\;\mathcal{H}^n(S(h,r))\;=\;\int_{A(h,r)}\sqrt{1+|\nabla u_\delta|^2}\,dx\,,\\ \tau(h,r)\;&:=\;\tau_\delta(h,r)\;:=\;\int_{S(h,r)}(\omega_\delta-h)^2\,d\mathcal{H}^n+\delta\int_{A(h,r)}\big(1+|\nabla u_\delta|^2\big)(\omega_\delta-h)^2\,dx\,.\end{aligned}$$

Proposition B.7. *Fix a ball B_{R_0} compactly contained in Ω. Then for arbitrary numbers $h > k \ge k_0(\psi)$ and $0 < r < R \le R_0$ the following estimates hold:*

a) $\tau(h,r) \;\le\; \frac{c_1}{(R-r)^2}\,\tau(h,R)\,a(h,r)^{\frac{2}{n}}$,

b) $a(h,r) \;\le\; \frac{1}{(h-k)^2}\,\tau(k,r)$.

Here c_1 is a constant independent of h, k, r, R, and δ.

Proof. Let $\eta \in C^1_0(B_R)$, $0 \le \eta \le 1$, $\eta \equiv 1$ on B_r and $|\nabla\eta| \le c(R-r)^{-1}$.
(i) Let us assume $n \ge 3$, the case $n = 2$ requires some non–essential modifications.

We have by Hölder's and Sobolev's inequality (see [GMS], Lemma 3.8, compare [LU])

$$
\begin{aligned}
\int_{S(h,r)} (\omega_\delta - h)^2 \, d\mathcal{H}^n &\le a(h,r)^{\frac{2}{n}} \Big(\int_{S(h,R)} \big\{\eta(\omega_\delta - h)\big\}^{\frac{2n}{n-2}} \, d\mathcal{H}^n \Big)^{\frac{n-2}{n}} \\
&\le c\, a(h,r)^{\frac{2}{n}} \int_{S(h,R)} \big|\mathcal{D}\big\{\eta(\omega_\delta - h)\big\}\big|^2 \, d\mathcal{H}^n \\
&\le c\, a(h,r)^{\frac{2}{n}} \int_{S(h,R)} \big\{(\omega_\delta - h)^2 \, |\mathcal{D}\eta|^2 + \eta^2 \, |\mathcal{D}\omega_\delta|^2\big\} \, d\mathcal{H}^n \\
&\le \frac{c}{(R-r)^2} \, a(h,r)^{\frac{2}{n}} \int_{S(h,R)} (\omega_\delta - h)^2 \, d\mathcal{H}^n \\
&\quad + c\, a(h,r)^{\frac{2}{n}} \int_{S(h,R)} \eta^2 \, |\mathcal{D}\omega_\delta|^2 \, d\mathcal{H}^n \, .
\end{aligned}
$$

(ii) On account of $\frac{n-1}{2n} < \frac{1}{2}$ we get in the same manner

$$
\begin{aligned}
&\int_{A(h,r)} \big(1 + |\nabla u_\delta|^2\big)(\omega_\delta - h)^2 \, dx \\
&\le a(h,r)^{\frac{2}{n}} \Big(\int_{A(h,R)} \Big\{\eta\big(1 + |\nabla u_\delta|^2\big)^{\frac{n-1}{2n}} (\omega_\delta - h)\Big\}^{\frac{2n}{n-2}} \, dx \Big)^{\frac{n-2}{n}} \\
&\le a(h,r)^{\frac{2}{n}} \Big(\int_{A(h,R)} \Big\{\eta\sqrt{1 + |\nabla u_\delta|^2} (\omega_\delta - h)\Big\}^{\frac{2n}{n-2}} \, dx \Big)^{\frac{n-2}{n}} \\
&\le c\, a(h,r)^{\frac{2}{n}} \int_{A(h,R)} \Big|\nabla\big\{\eta\sqrt{1 + |\nabla u_\delta|^2} \, (\omega_\delta - h)\big\}\Big|^2 \, dx \, ,
\end{aligned}
$$

thus

$$
\begin{aligned}
&\delta \int_{A(h,r)} \big(1 + |\nabla u_\delta|^2\big)(\omega_\delta - h)^2 \, dx \\
&\le c\, \delta\, a(h,r)^{\frac{2}{n}} \Big\{ \frac{1}{(R-r)^2} \int_{A(h,R)} \big(1 + |\nabla u_\delta|^2\big)(\omega_\delta - h)^2 \, dx \\
&\quad + \int_{A(h,R)} \eta^2 (\omega_\delta - h)^2 \, \big|\nabla^2 u_\delta\big|^2 \, dx + \int_{A(h,R)} \eta^2 \big(1 + |\nabla u_\delta|^2\big) \, |\nabla \omega_\delta|^2 \, dx \Big\} \, .
\end{aligned}
$$

