PDE and Boundary-Value Problems
Winter Term 2014/2015

Lecture 13

Saarland University

5. Januar 2015

Purpose of Lesson

- To interpretate the D'Alembert solution in the *xt*-plane.
- To illustrate how the D'Alembert solution can be used to find the wave motion of a semi-infinite-string problem.
- To illustrate how the boundary conditions is generally associated with the wave equation.

The Space-Time Interpretation of D'Alembert's Solution

We present an interpretation of the D'Alembert solution

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int\limits_{x-ct}^{x+ct} g(s)ds$$

in the *xt*-plane looking at two specific cases.

Case 1. (Initial position given; initial velocity zero)

- Suppose the string has initial conditions

$$u(x,0) = f(x)$$
$$u_t(x,0) = 0$$

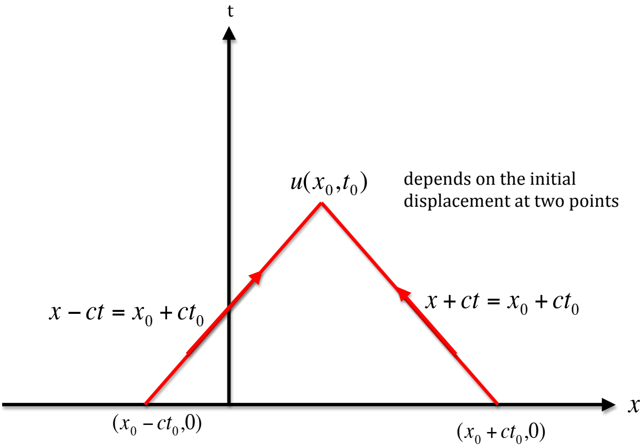
Here, the D'Alembert solution is

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right].$$

- The solution u at a point (x_0, t_0) can be interpreted as being the average of the initial displacement $f(x)$ at the points $(x_0 - ct_0, 0)$ and $(x_0 + ct_0, 0)$ found by backtracking along the lines (characteristic curves)

$$x - ct = x_0 - ct_0$$
$$x + ct = x_0 + ct_0$$

Fig.13.1 Interpretation of $u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right]$ in the xt-plane



For example, using this interpretation, the IVP

Problem 13-1

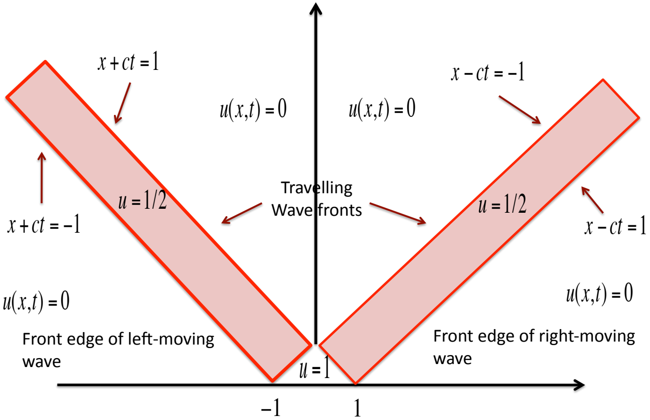
To find the function $u(x,t)$ that satisfies

$$\text{PDE:} \qquad u_{tt} = c^2 u_{xx}, \qquad \begin{array}{c} -\infty < x < \infty, \\ 0 < t < \infty \end{array}$$

$$\text{ICs:} \quad \begin{cases} u(x,0) = \begin{cases} 1, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases} \\ u_t(x,0) = 0 \end{cases} \qquad -\infty < x < \infty$$

would give us the solution in the xt-plane shown in Fig. 13.2.

Fig. 13.2 Solution of problem 13-1 in the *xt*-plane



Case 2. (Initial displacement zero; velocity arbitrary)

- Consider now the ICs

$$u(x,0) = 0$$
$$u_t(x,0) = g(x)$$

Here, the D'Alembert solution is

$$u(x,t) = \frac{1}{2c} \int\limits_{x-ct}^{x+ct} g(s)ds$$

- Hence, the solution u at (x_0, t_0) can be interpreted as integrating the initial velocity between $x_0 - ct_0$ and $x_0 + ct_0$ on the initial line $t = 0$.

