# Convex Variational Problems with Linear, Nearly Linear and/or Anisotropic Growth Conditions

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# Chapter I

# Introduction

One of the most fundamental problems arising in the calculus of variations is to minimize strictly convex energy functionals with respect to prescribed Dirichlet boundary data. Numerous applications for this type of variational problems are found, for instance, in mathematical physics or geometry.

Here we do not want to give an introduction to this topic — we just refer to the monograph of Giaquinta and Hildebrandt ([GH]), where the reader will find in addition an intensive discussion of historical facts, examples and references.

Let us start with a more precise formulation of the problem under consideration: given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and a variational integrand $f$: $\mathbb{R}^{nN} \to \mathbb{R}$ of class $C^2(\mathbb{R}^{nN})$ we consider the autonomous minimization problem

$$(\mathcal{P}) \qquad J[w] \;:=\; \int_\Omega f(\nabla w)\, dx \longrightarrow \min$$

among mappings $w$: $\Omega \to \mathbb{R}^N$, $N \geq 1$, with prescribed Dirichlet boundary data $u_0$. Depending on $f$, the comparison functions are additionally assumed to be elements of a suitable energy class $\mathbb{K}$. In the following, the variational integrand is always assumed to be strictly convex (in the sense of definition), thus we do not touch the quasiconvex case (compare, for instance, [Ev], [FH], [EG1], [AF1], [AF2], [CFM]).

The purpose of our studies is to establish regularity results for (maybe generalized and not necessarily unique) minimizers of problem $(\mathcal{P})$ under linear, nearly linear and/or anisotropic growth conditions on $f$ together with some appropriate notion of ellipticity: if $u$ denotes a suitable (weak) solution of $(\mathcal{P})$, then three different kinds of results are expected to be true.

**THEOREM 1** (REGULARITY IN THE SCALAR CASE)
*Assume that $N = 1$ and that $f$ satisfies some appropriate growth and ellipticity conditions. Then $u$ is of class $C^{1,\alpha}(\Omega)$ for any $0 < \alpha < 1$.*

According to an example of De Giorgi (see [DG3], compare also [GiM2], [Ne] and the recent example [SY]), there is no hope to prove an analogous result of this strength in the vectorial setting. Here we can only hope for

**THEOREM 2** (PARTIAL REGULARITY IN THE VECTORVALUED CASE)
*Assume that $N > 1$ and that $f$ satisfies some appropriate growth and ellipticity conditions. Then there is an open set $\Omega_0 \subset \Omega$ of full Lebesgue measure, i.e. $|\Omega - \Omega_0| = 0$, such that $u \in C^{1,\alpha}(\Omega_0;\mathbb{R}^N)$, $0 < \alpha < 1$.*

Finally, an additional structure condition might improve Theorem 2 to full regularity (see [Uh], earlier ideas are due to [Ur]):

**THEOREM 3** (FULL REGULARITY IN THE VECTORVALUED CASE WITH SOME ADDITIONAL STRUCTURE)
*Suppose that in the vectorial setting the integrand $f$ satisfies in addition $f(Z) = g(|Z|^2)$ for some function $g$: $[0,\infty) \to [0,\infty)$ of class $C^2$ (plus some Hölder condition for the second derivatives). Then $u$ is of class $C^{1,\alpha}(\Omega;\mathbb{R}^N)$, $0 < \alpha < 1$.*

As essential assumptions, the growth and ellipticity conditions on $f$ are involved in the above theorems. Hence, in order to make our discussion more precise and to summarize the various cases for which Theorems 1–3 are known to be true, we first introduce some brief classification of the integrands under consideration involving both growth and ellipticity properties. We also remark that in the cases A and B considered below existence (and uniqueness) of minimizers in suitable energy spaces is easily established.

Before going through the following list it should be emphasized that we do not claim to give an historical overview which is complete to any extent.

