PDE and Boundary-Value Problems
Winter Term 2014/2015

Lecture 1

Universität des Saarlandes

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- More details on the web page:

  http://www.math.uni-sb.de/ag/fuchs/ag-fuchs.html

What type of Lectures is it?

- Lectures (3h) with exercises (1h) (6 ECTS points)
- Time and Location: **Monday 12-14 c.t.** and **Thursday 08:30-10 c.t.**, Building E1.3, Lecture Room 003
- Exercises: Every second Thursday instead of a lecture
- First exercise: Thursday, November 06, 2014

What Prerequisites are required?

- Undergraduate knowledge in mathematics
(i.e., Calculus I and II, Linear Algebra)

- Passive knowledge of simple English.
(solutions in assignments or exam can be also submitted in German or in Russian)

- Basic knowledge of Maple

Assignments:

- Homework will be assigned bi-weekly.
- To qualify for the exam you need 50% of the points from these assignments.
- Working in groups of up to 2 people is permitted

Exams:

There will be an written-oral exam.

Contents:

1. Introduction to partial differential equations,
2. Parabolic-type problems,
3. Hyperbolic-type problems,
4. Elliptic-type problems

Script:

Course material would be available on the webpage in order to **support** the classroom teaching, **not to replace** it.

Additional organisational information, examples and explanations that may be relevant for your understanding and the exam are provided in the lectures.

It is solely **your** responsibility to make sure that you receive this infomation.

Literature:

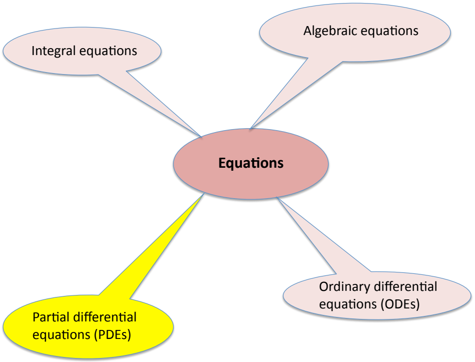
- L.C. Evans,
  Partial Differential Equations
  Amer. Math. Soc., Providence, RI (1998).
- M.A. Pinsky,
  Partial Differential Equations and Boundary-Value Problems with Applications.
  Amer. Math. Soc., Providence, RI, 2011
- S.J. Farlow, , Dover Publications, INC. New York, 1993.
  Partial Differential Equations for Scientists and Engineers.
  Dover Publications, INC. New York, 1993.

Chapter 1. Introduction to Partial Differential Equations

Purpose of Lesson:

- To show what PDEs are, why they are useful, and how they are solved.
- A brief discussion on how PDEs are classified as various kinds and types.

What are PDEs?



- Algebraic equations state relations between unknown number and its power, e.g.

$$x^3 - 7x^2 - 44x = 0.$$

Three solutions: $x_1 = 0$, $x_2 = -4$, $x_3 = 11$.

- Ordinary differential equations state relations between an unknown function of ONE variable and its derivatives, e.g.

$$u''(t) = 0.$$

Infinitely many solution: $u(t) = at + b \quad (a, b \in \mathbb{R})$.

- Partial differential equations state relations between an unknown function of SEVERAL variables and its partial derivatives, e.g.

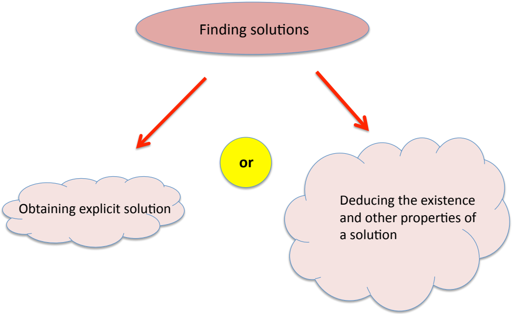
$$
\begin{aligned}
u_t(x,t) &= u_{xx}(x,t) \quad \text{(heat equation)}\\
u_t(x,y,t) &= u_{xx}(x,y,t) + u_{yy}(x,y,t) \quad \text{(heat equation)}\\
u_{tt}(x,y,z,t) &= u_{xx} + u_{yy} + u_{zz} \quad \text{(wave equation)}\\
u_{tt}(x,t) &= u_{xx} + \alpha u_t + \beta u \quad \text{(telegraph equation)}
\end{aligned}
$$

Examples of PDEs:

$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = 0$	Laplace's equation
$u_t - \Delta u = 0$	heat (or diffusion) equation
$u_{tt} - \Delta u = 0$	wave equation
$u_t - \Delta u - \sum_{i=1}^{n} \left(b^i u\right)_{x_i} = 0$	Fokker-Planck equation
$\operatorname{div}\left(\frac{Du}{\sqrt{1+\lvert Du\rvert^2}}\right) = 0$	minimal surface equation

How do you solve a PDE?

We solve the PDE if we find all functions verifying our PDE.


Finding solutions
or
Obtaining explicit solution
Deducing the existence and other properties of a solution

How do you solve a PDE?

The most important methods are those that change PDEs into ODEs. The useful techniques are:

- *Separation of Variables*. This technique reduce a PDE in n variables to n ODEs.
- *Integral Transforms*. This procedure reduces a PDE in n independent variables to one in $n-1$ variables; hence, a PDE in two variables could be changed to an ODE.
- *Eigenfunction Expansion*. This method attemts to find the solution of a PDE as an infinite sum of *eigenfunctions*. These eigenfunctions are found by solving what is known as an eigenvalue problem.
- *Numerical Methods*.