Again, using this interpretation, the solution to the IVP

Problem 13-2

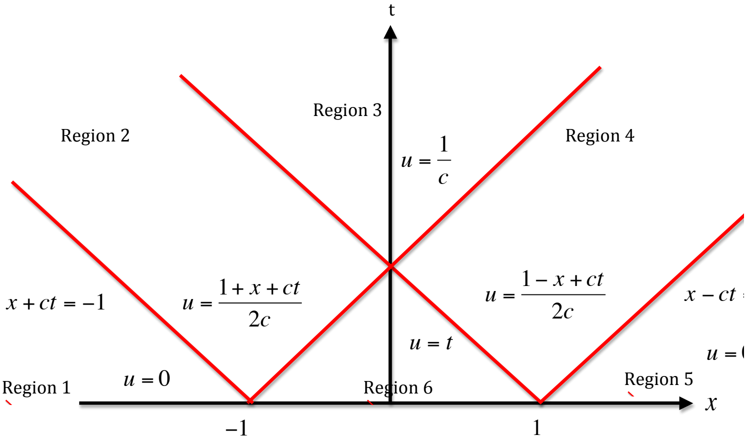
To find the function $u(x,t)$ that satisfies

$$\text{PDE:} \qquad u_{tt} = c^2 u_{xx}, \qquad \begin{array}{c} -\infty < x < \infty, \\ 0 < t < \infty \end{array}$$

$$\text{ICs:} \quad \begin{cases} u(x,0) = 0 \\ u_t(x,0) = \begin{cases} 1, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases} \end{cases} \qquad -\infty < x < \infty$$

has a solution in the *xt*-plane illustrated in Fig. 13.3.

Fig. 13.3 Solution of problem 13-3 in the *xt*-plane



Problem 13-2 corresponds to imposing an initial impulse (velocity $= 1$) on the string for $-1 < x < 1$ and watching the resulting wave motion (as in the piano string).

The solution is graphed at various values of times in Figures 13.4-13.4a.

Fig. 13.4 Solution of problem 13-2 for various values of time

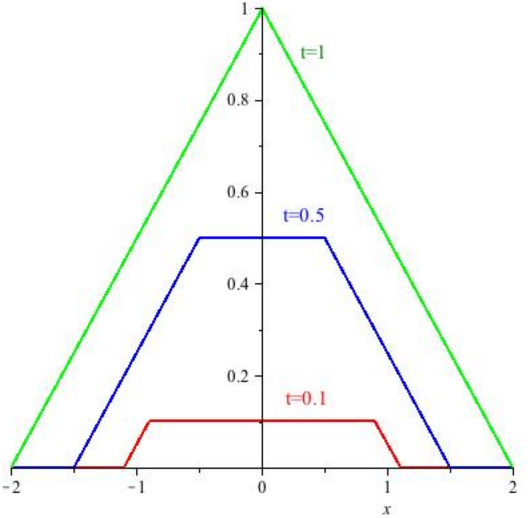
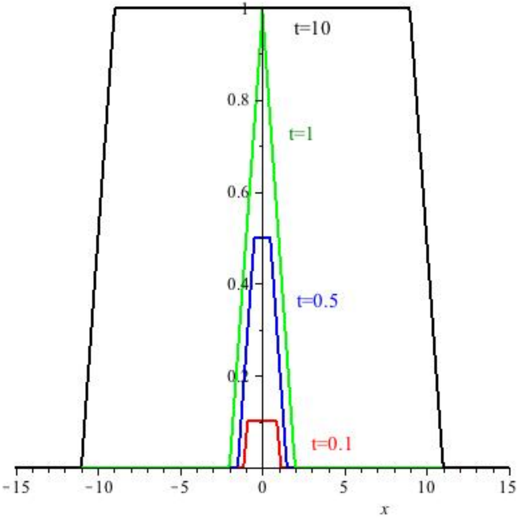


Fig. 13.4a Solution of problem 13-2 for various values of time



Solution of the Semi-infinite String via the D'Alembert Formula

We will solve the IBVP for the semi-infinite string

Problem 13-3

To find the function $u(x,t)$ that satisfies

$$\text{PDE:} \quad u_{tt} = c^2 u_{xx}, \qquad 0 < x < \infty, \quad 0 < t < \infty$$

$$\text{BC:} \quad u(0,t) = 0, \qquad 0 < t < \infty$$

$$\text{ICs:} \quad \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} \qquad 0 < x < \infty$$

by modifying the D'Alembert formula. To find the solution of problem 13-3, we proceed in a manner similar to that used with the infinite string.