A.1 POWER GROWTH

Having the standard example $f_p(Z) = (1+|Z|^2)^{p/2}$, $1 < p$, in mind, let us assume that the growth rates from above and below coincide, i.e. for some number $p > 1$ and with constants $c_1$, $c_2$, $C$, $\lambda$, $\Lambda > 0$ the integrand $f$ satisfies for all $Z$, $Y \in \mathbb{R}^{nN}$ (note that the second line of (1) implies the first one)

$$
(1)\qquad
\begin{aligned}
c_1|Z|^p - c_2 \;&\le\; f(Z) \;\le\; C\left(1+|Z|^p\right),\\
\lambda\left(1+|Z|^2\right)^{\frac{p-2}{2}}|Y|^2 \;&\le\; D^2f(Z)(Y,Y) \;\le\; \Lambda\left(1+|Z|^2\right)^{\frac{p-2}{2}}|Y|^2 .
\end{aligned}
$$

With the pioneering work of De Giorgi, Moser, Nash as well as of Ladyzhenskaya and Ural'tseva, Theorem 1 is well known in this setting and of course many other authors could be mentioned (see [DG1], [Mos], [Na] and [LU1] for a complete overview and a detailed list of references).

As already noticed above, the third theorem in this setting should be mainly connected with the name of Uhlenbeck (see [Uh], where the full strength of (1) is not needed which means that also degenerate ellipticity can be considered).

Without additional structure conditions in the vectorial case, the twodimensional case $n = 2$ differs substantially from the situation in higher dimensions: a classical result of Morrey ensures full regularity if $n = 2$ (here we like to refer to [Mor1], the first monograph on multiple integrals in the calculus of variations, where again detailed references can be found).

Finally, Theorem 2 is proved in any dimension and in a quite general setting by Anzellotti/Giaquinta ([AG2]), where the whole scale of integrands up to the limit case of linear growth is covered (with some suitable notion of relaxation). In addition, the assumptions on the second derivatives are much weaker than stated above, i.e. their partial regularity result is true whenever $D^2 f(Z) > 0$ holds for any matrix $Z$.

To keep the historical line, we like to mention the earlier contributions on partial regularity [Mor2], [GiM1], [Giu1] (compare also [DG2], [Alm], a detailed overview is found in [Gia1]).

## A.2 Anisotropic Power Growth

The study of anisotropic variational problems was forced by Marcellini [Ma2]–[Ma7] and is a natural extension of (1). To give some motivation we may consider the case $n = 2$, $2 \le p \le q$ and replace $f_p$ by

$$f_{p,q}(Z) \;=\; \big(1 + |Z|^2\big)^{\frac{p}{2}} + \big(1 + |Z_2|^2\big)^{\frac{q}{2}}\,, \qquad Z = (Z_1, Z_2) \in \mathbb{R}^{2N}\,,$$

hence $f$ is allowed to have different growth rates from above and from below. The natural generalization of the structure condition (1) is the requirement that $f$ satisfies (again the growth conditions on the second derivatives imply the corresponding growth rates of $f$)

$$\begin{array}{rcccl} c_1 |Z|^p - c_2 & \le & f(Z) & \le & C\big(1 + |Z|^q\big)\,, \\ \lambda \big(1 + |Z|^2\big)^{\frac{p-2}{2}} |Y|^2 & \le & D^2 f(Z)(Y,Y) & \le & \Lambda \big(1 + |Z|^2\big)^{\frac{q-2}{2}} |Y|^2 \end{array} \tag{2}$$

for all $Z$, $Y \in \mathbb{R}^{nN}$, where as usual $c_1$, $c_2$, $C$, $\lambda$, $\Lambda$ denote some positive constants and $1 < p \le q$.

If $p$ and $q$ differ too much, then it turns out that even in the scalar case singularities may occur (to mention only one famous example we refer to [Gia2]). However, following the work of Marcellini, suitable assumptions on $p$ and $q$ yield regular solutions (compare Section III.5 for a discussion of these conditions). Note that [Ma5] also covers the case $N > 1$ with some additional structure condition.

In the general vectorial setting only a few contributions are available, we like to refer to the papers of Acerbi/Fusco ([AF4]) and Passarelli Di Napoli/Siepe ([PS]), where partial regularity results are obtained under quite restrictive assumptions on $p$ and $q$ excluding any subquadratic growth (again see Section III.5).

