PDE and Boundary-Value Problems Winter Term 2014/2015

Lecture 11

Saarland University

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Purpose of Lesson

- To derive the fundamental solution of the heat equation and discuss the corresponding solutions of homogeneous and nonhomogeneous IVPs.

Fundamental solution of the heat equation:

Problem 11-1

To find the function $u(x, t)$ that satisfies

$$\text{PDE:} \quad u_t = \Delta u, \qquad x \in \mathbb{R}^n, \quad 0 < t < \infty$$

$$\text{IC:} \quad u(x, 0) = u_0, \qquad x \in \mathbb{R}^n$$

We will solve problem 11-1 by applying the exponential Fourier transform with respect the spatial variables x.

We define

$$U(t, \xi) = \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} u(t, x) e^{-i\xi \cdot x} dx, \quad \xi \in \mathbb{R}^n.$$

Step 1. (Transformation)

- We take n Fourier transforms with respect to all x_i-variables. As a result we get the following ODE in t

$$
\begin{aligned}
\text{ODE:} \quad & U_t(t) + |\xi|^2 U(t) = 0, \qquad 0 < t < \infty \\
\text{IC:} \quad & U(0) = \mathcal{F}[u_0]
\end{aligned}
\tag{11.1}
$$

Step 2. (Solving the transformed problem)

- Remember the new variable ξ is nothing more than a constant vector in this differential equation. So, the solution to problem (11.1) is

$$U(t, \xi) = \mathcal{F}[u_0](\xi) e^{-|\xi|^2 \cdot t}.$$

Step 3. (Finding the inverse transform)

- We compute

$$u(x,t) = \mathcal{F}^{-1}\left[U(t,\xi)\right] = \mathcal{F}^{-1}\left[\mathcal{F}[u_0](\xi)e^{-|\xi|^2\cdot t}\right]$$

- Due to the convolution property we can write

$$\begin{aligned} u(x,t) &= \mathcal{F}^{-1}\left[\mathcal{F}[u_0](\xi)\right] * \mathcal{F}^{-1}\left[e^{-|\xi|^2\cdot t}\right] \\ &= u_0(x) * \left[\frac{1}{(2t)^{n/2}}e^{-|x|^2/(4t)}\right] \\ &= \frac{1}{(4\pi t)^{n/2}}\int\limits_{\mathbb{R}^n} e^{-|x-y|^2/(4t)}u_0(y)dy \end{aligned}$$

Fundamental solution

The function

$$\Phi(x,t) := \begin{cases} \dfrac{1}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)}, & x \in \mathbb{R}^n,\ t > 0 \\ 0, & x \in \mathbb{R}^n,\ t < 0 \end{cases}$$

is called the fundamental solution of the heat equation.

The fundamental solution has the following properties:

- Φ is singular at the point $(0,0)$
- $\Phi(x,t) = \Phi(|x|,t)$, i.e., the fundamental solution is radial in the variable x.
- For each time $t > 0$ we have $\int\limits_{\mathbb{R}^n} \Phi(x,t)dx = 1$.
- $\Phi_t = \Delta\Phi, \quad x \in \mathbb{R}^n,\ t > 0.$

Remarks

- If u_0 is bounded, continuous, $u_0 \geqslant 0$, and $u_0 \not\equiv 0$, then

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-|x-y|^2/(4t)} u_0(y)dy$$

  is in fact positive for all points $x \in \mathbb{R}^n$ and times $t > 0$.

- We interpret the above observation by saying the heat equation forces infinite propagation speed for disturbances.

- If the initial temperature is nonnegative and is positive somewhere, the temperature at any later time is everywhere positive.

Now let us turn our attention to the nonhomogeneous IVP.

Problem 11-2

To find the function $u(x,t)$ that satisfies

$$\text{PDE:} \quad u_t - \Delta u = f, \qquad x \in \mathbb{R}^n, \quad 0 < t < \infty$$

$$\text{IC:} \quad u(x,0) = 0, \qquad x \in \mathbb{R}^n$$

Solving of problem 11-2:

- First of all, we recall that the mapping

$$(x,t) \mapsto \Phi(x-y, t-s)$$

is a solution of the heat equation (for given $y \in \mathbb{R}^n$, $0 < s < t$).