(iii) Putting together the estimates from (i) and (ii) and using the results from Propositions B.4 and B.6, we immediately deduce claim a) of Proposition B.7.

(iv) Claim b) is immediate:

$$
\begin{aligned}
a(h,r) &= \int_{A(h,r)} \sqrt{1 + |\nabla u_\delta|^2} \, dx \\
&\le \frac{1}{(h-k)^2} \int_{A(h,r)} (\omega_\delta - k)^2 \sqrt{1 + |\nabla u_\delta|^2} \, dx \\
&\le \frac{1}{(h-k)^2} \int_{A(k,r)} (\omega_\delta - k)^2 \sqrt{1 + |\nabla u_\delta|^2} \, dx \\
&= \frac{1}{(h-k)^2} \, \tau(k,r) \, .
\end{aligned}
$$

□

With notation introduced in and before Proposition B.7 we infer from Lemma 3.7 in [GMS] (compare [ST], Lemma 5.1): there exist numbers $d \geq k_0(\psi)$ and $c_2 = c_2(c_1, n)$ such that

$$a\left(d, \frac{R_0}{2}\right)\tau\left(d, \frac{R_0}{2}\right) \;=\; 0\,.$$

For d we have the estimate

$$d \;\leq\; c_2\,\tau(0,R_0)^{\frac{1}{2}} R_0^{-\frac{n}{2}\theta} a(0,R_0)^{\frac{\theta-1}{2}}\,, \quad \theta \;:=\; \frac{1}{2} + \sqrt{\frac{1}{4}+\frac{2}{n}}\,.$$

Remark B.8. In order to apply Lemma 3.7 of [GMS] it is sufficient to check the hypothesis just for $h > k \geq k_0(\psi)$.

By construction we deduce $A\big(d, \frac{R_0}{2}\big) = \emptyset$, i. e.

$$|\nabla u_\delta(x)|^2 \;\leq\; e^d \quad \text{on} \quad B_{R_0/2}\,,$$

thus it remains to estimate d which means that we have to control

$$\begin{aligned} T_1 \;&:=\; \int_{S(0,R_0)} \omega_\delta^2 \, d\mathcal{H}^n + \delta \int_{B_{R_0}} \big(1+|\nabla u_\delta|^2\big)\omega_\delta^2\,dx \quad \text{and} \\ T_2 \;&:=\; \int_{B_{R_0}} \sqrt{1+|\nabla u_\delta|^2}\,dx\,. \end{aligned}$$

For a suitable constant c we have

$$T_2 \;\leq\; c\,J_\delta(u_\delta) \;\leq\; c\,J_\delta\big(\max\{0,\psi\}\big) \;\leq\; c\,J_1\big(\max\{0,\psi\}\big)$$

if we assume without loss of generality $\delta \leq 1$. Thus it remains to find a suitable bound for T_1. Assuming $B_{2R_0} \subset\subset \Omega$ we let $\varphi = u_\delta \omega_\delta^2 \eta^2$ with $\eta = 1$ on B_{R_0}, $0 \leq \eta \leq 1$, $|\nabla \eta| \leq c/R_0$. Inserting φ as test function into the Euler equation from Lemma 3.2 and using the structural conditions for f we get (applying also Young's inequality)