If some additional boundedness condition is imposed, then the above results are improved by Esposito/Leonetti/Mingione ([ELM2]) and Choe ([Ch]). In [ELM2] higher integrability (up to a certain extent) is established ($N \geq 1$, $2 \leq p$) under a quite weak relation between $p$ and $q$. A theorem of the third type is found in [Ch].

### B.1 Growth Conditions Involving N-Functions

Studying the monograph of Fuchs and Seregin ([FS2]), it is obvious that many problems in mathematical physics are not within the reach of power growth models — the theories of Prandtl-Eyring fluids and of plastic materials with logarithmic hardening serve as typical examples. The variational integrands under consideration are now of nearly linear growth, for example we have to study the logarithmic integrand

$$f(Z) \;=\; |Z| \ln(1 + |Z|)$$

which satisfies none of the conditions (1) or (2).

The main results on integrands with logarithmic structure are proved by Frehse/Seregin ([FrS]: full regularity if $n = 2$), Fuchs/Seregin ([FS1]: partial regularity if $n \leq 4$), Esposito/Mingione ([EM2]: partial regularity in any dimension) and finally by Mingione/Siepe ([MS]: full regularity in any dimension).

### B.2 The First Extension of the Logarithm

As a first natural extension one may think of integrands which are bounded from above and below by the same quantity $A(|Z|)$, where $A\colon [0, \infty) \to [0, \infty)$ denotes some arbitrary N-function satisfying a $\Delta_2$-condition (see [Ad] for precise definitions). Although this does not imply some natural bounds (in terms of $A$) on the second derivatives, (1) and (2) suggest the following model: given a N-function $A$ as above, positive constants $c$, $C$, $\lambda$ and $\Lambda$, we assume that our integrand $f$ satisfies

$$
\begin{array}{rcccl}
 & c\,A(|Z|) & \leq & f(Z) & \leq & C\,A(|Z|)\,, \\
(3) & & & & & \\
 & \lambda\left(1+|Z|^2\right)^{-\frac{\mu}{2}}|Y|^2 & \leq & D^2f(Z)(Y,Y) & \leq & \Lambda\left(1+|Z|^2\right)^{\frac{q-2}{2}}|Y|^2
\end{array}
$$

for all $Z$, $Y \in \mathbb{R}^{nN}$ and for some real numbers $1 \leq \mu$, $1 < q \leq 2$, this choice being adapted to the logarithmic integrand which satifies (3) with $\mu = 1$ and $q = 1 + \varepsilon$ for any $\varepsilon > 0$. Note that the correspondence to (1) and (2) is only of formal nature: since we require $\mu \geq 1$, the $\mu$-ellipticity condition, i.e. the first inequality in the second line of (3), does not give any information on the lower growth rate of $f$ in terms of a power function with exponent $p > 1$.

A first investigation of variational problems with the structure (3) under some additional balancing conditions is due to Fuchs and Osmolovskii ([FO]), where Theorem 2 is shown to be true in the case that $\mu < 4/n$.

Theorems of type 1 and 3 are established by Fuchs and Mingione (see [FM]) — their assumptions on $\mu$ and $q$ are discussed in Section III.5.

C Linear Growth

It remains to discuss the case of variational problems with linear growth. On account of the lack of compactness in the non-reflexive Sobolev space $W^1_1(\Omega;\mathbb{R}^N)$, problem $(\mathcal{P})$ in general fails to have solutions. Thus one either has to introduce a suitable notion of generalized minimizers (possibility *i*)) or one must pass to the dual variational problem (possibility *ii*)).

ad *i*). Since the integrand $f$ under consideration is of linear growth, any $J$-minimizing sequence $\{u_m\}$, $u_m \in u_0 + \overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)$, is uniformly bounded in the space $BV(\Omega;\mathbb{R}^N)$. This ensures the existence of a subsequence (not relabelled) and a function $u$ in $BV(\Omega;\mathbb{R}^N)$ such that $u_m \to u$ in $L^1(\Omega;\mathbb{R}^N)$. Thus, one suitable definition of a generalized minimizer $u$ is to require $u \in \mathcal{M}$, where the set $\mathcal{M}$ is given by

$$
\mathcal{M} \;=\; \Big\{u \in BV\big(\Omega;\mathbb{R}^N\big):\; u \text{ is the } L^1\text{-limit of a } J\text{-minimizing sequence from } u_0 + \overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)\Big\}\,.
$$

Another point of view is to define a relaxed functional $\hat{J}$ on the space $BV(\Omega;\mathbb{R}^N)$ (a precise notion of relaxation is given in Appendix A). Then generalized solutions of problem $(\mathcal{P})$ are introduced as minimizers of a relaxed problem $(\hat{\mathcal{P}})$.