- Now for fixed s, the function

$$u = u(x,t,s) = \int\limits_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s) dy$$

solves the problem

$$\begin{cases} u_t(\cdot, s) - \Delta u(\cdot, s) = 0 \quad \text{in } \mathbb{R}^n \times (s, \infty) \\ \qquad\qquad\qquad u(\cdot, s) = f(\cdot, s) \text{ on } \mathbb{R}^n \times \{t = s\}. \end{cases} \tag{11.1s}$$

Solving of problem 11-2 (cont.):

- Observe that (11.1s) is an IVP of the form 11-1, with the starting time $t = 0$ replaced by $t = s$, and u_0 replaced by $f(\cdot, s)$.
- Duhamel's principle asserts that we can build a solution of problem 11-2 out of the solutions of (11.1s), by integrating with respect to s. The idea is to consider

$$u(x,t) = \int_0^t u(x,t,s)ds.$$

Solving of problem 11-2 (cont.):

- Rewriting, we have

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s) ds$$
$$= \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s) dyds,$$

for $x \in \mathbb{R}^n$, $t > 0$.

Combining the solutions of problems 11-1 and 11-2, we discover that

$$u(x,t) = \int\limits_{\mathbb{R}^n} \Phi(x-y,t)u_0(y)dy + \int\limits_0^t \int\limits_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)ds$$

is a solution of the nonhomogeneous problem

Problem 11-3

$$\text{PDE:} \quad u_t - \Delta u = f, \qquad x \in \mathbb{R}^n, \quad 0 < t < \infty$$

$$\text{IC:} \quad u(x,0) = u_0, \qquad x \in \mathbb{R}^n$$

Assume $U \subset \mathbb{R}^n$ is open and bounded, and fix a time $T > 0$.

Remarks

- We define the parabolic cylinder
$$U_T := U \times (0, T].$$
- The parabolic boundary of U_T is
$$\partial' U_T := \overline{U_T} \setminus U_T.$$
- We note that U_T includes the top $U \times \{t = T\}$. The parabolic boundary $\partial' U_T$ comprises the bottom and vertical sides of $U \times [0, T]$, but not the top.

Properties of solutions to the heat equation

1. (Strong maximum principle)

Assume $u \in C^{2,1}(U_T) \cap C(\overline{U_T})$ solves the heat equation in the parabolic cylinder U_T.

(i) Then

$$\max_{\overline{U_T}} u = \max_{\partial' U_T} u.$$

(ii) Furthermore, if U is connected and there exists a point $(x_0, t_0) \in U_T$ such that

$$u(x_0, t_0) = \max_{\overline{U_T}} u,$$

then

$$u \text{ is a constant in } \overline{U_{T_0}}.$$

Remarks

- Assertion (i) is the maximum principle for the heat equation and (ii) is the strong maximum principle.
- Similar assertions are valid with „min“ replacing „max“.
- So if u attains its maximum (or minimum) at the interior point, then u is constant at all earlier times. The solution may change at times $t > t_0$, provided the boundary conditions alter after t_0.

Properties of solutions to the heat equation (cont.)

2. (Uniqueness on bounded domains)

Let $g \in C(\partial' U_T)$ and $f \in C(U_T)$. Then there exists at most one solution

$$u \in C^{2,1}(U_T) \cap C(\overline{U_T})$$

of the problem

$$\begin{cases} u_t - \Delta u = f & \text{in } U_T \\ u = g & \text{on } \partial' U_T. \end{cases}$$

Properties of solutions to the heat equation (cont.)

3. (Smoothness)

Suppose $u \in C^{2,1}(U_T)$ solves the heat equation in U_T. Then

$$u \in C^{\infty}(U_T).$$

Remark

- The regularity assertion is valid even if u attains nonsmooth boundary values on $\partial' U_T$.