PDE and Boundary-Value Problems
Winter Term 2014/2015

Lecture 10

Saarland University

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Purpose of Lesson

- With the help of the Laplace transform, to illustrate an important concept known as Duhamel's principle.
- To show how the term u_x in the diffusion equation represents the phenomenon of convection. With the help of the Laplace transform, to solve the convection problem.
- To derive the fundamental solution of the heat equation and discuss the corresponding solutions of homogeneous and nonhomogeneous IVPs

Heat Flow within a Rod with Temperature Fixed on the Boundaries

Quite often, it is important to find the temperature inside a medium due to time-varying boundary conditions. For example, consider an insulated rod with temperature specified as $f(t)$ on the right end.

Problem 10-1

PDE: $u_t = u_{xx}, \qquad 0 < x < 1, \quad 0 < t < \infty$

BCs: $\begin{cases} u(0,t) = 0 \\ u(1,t) = f(t) \end{cases}, \qquad 0 < t < \infty$

IC: $u(x,0) = 0, \qquad 0 \leqslant x \leqslant 1$

Heat Flow within a Rod with Temperature Fixed on the Boundaries

We may think that the solution to problem 10-1 can be easily found once we know the solution to the simpler version (constant temperature on the boundaries)

Problem 10-2

$$\text{PDE:} \quad w_t = w_{xx}, \qquad 0 < x < 1, \quad 0 < t < \infty$$

$$\text{BCs:} \quad \begin{cases} w(0,t) = 0 \\ w(1,t) = 1 \end{cases}, \qquad 0 < t < \infty$$

$$\text{IC:} \quad w(x,0) = 0, \qquad 0 \leqslant x \leqslant 1$$

Heat Flow within a Rod with Temperature Fixed on the Boundaries

Solving problems 10-1 and 10-2 at the same time we have

Easy Problem 10-2 (constant BCs)	**Hard Problem 10-1** (time-varying BCs)
Transform problem 10-2 by the Laplace transform $$\frac{d^2}{dx^2}W - sW(x) = 0$$ $$W(0) = 0$$ $$W(1) = 1/s$$	Transform problem 10-1 by the Laplace transform $$\frac{d^2}{dx^2}U - sU(x) = 0$$ $$U(0) = 0$$ $$U(1) = F(s)$$

Easy Problem 10-2 (cont.) (constant BCs)	**Hard Problem 10-1 (cont.)** (time-varying BCs)
Solve the ODE	Solve the ODE
$W(x,s) = \frac{1}{s}\left[\frac{\sinh\left(x\sqrt{s}\right)}{\sinh\left(\sqrt{s}\right)}\right]$	$U(x,s) = F(s)\left[\frac{\sinh\left(x\sqrt{s}\right)}{\sinh\left(\sqrt{s}\right)}\right]$
Find the inverse transform	Don't invert yet - first multiply and divide by s
$w(x,t) = x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-(n\pi)^2 t}\sin\left(n\pi x\right)$	$U = F(s)\left\{s\left[\frac{\sinh\left(x\sqrt{s}\right)}{s\sinh\left(\sqrt{s}\right)}\right]\right\}$

Easy Problem 10-2 (cont.) (constant BCs)	Hard Problem 10-1 (cont.) (time-varying BCs)
	Using the relationship $\mathcal{L}[w_t] = sW - w(x,0)$ we get $U(x,s) = F(s) \cdot \mathcal{L}[w_t]$ $\Downarrow$ $u(x,t) = \mathcal{L}^{-1}\{F(s) \cdot \mathcal{L}[w_t]\} = \mathcal{L}^{-1}[F(s)] * w_t = f(t) * w_t(t)$

In other words,

$$
\begin{aligned}
u(x,t) &= \int_0^t w_t(x,t-\tau)f(\tau)d\tau \\
&= \int_0^t w(x,t-\tau)f'(\tau)d\tau + f(0)w(x,t).
\end{aligned}
\tag{10.1}
$$

Remarks

- Equation (10.1) is known as Duhamel's principle.
- Often, even the easy problem (constant BCs) cannot be solved analytically. However, we can observe the solution experimentally.
- When we have this experimental data, we can solve the problem with arbitrary BCs using Duhamel's formula.

The Convection term u_x in the Diffusion Problems

- Consider now the problem of finding the concentration of a substance upwards from the surface of the earth where the substance both diffuses through the air and is carried upward (convected)by moving currents (moving with velocity V).
- Diffusion is mixing the substance through the air, while convection is the movement of the substance by means of the air (the movement of the medium).
- It is possible for the convection of the substance to contribute more of a movement in the substance than the diffusion itself.

Derivation of the Diffusion-Convection Equation

To derive this equation for a concentration $u(x, t)$ of a substance, we use two basic facts

1. Flux due to convection

   The flux of the material (from left to right) across a point due to convection is given by $Vu(x, t)$, where V is the linear velocity of the medium (cm/sec) and $u(x, t)$ is the linear concentration (g/sm)

2. Flux due to diffusion

   The flux of material (from left to the right) across a point due to diffusion is given by $-Du_x(x, t)$, where D is the diffusion coefficient

Derivation of the Diffusion-Convection Equation

If we substitute these two expressions into the conservation equation (see Lecture 4), we get the basic PDE

$$\boxed{u_t = Du_{xx} - Vu_x}$$

Solution of the convection equation

- First, we consider the problem that is pure convection (the diffusion term is zero).
- A typical problem would be dumping a substance into a clean river (moving with velocity V) and observing the concentration of the substance downstream.