**REMARK I.1** *We already like to mention that these formally different points of view in fact lead to the same set of functions. Moreover, the third approach to the definition of generalized minimizers given in [Se1], [ST] also leads to the same class of minimizing objects.*

ad *ii*). Following [ET] we write

$$J[w] = \sup_{\tau \in L^\infty(\Omega;\mathbb{R}^{nN})} l(w,\tau)\,, \qquad w \in u_0 + \overset{\circ}{W}{}^1_1\big(\Omega;\mathbb{R}^N\big)\,,$$

where $l(w,\tau)$ denotes some natural Lagrangian (see Section II.1.1). If we let

$$R:\quad L^\infty\big(\Omega;\mathbb{R}^{nN}\big) \to \overline{\mathbb{R}}\,,$$

$$R(\tau) := \inf_{u \in u_0 + \overset{\circ}{W}{}^1_1(\Omega;\mathbb{R}^N)} l(u,\tau) = \begin{cases} -\infty\,, & \text{if} \quad \operatorname{div}\tau \neq 0\,,\\[2ex] l(u_0,\tau)\,, & \text{if} \quad \operatorname{div}\tau = 0\,,\end{cases}$$

then the dual problem reads as

$(\mathcal{P}^*)$ to maximize $R$ among all functions in $L^\infty\big(\Omega;\mathbb{R}^{nN}\big)$,

where the existence of solutions easily is established.

In any of the above definitions the set of generalized minimizers of problem $(\mathcal{P})$ may be very "large". In contrast to this fact, the solution of the dual problem is unique (see the discussion of Section II.2). Moreover, the dual solution $\sigma$ admits a clear physical or geometrical interpretation, for instance as a stress tensor or the normal to a surface. Hence, in the linear growth situation we wish to complete the above theorems by analogous regularity results for $\sigma$.

### C.1 GEOMETRIC PROBLEMS OF LINEAR GROWTH

One of the most important (scalar) examples is the minimal surface case $f(Z) = \sqrt{1+|Z|^2}$. A variety of references is available for the study of this variational integrand, let us mention the monographs of Giusti ([Giu2]) and Giaquinta/Modica/Souček ([GMS2]) at this point.

At first sight, ellipticity now is very bad since the inequalities in the second line of (3) just hold for the choices $\mu = 3$ and $q = 1$. On the other hand, this rough estimate is not needed because it is possible to benefit from the geometric structure of the problem (see Remark IV.0.3). A class of integrands with this structure is studied, for instance, in [GMS1] following the apriori gradient bounds given in [LU2]. It turns out that in the minimal surface case generalized $\hat{J}$-minimizers are of class $C^{1,\alpha}(\Omega)$ and that we have uniqueness up to a constant.

### C.2 Linear Growth Problems Without Geometric Structure

The theory of perfect plasticity provides another famous variational integrand of linear growth. In this case the assumptions of smoothness and strict convexity imposed on $f$ are no longer satisfied. Nevertheless, the example should be included in our discussion since we will benefit in Chapter II from the studies of Seregin ([Se1]–[Se6]) on this topic (compare the recent monograph [FS2]).

The quantity of physical interest is the stress tensor $\sigma$, which is only known to be partially regular (compare [Se4]). Even in the twodimensional setting $n = 2$ we just have some additional information on the singular set (see [Se6]) and the model of plastic materials with logarithmic hardening (as decribed in B.1) serves as a regular approximation.