- We find the general solution to the PDE

$$u(x,t) = \phi(x-ct) + \psi(x+ct).$$

- Substituting this general solution into ICs we arrive at

$$\begin{aligned} \phi(x-ct) &= \frac{1}{2} f(x-ct) - \frac{1}{2c} \int\limits_{x_0}^{x-ct} g(s) ds \\ \psi(x+ct) &= \frac{1}{2} f(x+ct) + \frac{1}{2c} \int\limits_{x_0}^{x+ct} g(s) ds \end{aligned} \tag{13.1}$$

We now have a problem that we didn't encounter when dealing with the infinite string.

- Since we are looking for the solution $u(x, t)$ for $x > 0$ and $t > 0$, it is obvious that we must find

$$\begin{aligned} &\phi(x - ct) \quad \forall \quad -\infty < x - ct < \infty \\ &\psi(x + ct) \quad \forall \quad 0 < x + ct < \infty. \end{aligned}$$

- Unfortunately, the first equation in (13.1) only gives us $\phi(x - ct)$ for $x - ct \geqslant 0$, since our initial data $f(x)$ and $g(x)$ are only known for positive arguments.

- As long as $x - ct \geqslant 0$, we have

$$\begin{aligned} u(x,t) &= \phi(x-ct) + \psi(x+ct) \\ &= \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int\limits_{x-ct}^{x+ct} g(s)ds. \end{aligned}$$

- The question is, what to do when $x < ct$?

- When $x < ct$, we use our BC. Substituting the general solution u into the BC $u(0,t) = 0$ gives

$$\phi(-ct) = -\psi(ct)$$

- Hence, by functional substitution

$$\phi(x - ct) = -\psi(ct - x) = -\frac{1}{2} f(ct - x) - \frac{1}{2c} \int\limits_{x_0}^{ct-x} g(s)ds$$

- Substituting this value of ϕ into the general solution gives

$$u(x,t) = \frac{1}{2}\left[f(x + ct) - f(ct - x)\right] + \frac{1}{2c} \int\limits_{ct-x}^{x+ct} g(s)ds \quad 0 < x < ct.$$

- Combining the solutions for $x < ct$ and $x > ct$ we have our result

$$\boxed{u(x,t) = \begin{cases} \dfrac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \dfrac{1}{2c}\displaystyle\int\limits_{x-ct}^{x+ct} g(s)ds, & x \geqslant ct \\ \dfrac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \dfrac{1}{2c}\displaystyle\int\limits_{ct-x}^{x+ct} g(s)ds, & x < ct \end{cases}}$$

Remarks

- Solution of problem 13-3 would not be the same if the BC $u(0,t) = 0$ were changed. Solutions can also be found with other BCs, such as

$$u(0,t) = f(t) \quad \text{or} \quad u_x(0,t) = 0.$$

- The straight lines

$$\begin{aligned} x + ct &= \text{ constant} \\ x - ct &= \text{ constant} \end{aligned}$$

are known as characteristics, and it is along these lines that disturbances are propagated. Characteristics are generally asociated with hyperbolic equations.

Boundary Conditions Associated with the Wave Equation

- We have discussed the one-dimensional transverse vibrations of a string. A few other types of important vibrations are:
  1. Sound waves (longitudinal waves)
  2. Electromagnetic waves of light and electricity
  3. Vibrations in solids (longitudinal, transverse, and torsional)
  4. Probability waves in quantum mechanics
  5. Water waves (transverse waves)
  6. Vibrating string (transverse waves)
- We will discuss some of the various types of BCs that are associated with physical problems of this kind.