$$\begin{aligned} &\nu_1 \int_{B_{2R_0}} \sqrt{1+|\nabla u_\delta|^2}\,\omega_\delta^2\eta^2\,dx + \delta \int_{B_{2R_0}} \omega_\delta^2\,|\nabla u_\delta|^2\eta^2\,dx - \nu_2 \int_{B_{2R_0}} \omega_\delta^2\eta^2\,dx \\ \leq\; & C\left\{ \frac{1}{R_0}\sup_\Omega |u_\delta| \int_{B_{2R_0}} \omega_\delta^2\,dx \right. \\ &\quad + \sup_\Omega |u_\delta| \int_{B_{2R_0}} \eta^2\left(\varepsilon\omega_\delta^2\sqrt{1+|\nabla u_\delta|^2} + \varepsilon^{-1}\frac{1}{\sqrt{1+|\nabla u_\delta|^2}}\,|\nabla\omega_\delta|^2\right)dx \\ &\quad + \delta \sup_\Omega |u_\delta| \int_{B_{2R_0}} \Big\{2\varepsilon\eta^2\,|\nabla u_\delta|^2\,\omega_\delta^2 + \varepsilon^{-1}\big[\omega_\delta^2\,|\nabla\eta|^2 + |\nabla\omega_\delta|^2\eta^2\big]\Big\}\,dx \\ &\quad \left. + \int_{B_{2R_0}} g_\delta u_\delta \omega_\delta^2 \eta^2\,dx \right\}. \end{aligned}$$

Since ω_δ^2 is bounded by $\sqrt{1+|\nabla u_\delta|^2}$, we can apply the same reasoning as for T_2 to get a uniform bound for $\int_{B_{2R_0}} \omega_\delta^2\,dx$. Moreover we have

$$\int_{B_{2R_0}} g_\delta u_\delta \omega_\delta^2 \eta^2\,dx \;\leq\; C\sup_\Omega |u_\delta| \int_{B_{2R_0}} \omega_\delta^2\,dx\,,$$

hence the desired bound for T_1 follows after appropriate choice of ε and η as soon we can control $\sup_\Omega |u_\delta|$ and

$$T_3 \;:=\; \int_{B_{2R_0}} \eta^2 \frac{|\nabla\omega_\delta|^2}{\sqrt{1+|\nabla u_\delta|^2}}\,dx + \delta\int_{B_{2R_0}} \eta^2\,|\nabla\omega_\delta|^2\,dx\,.$$

Let $v := \min\big\{u_\delta, \sup_\Omega|\psi|\big\}$. Then the minimality of u_δ implies (recall $f\geq 0$)

$$\int_\Omega f_\delta(\nabla u_\delta)\,dx \;\leq\; \int_\Omega f_\delta(\nabla v)\,dx \;=\; \int_{\left[u_\delta \leq \sup_\Omega|\psi|\right]} f_\delta(\nabla u_\delta)\,dx\,,$$

so that $\nabla u_\delta = 0$ on $\big[u_\delta > \sup_\Omega|\psi|\big]$, i.e. $u_\delta \leq \sup_\Omega|\psi|$. In order to discuss T_3 we observe $|\nabla\omega_\delta|^2 \leq |\mathcal{D}\omega_\delta|^2\big(1+|\nabla u_\delta|^2\big)$, therefore we are going to consider

$$T_3' \;:=\; \int_{B_{2R_0}} \eta^2\,|\mathcal{D}\omega_\delta|^2\sqrt{1+|\nabla u_\delta|^2}\,dx + \delta\int_{B_{2R_0}} \eta^2\,|\nabla\omega_\delta|^2\,dx\,.$$

To this purpose we let $\varphi := \partial_s\big(\eta^2\partial_s u_\delta\big)$ in the Euler equation for u_δ, thus

$$\begin{aligned}
&\int_{B_{2R_0}} D^2 f_\delta(\nabla u_\delta)\big(\nabla\partial_s u_\delta, \nabla\partial_s u_\delta\big)\eta^2\,dx + \int_{B_{2R_0}} D^2 f_\delta(\nabla u_\delta)\big(\nabla\partial_s u_\delta, \nabla\eta^2\big)\partial_s u_\delta\,dx\\
&= \;-\int_{B_{2R_0}} g_\delta\partial_s\big(\eta^2\partial_s u_\delta\big)\,dx\\
&= \;-\int_{B_{2R_0}} g_\delta\partial_s\big(\eta^2\partial_s \psi\big)\,dx\,,
\end{aligned}$$

and the right–hand side is bounded independent of δ. The first integral on the left–hand side is bounded from below by

$$\mu_1\int_{B_{2R_0}} \eta^2\frac{1}{\sqrt{1+|\nabla u_\delta|^2}}\,|\mathcal{D}\partial_s u_\delta|^2\,dx + \delta\int_{B_{2R_0}} \big|\nabla^2 u_\delta\big|^2\eta^2\,dx\,,$$

