PDE and Boundary-Value Problems
Winter Term 2014/2015

Lecture 14

Saarland University

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Purpose of Lesson

- To show how transverse vibrations of a finite string can be found by the standard technique of separation of variables and to show how the solution $u(x, t)$ can be interpreted as the infinite sum

$$u(x, t) = \sum_{n=1}^{\infty} X_n(x) T_n(t)$$

of simple vibrations where the shape $X_n(x)$ of these fundamental vibrations are solutions (eigenfunctions) of a certain Sturm-Liouville BVP.

Purpose of Lesson (cont.)

- To illustrate how higher-order PDEs come about in the study of vibrating-beam problems.
- To solve the problem of a vibrating beam with simply supported ends by separation of variables.
- To compare the vibrations of the beam with the vibrations of the violin string.

The Finite Vibrating String (Standing Waves)

- So far, we have studied the wave equation $u_{tt} = c^2 u_{xx}$ for the unbounded domain $-\infty < x < \infty$ and have found (D'Alembert's solution) solutions to be certain travelling waves (moving in opposite directions).
- When we study the same wave equation in a bounded region of space $0 < x < L$, we find that the waves no longer appear to be moving due to their repeated interaction with the boundaries and, in fact, often appear to be what are known as standing waves.

Consider what happens when a guitar string (fixed at both ends $x = 0, L$) described by the simple hyperbolic IBVP

Problem 14-1

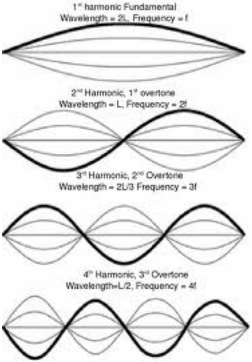
To find the function $u(x,t)$ that satisfies

$$\text{PDE:} \qquad u_{tt} = c^2 u_{xx}, \qquad 0 < x < L, \quad 0 < t < \infty$$

$$\text{BCs:} \qquad \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases} \qquad 0 < t < \infty$$

$$\text{ICs:} \qquad \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} \qquad 0 \leqslant x \leqslant L$$

What happens is that the travelling-wave solution to the PDE and IC keeps reflecting from the boundaries in such a way that the wave motion does not appear to be moving, but, in fact, appears to be vibrating in one position.



If we knew the shapes $X_n(x)$ of these standing waves and how each one of them vibrated $T_n(t)$, then all we would have to do to find the solution of the vibrating guitar string is sum the simple vibrations $X_n(x)T_n(t)$

$$u(x,t) = \sum_{n=1}^{\infty} c_n X_n(x) T_n(t)$$

in such a way (find the coefficients c_n) that the sum agrees with the ICs

$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$

Separation-of-Variables Solution to the Finite Vibrating String

We solve problem 14-1 by breaking it into several steps:

Step 1. (Separation of Variables)

- We start by seeking solutions to the PDE of the form

$$u(x,t) = X(x)T(t)$$

- Substituting this expression into the wave equation and separating variables gives us the two ODEs

$$\begin{aligned} T'' - c^2\lambda T &= 0 \\ X'' - \lambda X &= 0 \end{aligned}$$

where the constant λ can now be any number $-\infty < \lambda < \infty$.

Step 2. (Solving ODEs)

Possible values of λ

$\lambda < 0$
$(\lambda = -\beta^2)$

$\lambda = 0$

$\lambda > 0$
$(\lambda = \beta^2)$

$$T(t) = A\sin(c\beta t) + B\cos(c\beta t)$$
$$X(x) = C\sin(\beta x) + D\cos(\beta x)$$

$$T(t) = At + B$$
$$X(x) = Cx + D$$

$$T(t) = Ae^{c\beta t} + Be^{-c\beta t}$$
$$X(x) = Ce^{\beta x} + De^{-\beta x}$$

$$u(x,t) = X(x)T(t)$$

Step 3. (Substituting into BCs)

- The idea now is to prune away all those standing waves that either are unbounded as $t \to \infty$ or else yield only the zero solution when substituted into the BCs.
- Only negative values of λ give nonzero and bounded solutions. Hence,

$$u(x,t) = [C\sin(\beta x) + D\cos(\beta x)]\,[A\sin(c\beta t) + B\cos(c\beta t)]$$