It is already mentioned above that the vectorvalued linear growth situation is covered by [AG2], provided that we restrict ourselves to smooth and stricty convex integrands. Anzellotti and Giaquinta prove Theorem 2 for generalized $\hat{J}$-minimizers, hence the same regularity result turns out to be true for any $u \in \mathcal{M}$ (see Section II.3.1 for details). It remains to study the properties of the dual solution which for linear growth problems is a quantity of particular interest.

Before we summarize this brief overview in the table on the next page, we like to mention that of course there is a variety of further contributions where the class of admissible energy densities is equipped with some additional structure (see [AF4], [Lie2], [UU] and many others).

## Some Known Results on Regularity in the Convex Case

| | $N = 1$ | $N > 1$ |
|---|---|---|
| A.1 | (1) De Giorgi, Moser, Nash, Ladyzhenskaya/Ural'tseva $\leq$ '65 | (2) Anzellotti/Giaquinta '88<br>(3) Uhlenbeck, '77 |
| A.2 | (1) $1 < p \leq q < \dots$ Marcellini $\approx$ '90 | (2) $2 \leq p \leq q < \dots$ Acerbi/Fusco '94,<br>(3) bounded ..., Choe '92 |
| B.1 | see $N > 1$ | (3) $n = 2$: Frehse/Seregin '98<br>(2) $n \leq 4$: Fuchs/Seregin '98<br>(2) Esposito/Mingione '00<br>(3) Mingione/Siepe '99 |
| B.2 | (1) $\mu < 1 + 2/n$, $q < \dots$ Fuchs/Mingione '00 | (2) $\mu \leq 4/n$, "balanced" Fuchs/Osmolovskii '98<br>(3) [FM] (see $N = 1$) |
| C.1 | $(1)_{\hat{J}}$ Giaquinta/Modica/ Souček '79 | — |
| C.2 | — | $(2)_{\hat{J}}$ [AG] (see A.1, $N > 1$)<br>$(\mathrm{P})_{\sigma,\mathrm{pl}}$ Seregin $\approx$ '90 |

(1), (2), (3): Theorems 1—3, respectively

$(1)_{\hat{J}}$, $(2)_{\hat{J}}$: corresponding results for generalized $\hat{J}$-minimizers

$(\mathrm{P})_{\sigma,\mathrm{pl}}$: partial regularity for the stress tensor in the theory of perfect plasticity

In the following we are going to

- have a close look at linear growth problems;
- unify the results of A and B by the way including new classes of integrands;
- discuss the substantial extensions which follow in cases A, B and C from a natural boundedness condition.

Our main line skips from linear to superlinear growth and vice versa: despite of the essential differences, these two items are strongly related by the form of the structure conditions imposed on the integrands under consideration (see Definition III.2.1 and Assumption IV.0.1), by the applied techniques and up to a certain extent by the obtained results. In particular, this relationship becomes evident while studying scalar variational problems with

- mixed anisotropic linear/superlinear growth conditions.

As the first centre of interest, the discussion starts in Chapter II by considering the general linear situation. Here no uniqueness results for generalized minimizers can be expected and we concentrate on the dual solution $\sigma$, which, according to the above remarks, is also a reasonable physical point of view. The main contributions are

*i*) uniqueness of the dual solution under very weak assumptions;

*ii*) partial $C^{1,\alpha}$-regularity for weak cluster points of $J$-minimizing sequences and, as a consequence, partial $C^{0,\alpha}$-regularity for $\sigma$;

*iii*) a proof of the duality relation $\sigma = \nabla f(\nabla^a u^*)$ for a class of degenerate variational problems with linear growth. Here $\nabla^a u^*$ denotes the absolutely continuous part of $\nabla u^*$ with respect to the Lebesgue measure.

ad *i*). Following standard arguments of convex analysis (compare [ET]), the uniqueness of the dual solution usually is ensured by assuming the conjugate function $f^*$ to be strictly convex. We do not want to impose this condition since it is formulated in terms of $f^*$, hence there might be no easy way to check this assumption. In fact, using more or less elementary arguments, it is proved in Section II.2 that there is no need to involve the conjugate function in an uniqueness theorem for the dual solution.