further we have (for any $\varepsilon > 0$)

$$\begin{aligned}
&\left|\int_{B_{2R_0}} D^2 f_\delta\big(\nabla\partial_s u_\delta, \nabla\eta^2\big)\partial_s u_\delta\,dx\right|\\
\leq\;& 2\int_{B_{2R_0}} |\nabla u_\delta|\Big(D^2 f(\nabla u_\delta)\big(\nabla\partial_s u_\delta,\nabla\partial_s u_\delta\big)\Big)^{\frac12}\Big(D^2 f(\nabla u_\delta)\big(\nabla\eta,\nabla\eta\big)\Big)^{\frac12}\eta\,dx\\
&+\varepsilon\,\delta\int_{B_{2R_0}} \eta^2\,\big|\nabla^2 u_\delta\big|^2\,dx + c\,\frac{\delta}{\varepsilon}\int_{B_{2R_0}} |\nabla\eta|^2\,|\nabla u_\delta|^2\,dx\\
\leq\;& c\,\mu_2\int_{B_{2R_0}} \eta\sqrt{\mathcal{D}\partial_s u_\delta\cdot\mathcal{D}\partial_s u_\delta}\,|\mathcal{D}\eta|\,dx + \varepsilon\,\delta\int_{B_{2R_0}} \eta^2\,\big|\nabla^2 u_\delta\big|^2\,dx\;+
\end{aligned}$$

$$
\begin{aligned}
&+ c\,\frac{\delta}{\varepsilon}\int_{B_{2R_0}} |\nabla\eta|^2\,|\nabla u_\delta|^2\,dx \\
\leq\;& \varepsilon\, c\,\mu_2 \int_{B_{2R_0}} \eta^2 \frac{|\mathcal{D}\partial_s u_\delta|^2}{\sqrt{1+|\nabla u_\delta|^2}}\,dx + \frac{c\,\mu_2}{\varepsilon}\int_{B_{2R_0}} \sqrt{1+|\nabla u_\delta|^2}\,|\mathcal{D}\eta|^2\,dx \\
&+ \varepsilon\,\delta \int_{B_{2R_0}} \eta^2 \left|\nabla^2 u_\delta\right|^2 dx + c\,\frac{\delta}{\varepsilon}\int_{B_{2R_0}} |\nabla\eta|^2\,|\nabla u_\delta|^2\,dx\,.
\end{aligned}
$$

Appropriate choice of ε yields after absorbing terms

$$
\begin{aligned}
&\int_{B_{2R_0}} \frac{1}{\sqrt{1+|\nabla u_\delta|^2}}\,|\mathcal{D}\partial_s u_\delta|^2\eta^2\,dx + \delta\int_{B_{2R_0}} \left|\nabla^2 u_\delta\right|^2\eta^2\,dx \\
\leq\;& C(R_0)\left\{1 + \int_{B_{2R_0}} \sqrt{1+|\nabla u_\delta|^2}\,dx + \delta\int_{B_{2R_0}} |\nabla u_\delta|^2\,dx\right\}.
\end{aligned}
$$

The boundedness of $\delta\int_{B_{2R_0}} |\nabla u_\delta|^2\,dx$ is immediate (see Section 3), hence

$$
\int_{B_{2R_0}} \frac{1}{\sqrt{1+|\nabla u_\delta|^2}}\,|\mathcal{D}\partial_s u_\delta|^2\eta^2\,dx + \delta\int_{B_{2R_0}} \left|\nabla^2 u_\delta\right|^2\eta^2\,dx \;\leq\; C\,,
$$

where C — as before — denotes a local constant not depending on δ. Finally we recall

$$
|\mathcal{D}\omega_\delta|^2 \;\leq\; \frac{c}{1+|\nabla u_\delta|^2}\,|\mathcal{D}\partial_s u_\delta|^2 \quad\text{and}\quad |\nabla\omega_\delta|^2 \;\leq\; c\left|\nabla^2 u_\delta\right|^2,
$$

thus T_3' is bounded. Putting together all our results we arrive at

$$
|\nabla u_\delta(x)| \;\leq\; C \;<\; \infty
$$

for all $x \in B_{R_0/2}$, C being independent of δ. From this the claim of Lemma B.2 follows by passing to the limit $\delta \downarrow 0$ and a covering argument. $\square$

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