We will stick to one-dimensional problems where the BCs (linear ones) are generally grouped in to one of three kinds:

1. Controlled end points (first kind)

$$u(0,t) = g_1(t)$$
$$u(L,t) = g_2(t)$$

2. Force given on the boundaries (second kind)

$$u_x(0,t) = g_1(t)$$
$$u_x(L,t) = g_2(t)$$

3. Elastic attachment on the boundaries (third kind)

$$u_x(0,t) - \gamma_1 u(0,t) = g_1(t)$$
$$u_x(L,t) - \gamma_2 u(L,t) = g_2(t)$$

1. Controlled End Points

We are now involved with problems like

Problem 13-4

To find the function $u(x,t)$ that satisfies

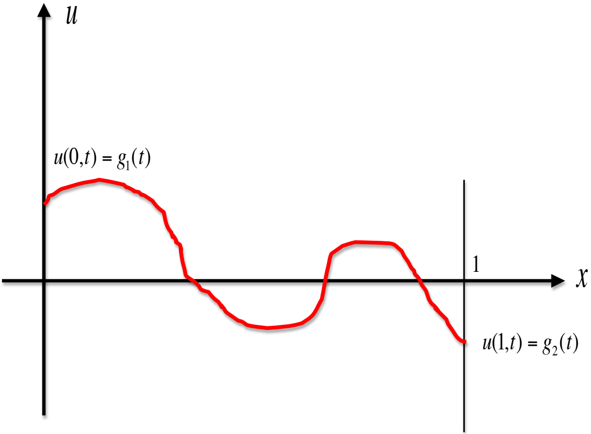
$$\text{PDE:} \qquad u_{tt} = c^2 u_{xx}, \qquad 0 < x < 1, \quad 0 < t < \infty$$

$$\text{BCs:} \quad \begin{cases} u(0,t) = g_1(t) \\ u(1,t) = g_2(t) \end{cases} \quad 0 < t < \infty$$

$$\text{ICs:} \quad \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} \quad 0 \leqslant x \leqslant 1$$

where we control the end points so that they move in a given manner.

Fig. 13.5 Controlling the ends of a vibrating string

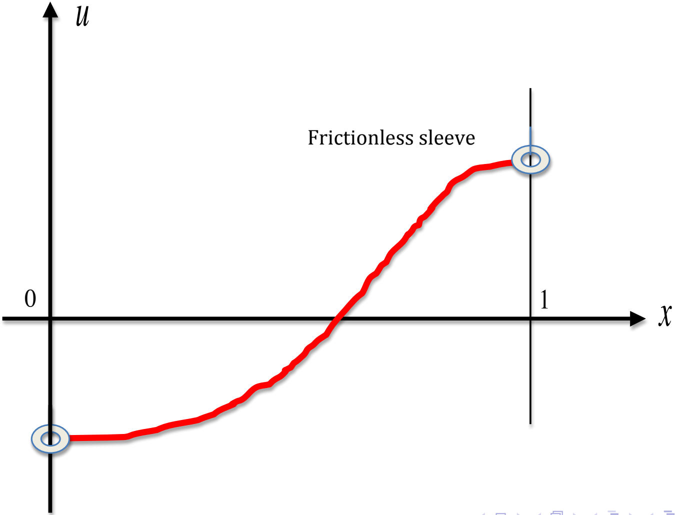


$\boxed{2.}$ Force Given on the Boundaries

Inasmuch as the vertical forces on the string at the left and right ends are given by $Tu_x(0, t)$ and $Tu_x(L, t)$, respectively, by allowing the ends of the string to slide vertically on frictionless, the BCs become

$$\begin{aligned} u_x(0, t) &= 0 \\ u_x(L, t) &= 0 \end{aligned} \tag{13.2}$$

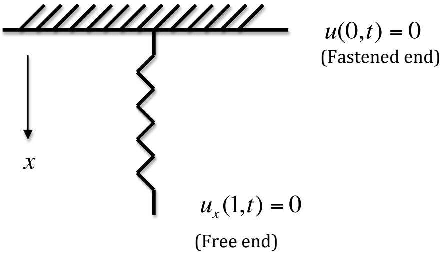
Fig. 13.6 Free BC on the string



BCs similar to (13.2) are presented in the following two examples:

$\boxed{a)}$ Free end of a longitudinally vibrating spring

Consider a vibrating spring with the bottom end unfastened



$\boxed{b)}$ Forced end of a vibrating spring

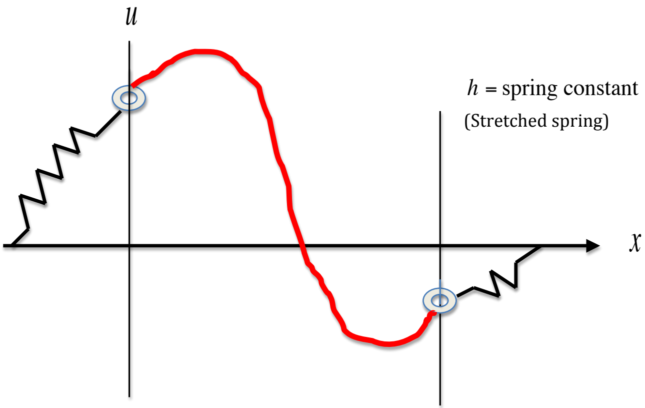
- If a force of $v(t)$ dynes is applied at the end $x = 1$ (a positive force is measured downward), then the BC would be