ad *ii*). Following the lines of [GMS1], any weak cluster point $u \in \mathcal{M}$ is seen to minimize the relaxed problem $(\hat{\mathcal{P}})$ associated to the original problem (see Appendix A.1). Alternatively, a local approach is prefered in Section II.3.1 (see Remark II.3.1 for a brief comment). In any case, the results of Anzellotti and Giaquinta apply and $u$ is seen to be of class $C^{1,\alpha}$ on the non-degenerate regular set $\Omega_u$ (see (2), Section II.3). As a next step, the duality relation $\sigma = \nabla f(\nabla u^*)$, $x \in \Omega_{u^*}$, is shown for a particular solution $u^*$ and $\sigma$ is seen to be of class $C^{0,\alpha}$ on this set.

ad *iii*). The duality relation is proved using local $C^{1,\alpha}$-results for some $u^*$ as above. As a consequence, information on the behaviour of $\sigma$ is only obtained on the $u^*$-regular set. In Section II.4, the almost everywhere identity $\sigma = \nabla f(\nabla^a u^*)$ is established for a class of degenerate problems by the way proving intrinsic regularity results in terms of $\sigma$. Note that the applied technique completely differs from the previous considerations since we can not rely on regularity results: arguments from measure theory are combined with the construction of local comparison functions (see Appendix B.3).

Chapter III deals with the nearly linear and/or anisotropic situation. Here

*i*) we introduce the notion of integrands with $(s, \mu, q)$-growth;

and give a unified and extended approach to

*ii*) the results of type (1) and (3) given in the above scheme;

*iii*) the corresponding theorems (2).

Finally, reducing the generality of the previous sections, a theorem on

*iv*) full $C^{1,\alpha}$-regularity of solutions of twodimensional vectorvalued problems with anisotropic power growth

completes Chapter III.

ad *i*). The main observation is clearyfied in Example III.2.4. Three independent quantities occurring in the structure and growth conditions imposed on the integrand $f$ determine the behaviour of solutions, which now uniquely exist in an appropriate energy class: the growth rate $s$ of the integrand $f$ under consideration, the $\mu$-ellipticity and the upper bound for the second derivative of $f$ given in terms of $q$. This leads to the notion of integrands with $(s, \mu, q)$-growth, which includes and extends the list given in A and B in a natural way. Note that related structure conditions for variational integrands with superquadratic growth are introduced in [Ma5]–[Ma7] (see Section III.5 for a brief discussion).

ad *ii*). Since regular solutions can not be expected for the whole range of $s$, $\mu$ and $q$ (we already mentioned [Gia2]) we impose the so called $(s, \mu, q)$-condition. Observe that we do not loose information in comparison with known results (see Section III.5).

As a next step, uniform apriori $L^q_{loc}$-estimates for the gradients of a regularizing sequence are proved. This enables us to apply De Giorgi-type arguments with uniform local apriori gradient bounds as the result. The conclusion then follows in a well known manner.

It should be emphasized that the proof covers the whole scale of $(s, \mu, q)$-integrands without distinguishing any subcases.

ad *iii*). Here a blow-up procedure (compare [Ev], [CFM]) is used to prove partial regularity in the above setting. This generalizes the known results to a large extent (see Section III.5).

ad *iv*). With the higher integrability results of the previous sections it is possible to refer to a lemma due to Frehse and Seregin.

In Chapter IV we return to problems with linear growth, where we first benefit from some of the techniques outlined in Chapter III, i.e.

*i*) a regular class of $\mu$-elliptic integrands with linear growth is introduced.

Then the results are substantially improved by

*ii*) studying bounded solutions (in some natural sense);

*iii*) considering twodimensional problems.