- Substitution expression of u into BCs gives

$$D = 0$$
$$\beta_n = \frac{\pi n}{L}, \quad n = 0, 1, 2 \dots$$

Step 3. (Substituting into BCs)

- We have found a sequence of simple vibrations (which we subscript with n)

$$\begin{aligned} u_n(x,t) &= \sin\left(\frac{n\pi x}{L}\right)\left[a_n \sin\left(\frac{n\pi ct}{L}\right) + b_n \cos\left(\frac{n\pi ct}{L}\right)\right] \\ &= R_n \sin\left(\frac{n\pi x}{L}\right)\cos\left[\frac{n\pi c(t-\delta_n)}{L}\right], \end{aligned}$$

  where the constants a_n, b_n, R_n and δ_n are arbitrary. These simple vibrations satsify the wave equations and the BCs.

Step 4. (Substituting into ICs)

- Since any sum of these vibrations is also a solution to the PDE and BCs (since PDE and BCs are linear and homogeneous), we add them together in such a way that the resulting sum also agrees with the ICs.

- Substituting the sum

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[a_n \sin\left(\frac{n\pi ct}{L}\right) + b_n \cos\left(\frac{n\pi ct}{L}\right)\right]$$

into the ICs gives the two equations

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

$$\sum_{n=1}^{\infty} a_n \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi x}{L}\right) = g(x)$$

Step 4. (Substituting into ICs)

- Using the orthogonality condition

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0, & m \neq n \\ \frac{L}{2}, & m = n \end{cases}$$

we can find the coefficients a_n and b_n

$$\begin{aligned} a_n &= \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \\ b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \end{aligned} \tag{14.1}$$

Step 4. (Substituting into ICs)

- The solution is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[a_n \sin\left(\frac{n\pi ct}{L}\right) + b_n \cos\left(\frac{n\pi ct}{L}\right)\right],}$$

where the coefficients a_n and b_n are given by (14.1).

Remarks

- If the initial velocity of the string is zero, then the solution takes the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right)$$

and has the following interpretation. Suppose we break the initial string position into simple sine components

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

and let each sine term vibrate on its own according to

$$u_n(x,t) = b_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right)$$

Remarks (cont.)

- If we now add each individual vibration of the type (this is a fundamental vibration)

$$u_n(x,t) = b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right),$$

  we will get the solution of our problem.

- The *n*-th term in the solution

$$\sin\left(\frac{n\pi x}{L}\right)\left[a_n \sin\left(\frac{n\pi ct}{L}\right) + b_n \cos\left(\frac{n\pi ct}{L}\right)\right]$$

  is called *n*-th mode of vibration or the *n*-th harmonic.

Remarks (cont.)

- By using a trigonometric identity, we can write this harmonic as

$$R_n \sin\left(\frac{n\pi x}{L}\right)\cos\left[\frac{n\pi c(t-\delta_n)}{L}\right],$$

  where R_n and δ_n are the new arbitrary constants (amplitude and phase angle). This new form of the n-th mode is more useful for analyzing the vibrations.

- The frequency ω_n (rad / sec) of the n-th mode is

$$\omega_n = \frac{n\pi c}{L} = \frac{n\pi}{L}\sqrt{\frac{T}{\rho}}$$

  (T, ρ are tension and density of the string, respectively).

Remarks (cont.)

- The frequency ω_n is n times the fundamental frequency ω_1.

$$\omega_n = n \cdot \omega_1$$

- The property that all sound frequencies are multiples of a basic one is not shared by all types of vibrations.

  This has something to do with the pleasing sound of a violin or guitar string in contrast to a drumhead, where the higher-order frequencies are not multiple frequencies of the fundamental one.

The Vibrating Beam (Fourth-Order PDE)

- The major difference between the transverse vibrations of a violin string and the transverse vibrations of a thin beam is that the beam offers resistance to bending.
- The resistance is responsible for changing the wave equation to the fourth-order beam equation

$$u_{tt} = -\alpha^2 u_{xxxx},$$

where

$$\begin{aligned} \alpha^2 &= K/\rho \\ K &= \text{rigidity constant} \\ \rho &= \text{linear density of the beam} \end{aligned}$$

The Simply Supported Beam

- Consider the small vibrations of a thin beam whose ends are simply fastened to two foundations.
- By „simply fastened", we mean that the ends of the beam are held stationary, but the slopes at the end points can move (the beam is held by a pin-type arrangement).
- It seems clear that the BCs at the ends of the beam should be

$$u(0,t) = 0$$
$$u(1,t) = 0$$

but what isn't so obvious is that the two BCs

$$u_{xx}(0,t) = 0$$
$$u_{xx}(1,t) = 0$$

also hold at the two ends.