Problem 10-3

PDE:	$u_t = -Vu_x,$	$0 < x < \infty,$	$0 < t < \infty$
BC:	$u(0,t) = P,$	$\leftarrow$ Constant input	of the substance
IC:	$u(x,0) = 0,$	$\leftarrow$ Initially a clean	river

Step 1. (Transformation)

- Using the Laplace transform we get an ODE in x

$$\text{ODE:} \quad sU(x) = -V\frac{d}{dx}U, \qquad 0 < x < \infty$$

$$\text{BC:} \qquad U(0) = \frac{P}{s},$$

Step 2. (Solving the problem for ODE)

$$U(x) = \frac{P}{s}e^{-(sx/V)}$$

Step 3. (Inverse transform)

$$u(x,t) = \mathcal{L}^{-1}[U] = PH(t - x/V) = \begin{cases} 0, & t < x/V \\ P, & t \geqslant x/V \end{cases}$$

Solution of the convection equation

- The situation becomes more interesting when the solute (pollutant) diffuses with the medium.

Problem 10-4

$$\text{PDE:} \quad u_t = Du_{xx} - Vu_x, \qquad -\infty < x < \infty$$

$$\text{IC:} \quad u(x,0) = 1 - H(x), \qquad -\infty < x < \infty$$

- In problem 10-4, we have moved the boundary to $-\infty$ (we have an IVP).
- To solve problem 10-4, we could use the Laplace transform in t or the Fourier transform in x. Alternative approach is introducing a new coordinate ξ.

So, our goal is to transform the IVP 10-4 into a new one in the moving coordinate system, solve it, and then transform back to get the solution in terms of the original coordinates (x, t).

Step 1. (Change of variables)

- In place of the old coordinates (x, t), we introduce new ones (ξ, τ)

$$\xi = x - Vt$$
$$\tau = t$$

- To rewrite the PDE in terms of (ξ, τ), we use the chain rule

$$u_t = u_\xi \xi_\tau + u_\tau \tau_t = -Vu_\xi + u_\tau$$
$$u_x = u_\xi \xi_x = u_\xi$$
$$u_{xx} = (u_\xi)_x = u_{\xi\xi}\xi_x = u_{\xi\xi}$$

- Substituting our computed u_t, u_x and u_{xx} into the PDE, we get the new IVP in terms of ξ and τ

Problem 10-5

$$\text{PDE:} \quad u_t = Du_{\xi\xi}, \qquad -\infty < \xi < \infty$$
$$\text{IC:} \quad u(\xi, 0) = 1 - H(\xi), \qquad -\infty < \xi < \infty$$

Step 2. (Solving the transformed problem)

- The problem 10-5 has already been solved in Lecture 8 by the Fourier transform and has the solution

$$u(\xi,\tau) = \frac{1}{2\sqrt{D\pi\tau}} \int_{-\infty}^{\infty} \phi(\beta) e^{-(\xi-\beta)^2/(4D\tau)} d\beta,$$

  where $\phi(\beta)$ is the initial condition.

- Hence, in our case, we have

$$u(\xi,\tau) = \frac{1}{2\sqrt{D\pi\tau}} \int_{-\infty}^{0} e^{-(\xi-\beta)^2/(4D\tau)} d\beta$$

Step 2. (Solving the transformed problem)

- By letting

$$\overline{\beta} = \frac{\xi - \beta}{2\sqrt{D\tau}} \qquad d\overline{\beta} = \frac{-1}{2\sqrt{D\tau}} d\beta$$

we get

$$\begin{aligned} u(\xi,\tau) &= \frac{1}{2}\left[\frac{2}{\sqrt{\pi}} \int\limits_{\frac{\xi}{(2\sqrt{D\tau})}}^{\infty} e^{-\overline{\beta}^2} d\overline{\beta}\right] \\ &= \begin{cases} \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{-\xi}{2\sqrt{D\tau}}\right)\right], \ \xi < 0 \\ \frac{1}{2}\operatorname{erfc}\left(\frac{\xi}{2\sqrt{D\tau}}\right), \ \xi \geqslant 0 \end{cases} \end{aligned} \tag{10.2}$$

Step 3. (Change to the original variables)

- Finally, to get the solution of problem 10-4 in terms of the coordinates x and t, we substitute

$$\xi = x - Vt$$
$$\tau = t$$

into equation (10.2) to get

$$u(x,t) = \begin{cases} \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{Vt - x}{2\sqrt{D\tau}}\right)\right], & Vt > x \\ \operatorname{erfc}\left(\frac{x - Vt}{2\sqrt{D\tau}}\right), & Vt \leqslant x \end{cases}$$