We finish the studies of linear growth problems by proving the

*iv*) sharpness of the results.

ad *i*). Example III.2.6 also provides a class of $\mu$-elliptic integrands with linear growth in the sense that for all $Z$, $Y \in \mathbb{R}^{nN}$

$$(4) \qquad \lambda \big(1 + |Z|^2\big)^{-\frac{\mu}{2}} |Y|^2 \;\le\; D^2 f(Z)(Y, Y) \;\le\; \Lambda \big(1 + |Z|^2\big)^{-\frac{1}{2}} |Y|^2$$

holds for some $\mu > 1$ and with constants $\lambda$, $\Lambda$. If $\mu < 1 + 2/n$, then this class is called a regular one since generalized minimizers are unique up to a constant and since Theorems 1 and 3 for functions $u \in \mathcal{M}$ will be established following the arguments of Chapter III. Let us shortly discuss the limitation $\mu < 1 + 2/n$. Given a suitable regularization $u_\delta$, it is shown that

$$\omega_\delta \;:=\; \big(1 + |\nabla u_\delta|^2\big)^{\frac{2-\mu}{4}}$$

is uniformly bounded in the class $W^1_{2,loc}(\Omega)$. This provides no information at all if the exponent is negative, i.e. if $\mu > 2$. An application of Sobolev's inequality, which needs the bound $\mu < 1 + 2/n$, proves uniform local higher integrability of the gradients. Also the De Giorgi-type arguments which follow the first step will lead to the same limitation on the ellipticity exponent $\mu$.

ad *ii*). The minimal surface integrand can be interpreted as a $\mu$-elliptic example with limit exponent $\mu = 3$ (recall that in the minimal surface case the regularity of solutions is obtained by using the geometric structure).

Section IV.2 is devoted to the question, whether the limit $\mu = 3$ is also of some relevance if the geometric structure condition is dropped. To this purpose some examples are discussed and — imposing a kind of natural boundedness condition — we prove even in the vectorvalued setting without assuming additional structure that a generalized minimizer $u^*$ of class $W^1_1(\Omega; \mathbb{R}^N)$ exists. Moreover, $u^*$ uniquely (up to a constant) determines the solutions of problem

$$(\mathcal{P}') \quad \int_\Omega f(\nabla w)\, dx + \int_{\partial\Omega} f_\infty\big((u_0 - w) \otimes \nu\big)\, d\mathcal{H}^{n-1} \ \to \ \min \ \text{ in } \ W^1_1\big(\Omega; \mathbb{R}^N\big)\,.$$

If — as a substitute for the geometric structure — $\mu < 3$ is assumed, then uniqueness of generalized minimizers up to a constant as well as Theorems 1 and 3 are seen to be true. As indicated above, the proof of *i*) does not extend to these results: in Sections IV.2.2.1 and IV.2.3 we do not differentiate the Euler equation by the way avoiding Sobolev's inequality. In the case $\mu < 3$, a preliminary iteration gives uniform $L^p_{loc}$-gradient bounds for any $p$. This is the reason why we may use Hölder's inequality and finally adjust the De Giorgi iteration exponent to get the conclusion.

ad *iii*). It turns out that a boundedness condition is superfluous to establish the results of *ii*) in the twodimensional case $n = 2$ (with the usual structure in the vectorvalued setting). Note that, once more, $\mu = 3$ is exactly the limit case within reach.

ad *iv*). Extending the ideas of [GMS1], an example is given which shows that problem $(\mathcal{P}')$ in general does not admit a $W^1_1$-solution if ellipticity holds for some $\mu > 3$. Since the energy density under consideration is explicitely depending on $x$, we have to show first in Section IV.2.2.2 (as a model case) that a smooth $x$-dependence does not affect the above mentioned theorems, thus our example is really a counterexample.

Chapter V once more deals with the study of superlinear growth problems, where a boundedness condition analogous to Chapter IV.2 is supposed to be true. We prove

*i*) higher integrability and, as a corollary, a theorem of type (2) for variational integrands with a wide range of anisotropy.

Then, as a model case,

*ii*) scalar obstacle problems are studied for this class of energy densities by the way proving a theorem corresponding to type (1).

