Universität des Saarlandes
Fachrichtung 6.1 – Mathematik
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PDE and Boundary-Value Problems (Winter Term 2014/2015)
Exam-Part W

Problem E.1 (10 Points)

Solve the problem

$$\begin{array}{rll} \text{PDE} & u_t = u_{xx} - u + x, & 0 < x < 1, \quad 0 < t < \infty \\ \text{BCs} & \begin{cases} u(0,t) = 0 \\ u(1,t) = 1 \end{cases} & 0 < t < \infty \\ \text{IC} & u(x,0) = 0 & 0 \leqslant x \leqslant 1 \end{array}$$

by

(a) changing the nonhomogeneous BCs to homogeneous ones.

(b) transforming into a new equation without the term $-u$.

(c) solving the resulting problem.

Problem E.2 (10 Points)

We seek the wave amplitude $u(x,t)$ for transverse wave motion along a taut string over the semi-infinite interval $I = \{x : 0 < x < \infty\}$. The left end of the string is constarined to move in accordance with the function $q(t) = \sin t$. The string has an initial displacement distribution $f(x) = 0$ and an initial speed distribution $g(x) = 0$. There is no external source function, and the is no damping in the system. The wave speed is $c = 2$.

Using Maple solve the problem by means of the Laplace transform. Show the animation of the spatial-time wave aplitude distribution $u(x,t)$ of the string for $0 \leqslant x \leqslant 10$ and $0 \leqslant t \leqslant 10$.

Problem E.3 (10 Points)

(a) Find the finite-difference solution of the heat-conduction problem

$$
\begin{array}{lll}
\text{PDE} & u_t = u_{xx}, & 0 < x < 1, \quad 0 < t < \infty \\
\text{BCs} & \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} & 0 < t < \infty \\
\text{IC} & u(x,0) = sin(\pi x) & 0 \leqslant x \leqslant 1
\end{array}
$$

for $t = 0.005$, 0.010, 0.015 by the explicit method. Let $h = \Delta x = 0.1$. Plot the solution at $x = 0,\ 0.1,\ 0.2,\ 0.3,\ \ldots,\ 0.9,\ 1$ for $t = 0.015$.

(b) Solve the problem form part (a) analytically (separation of variables) and evaluate the analytical solution at the grid points: $x = 0,\ 0.1,\ 0.2,\ 0.3,\ \ldots,\ 0.9,\ 1$ for $t = 0.015$. Compare these results to your numerical solution. (You may wish to write a small or use a calculator to evaluate the separation-of-variables solution).

Problem E.4 (10 Points)

Find the Green's function for the first quadrant

$$Q = \big\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 > 0\big\}$$

by repeated reflection. Use your answer to solve the Dirichlet problem

$$
\begin{array}{lll}
\text{PDE} & u_{x_1x_1} + u_{x_2x_2} = 0, & (x_1, x_2) \in Q \\
\text{BCs} & \begin{cases} u(0, x_2) = g(x_2) \\ u(x_1, 0) = h(x_1) \end{cases} & x_1 > 0,\ x_2 > 0.
\end{array}
$$

Problem E.5* (7+5+3=15 Points)

We want to know how much time is needed to cook an egg. Assume that the egg is a ball of radius 10 and is homogeneous. Its initial temperature is $T_0 = 30°C$, and it is cooked by immersion in boiling water at time $t = 0$ which keeps the surface temperature constant at $100°C$ thereafter. Let $u = u(r,t)$ be the temperature of the egg in the spherical shell of radius $r < 10$ at time t. We know that u can be modelled by the following IBVP:

$$
(5.1)\qquad \left[\begin{array}{lll}
\text{PDE} & u_t = \varkappa r^{-2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u}{\partial r}\right), & 0 < r < 10, \quad 0 < t < \infty \\
\text{BCs} & \begin{cases} u(10,t) = 100 \\ |u(0,t)| < \infty \end{cases} & 0 < t < \infty \\
\text{IC} & u(r,0) = 30 & 0 \leqslant r \leqslant 10
\end{array}\right.
$$

(a) Find the temperature u by going through the following steps:

(i) Assume $v = v(r,t)$ satisfy

$$
(5.2)\qquad \left[\begin{array}{lll}
\text{PDE} & v_t = \varkappa r^{-2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u}{\partial r}\right), & 0 < r < 10, \quad 0 < t < \infty \\
\text{BCs} & \begin{cases} v(10,t) = 0 \\ |v(0,t)| < \infty \end{cases} & 0 < t < \infty
\end{array}\right.
$$

Let $v(r,t) = R(r)T(t)$. Write down the equations and boundary conditions (if there is any) $R(r)$ and $T(t)$ satisfy.

(ii) Let $S(r) = rR(r)$. Derive the equation and boundary conditions that $S(r)$ satisfies.

(iii) Solve the obtained problem for $S(r)$. Find the solution for problem (5.2).

(iv) Find the solution for problem (5.1).

(b) Now you have your series solution. Notice that almost all terms in your solution decay exponentially in time, so among the decaying terms we will only keep the term that has the slowest decay in time. Write down the non-decaying terms and the term in your series solution that has the slowest decay in time, and use it to answer the following question:

- How long does it take for the center of the egg to reach $80°C$?
  (Hint: $\lim_{r\to 0}\dfrac{\sin r}{r} = 1$.)

(c) If you cook the egg for a long time in $100°C$ water, is it possible to heat the middle of the egg to reach a temperature above $100°C$? Explain.

(Hint: you may want to use your intuition here.)

Deadline for submission: Wednesday, April 1