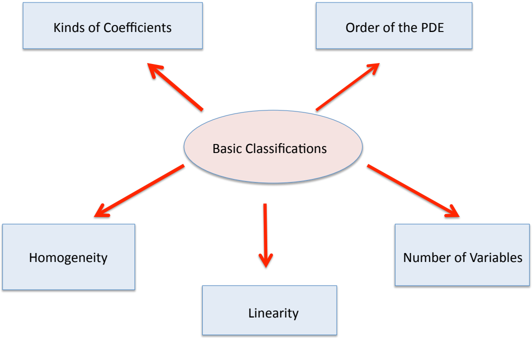
Remarks.

- There is no general theory known conerning the solvability of all PDEs.
- Such a theory is extremely unlikely to exist, given a rich variety of physical, geometric and probabilistic phenomena which can be modelled by PDEs.
- Instead, research focuses on various particular PDEs that are important for applications.

Kinds of PDEs.

PDE are classified according many things.

Classification is an important concept because the general theory and methods usually apply only to a given class of equations.


Kinds of Coefficients
Order of the PDE
Basic Classifications
Homogeneity
Linearity
Number of Variables

The basic classifications are:

- *Order of PDE*. The order of a PDE is the order of the highest partial derivative in the equation.
- *Number of Variables*. The number of variables is the number of independent variables.
- *Linearity*. PDEs are either linear or nonlinear. A PDE is said to be linear if it can be written in the form

$$\mathcal{L}u = g,$$

where g is a given function and $\mathcal{L}$ is a linear differential operator, i.e.,

$$\mathcal{L}(u+v) = \mathcal{L}u + \mathcal{L}v, \qquad \mathcal{L}(cu) = c\mathcal{L}u, \quad c \in \mathbb{R}.$$

- *Homogeneity*. A linear PDE is said to be homogeneous if it can be written in the form
$$\mathcal{L}u = g,$$
where g is a given function and $\mathcal{L}$ is a linear differential operator, and $g \equiv 0$. If a function g is not identically zero, the our linear PDE is called nonhomogeneous.
- *Kind of Coefficients*. If the coefficints of differential operator $\mathcal{L}$ are constants, then equation $\mathcal{L}u = g$ is said to have constant coefficients (otherwise, variable coefficients).

Remarks.

- PDE theory is (mostly) not restricted to two independent variables.
- Many interesting equations are nonlinear.

First we classify the 2nd order PDE in two independent variables which is linear with respect to its second-order derivatives:

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} + f(x,y,u,u_x,u_y) = 0. \qquad (1.1)$$

Here a, b, c, f are given differentiable functions.

- If $b^2 - 4ac < 0$, then PDE (1.1) is called elliptic.
- If $b^2 - 4ac = 0$, then PDE (1.1) is called parabolic.
- If $b^2 - 4ac > 0$, then PDE (1.1) is called hyperbolic.

Remark

All PDEs like (1.1) (in 2 independent variables!!!) are either

- parabolic
- elliptic
- hyperbolic.

Remark

In three and more dimensions, 2nd order PDEs can be one type in one pair of variables and of another type in other variables, e.g., there can occur elliptic-hyperbolic equations, ultra-hyperbolic equations, etc.

Remark

The nature of PDE depends only on the coefficients of the second order terms. First order terms and zero order terms do not play a role here.

Remark

The type of the 2nd order PDE can be different in different regions.

- Parabolic equations describe heat flow and diffusion processes.
- Hyperbolic equations describe vibrating systems and wave motion.
- Elliptic equations describe steady-state phenomena.

Normal forms of 2nd order PDEs in two independent variables:

Using a suitable transformation of independent variables

$$\xi = \xi(x, y), \qquad \eta = \eta(x, y)$$

we can always reduce equation

$$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} + f(x, y, u, u_x, u_y) = 0$$

to one of the following three NORMAL FORMs:

- for hyperbolic equations

$$u_{\xi\eta} = F(\xi, \eta, u, u_\xi, u_\eta), \quad \text{or} \quad u_{\xi\xi} - u_{\eta\eta} = F(\xi, \eta, u, u_\xi, u_\eta);$$

- for parabolic equations

$$u_{\eta\eta} = F(\xi, \eta, u, u_\xi, u_\eta),$$

  where F **must** depend on u_ξ: otherwise the equation degenerates into an ODE;

- for elliptic equations

$$u_{\xi\xi} + u_{\eta\eta} = F(\xi, \eta, u, u_\xi, u_\eta).$$

The classification (elliptic, parabolic etc.) can be extended to equations depending on more than 2 variables.

Consider the 2nd order PDE depeding on n variables,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu + g = 0. \tag{1.2}$$

The coefficient matrix (a_{ij}) should be symmetrized because

$$\frac{\partial^2}{\partial x_i \partial x_j} = \frac{\partial^2}{\partial x_j \partial x_i}, \qquad \text{for any} \quad i \text{ and } j \text{ in } [1, n].$$

The classification is as follows:

- hyperbolic for ($Z = 0$ and $P = 1$) or ($Z = 0$ and $P = n - 1$)
- parabolic for $Z > 0$ ($\Leftrightarrow \det(a_{ij}) = 0$)
- elliptic for ($Z = 0$ and $P = n$) or ($Z = 0$ and $P = 0$)
- ultra-hyperbolic for ($Z = 0$ and $1 < P < n - 1$)

where

$$\begin{aligned} Z &= \text{ number of zero eigenvalues } (a_{ij}), \\ P &= \text{ number of strictly positive eigenvalues of } (a_{ij}). \end{aligned}$$