ad *i*). Recalling the ideas of Chapter IV we expect that these techniques may be applied to improve the results of Sections III.3 and III.4 for bounded solutions of variational integrals with superlinear growth. If we consider integrands with anisotropic $(p,q)$-growth, then the corresponding relation between $p$ and $q$ should read as $q < p+2$. However, as proved in [BF5], the "linear growth techniques" just yield $W^1_{q,loc}$-solutions whenever $q < p + 2/3$. The reason for this "lack of anisotropy" is the following: in Section IV.2 we could benefit from the growth rate $1 = q$ of the main quantity $\nabla f(Z) : Z$ under consideration. In the anisotropic superlinear case however, we just have the lower bound $p < q$ of this quantity. This is the reason why we change methods again and give a refined study of an Ansatz which goes back to [Ch]. As a result, the full correspondence to the linear growth situation is established, i.e. with the assumption

$$\lambda\left(1+|Z|^2\right)^{-\frac{\mu}{2}}|Y|^2 \;\leq\; D^2f(Z)(Y,Y) \;\leq\; \Lambda\left(1+|Z|^2\right)^{\frac{q-2}{2}}|Y|^2$$

for all $Z$, $Y \in \mathbb{R}^{nN}$, with positive constants $\lambda$, $\Lambda$ and for exponents $\mu < 3$, $q > 1$, higher local integrability follows from $q < 4-\mu$. This provides (together with some natural hypothesis) a corollary on partial regularity.

ad *ii*). Here, as a model case, we include the study of scalar obstacle problems. The methods as described in *i*) yield full $C^{1,\alpha}_{loc}$-regularity under the same conditon $q < 4-\mu$ which is quite weak (recall the counterexample of Section IV.4).

Chapter VI closes the line with the consideration of

- scalar variational problems with mixed anisotropic linear/superlinear growth conditions.

Here, on one hand, we essentially have to rely on the wide range of anisotropy which is admissible on account of Chapter V. On the other hand, a refined study of the dual problem is needed since a dual solution may even fail to exist. This is caused by a possible anisotropic behaviour of the superlinear part itself. Nevertheless, we obtain locally regular and uniquely determined (up to a constant) generalized minimizers which in turn provide a "local stress tensor".

We finish our studies with two appendices:

the first one identifies the different ways to define generalized minimizers (recall Remark I.1). The main Theorem A.2.3 proves, as a corollary, the uniqueness results applied in Chapter IV which are based on the different approaches, respectively.

In Appendix B some density results are collected, where either a rigorous proof is hardly found in the literature or the claims have to be adjusted to the situation at hand. Maybe, the construction of local comparison functions given in Section B.3 is the only unknown result. This helpful lemma is used several times studying linear growth problems.

Up to now, the above mentioned results are just partially published:

the higher integrability and partial regularity theorems of Chapter II are outlined in [BF1]. Uniqueness of the dual solution as described above is established in [Bi1], whereas degenerate problems with linear growth are studied in the recent paper [Bi2].

The apriori gradient bounds of Chapter III for scalar variational problems with $(s, \mu, q)$-growth are proved in [BFM], partial regularity in the vectorial setting is due to [BF2]. The twodimensional considerations are given in [BF6].

Chapter IV starts with a brief discussion of the joint work [BF3]. The main contributions however, i.e. the study of bounded solutions in the linear growth situation as well as the twodimensional considerations are not published yet. This is also the case for the example given in Section IV.4 which shows that $\mu = 3$ is the worst case of ellipticity we can consider in Theorem IV.2.4.

Chapter V provides completely new results improving substantially [BF5] and the known results in the scalar case.

To our knowledge, integrands with mixed anisotropic linear/superlinear growth conditions as discussed in Chapter VI are the first time under consideration.

The observations on relaxation are found in [BF4].

**Acknowlegement.** Let me finally mention just a few names standing for the long list of persons who contributed to this thesis in one or the other way.

Once more, I like to thank Prof. S. Hildebrandt and Prof. M. Grüter who led me to the Diploma and PhD, respectively.

Prof. G. Mingione already appeared as one of the authors of [BFM]. The valuable discussions on variational problems with non standard growth conditions go much beyond this. Prof. G. Seregin took this part in the case of variational problems with linear growth.

As indicated above, large parts of this thesis ([BF1]–[BF6], [BFM]) are joint work with Prof. M. Fuchs which, in the best possible sense, needs no further comments. Moreover, neither the numerous discussions nor the helpful suggestions can be fixed to any extent. Most important, it is due to him that my joy in mathematics was growing stronger than the technical difficulties arising in the following chapters.

Before going through the mathematical details it should be emphasized that this thesis would be non-existent without the help of my parents, my whole family and, of course, Christina.

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