$$u_x(1, t) = \frac{1}{k} v(t) \qquad (k \text{ is Young's modulus})$$

- In the case of a forced BC, the ends of the string (or spring) are not required to maintain a given position, but the force that's applied tends to move the boundaries in the given direction.

3. Elastic Attachment on the Boundaries

Consider finally a violin string whose ends are attached to an elastic arrangement



$\boxed{3.}$ Elastic Attachment on the Boundaries

The spring attachments at each end give rise to vertical forces proportional to the displacements

$$\begin{aligned} \text{Displacement at the left end} &= u(0,t) \\ \text{Displacement at the right end} &= u(L,t) \end{aligned}$$

Setting the vertical tensions of the spring at the two ends

$$\left.\begin{aligned} \text{Upward tension at the left end} &= Tu_x(0,t) \\ \text{Upward tension at the right end} &= -Tu_x(L,t) \end{aligned}\right. \qquad (T = \text{string tension})$$

equal to these displacements (multiplied by the spring constant h) gives us our desired BCs:

$\boxed{3.}$ Elastic Attachment on the Boundaries

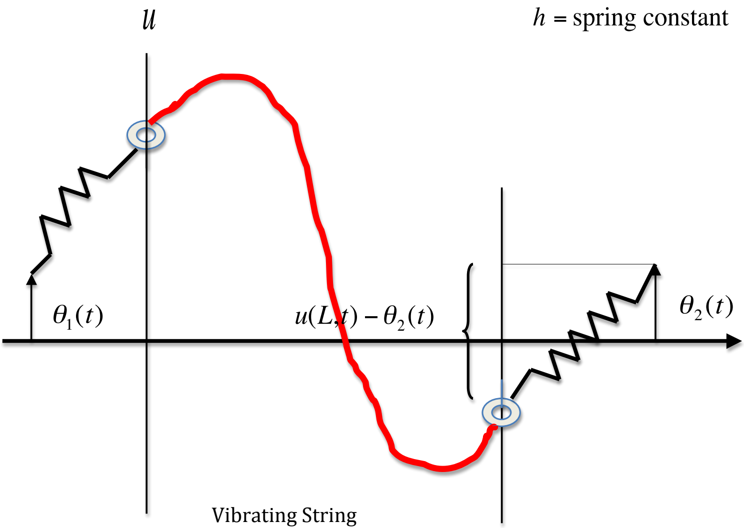
$$
\begin{aligned}
u_x(0,t) &= \frac{h}{T}u(0,t)\\
u_x(L,t) &= -\frac{h}{T}u(L,t)
\end{aligned}
$$

Remark

Note that $u(0,t)$ positive means that $u_x(0,t)$ is positive, while if $u(L,t)$ is positive, then $u_x(L,t)$ is negative.

If the two spring attachments are displaced according to the functions $\theta_1(t)$ and $\theta_2(t)$, we would have the nonhomogeneous BCs

$$u_x(0,t) = \frac{h}{T}\left[u(0,t) - \theta_1(t)\right]$$
$$u_x(L,t) = -\frac{h}{T}\left[u(L,t) - \theta_2(t)\right]$$



Vibrating String

Remarks

- Another BC not discussed today occurs when the vibrating string experiences a force at the ends proportional to the string velocity (and in the opposite direction). Here, we have the BC (at the left end)

$$Tu_x(0,t) = -\beta u_t(0,t)$$

- A nonlinear elastic attachment at the left end of the string would be

$$Tu_x(0,t) = \phi\left[u(0,t)\right]$$

where $\phi(u)$ is an arbitrary function of u; for example

$$Tu_x(0,t) = -hu^3(0,t)$$

Remarks (cont.)

- If a mass m is attached to the lower end of a longitudinally vibrating string, the BC would be

$$mu_{tt}(L,t) = -ku_x(L,t) + mg$$