The Simply Supported Beam

- Hence, the vibrating beam can be described by the following IVBP (α is set equal to one for simplicity)

Problem 14-2

To find the function $u(x, t)$ that satisfies

$$\text{PDE:} \qquad u_{tt} = -u_{xxxx}, \qquad 0 < x < 1, \quad 0 < t < \infty$$

$$\text{BCs:} \quad \begin{cases} u(0, t) = 0 \\ u_{xx}(0, t) = 0 \\ u(1, t) = 0 \\ u_{xx}(1, t) = 0 \end{cases} \qquad 0 < t < \infty$$

$$\text{ICs:} \quad \begin{cases} u(x, 0) = f(x) \\ u_t(x, 0) = g(x) \end{cases} \qquad 0 \leqslant x \leqslant 1$$

To solve problem 14-2, we use the separation of variables method and look for arbitrary periodic solutions; that is, vibrations of the form

$$u(x,t) = X(x)\left[A\sin(\omega t) + B\cos(\omega t)\right]. \tag{14.2}$$

Remark

By choosing the solution in the form (14.2), we are essentially saying that the separation constant in the separation of variables method has been chosen to be negative.

Step 1. (Separation of Variables)

- Substituting

$$u(x,t) = X(x)\left[A\sin(\omega t) + B\cos(\omega t)\right]$$

into the beam equation to get the ODE in $X(x)$

$$X^{(iv)} + \omega^2 X = 0$$

which has the general solution

$$X(x) = C\cos(\sqrt{\omega}x) + D\sin(\sqrt{\omega}x) + E\cosh(\sqrt{\omega}x) + F\sinh(\sqrt{\omega}x)$$

Step 2. (Substituting into BCs)

- Calculation of $X''(x)$ gives

$$X''(x) = -\omega C \cos(\sqrt{\omega} x) - \omega D \sin(\sqrt{\omega} x) + \omega E \cosh(\sqrt{\omega} x) + \omega F \sinh(\sqrt{\omega} x)$$

- Further, substitution of the expression for u into the BCs provides

$$\left.\begin{aligned} u(0,t) &= T(t)\,[C+E] = 0 \\ u_{xx}(0,t) &= T(t)\,[-\omega C + \omega E] = 0 \end{aligned}\right. \quad \Rightarrow \quad \boxed{C = E = 0}.$$

$$\begin{aligned} u(1,t) &= T(t)\left[D \sin(\sqrt{\omega}) + F \sinh(\sqrt{\omega})\right] = 0 \\ u_{xx}(1,t) &= T(t)\left[-\omega D \sin(\sqrt{\omega}) + \omega F \sinh(\sqrt{\omega})\right] = 0 \end{aligned}$$

$$\Rightarrow \quad \boxed{\begin{array}{c} F = 0 \\ \sin(\sqrt{\omega}) = 0 \end{array}}$$

Step 2. (Substituting into BCs)

- Substituting the expression for u into the BCs, giving

$$C = E = F = 0$$
$$\omega = (\pi n)^2 \qquad n = 1, 2, \dots$$

- Therefore, the fundamental solutions u_n (solutions of the PDE and BCs) are

$$\begin{aligned} u_n(x,t) &= X_n(x) T_n(t) \\ &= \left[a_n \sin{(\pi n)^2 t} + b_n \cos{(\pi n)^2 t}\right] \sin{(\pi n x)} \end{aligned}$$

- Since the PDE and BCs are linear and homogeneous, we can conclude that the sum

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left[a_n \sin{(\pi n)^2 t} + b_n \cos{(\pi n)^2 t}\right] \sin{(\pi n x)}}$$

Step 3. (Substituting into ICs)

- Substituting the expression for u into the ICs gives us

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin(\pi n x)$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} (\pi n)^2 a_n \sin(\pi n x)$$

- Using the fact that the family $\{\sin(\pi n x)\}$ is orthogonal on the interval $[0,1]$ we arrive at

$$a_n = \frac{2}{(\pi n)^2} \int_0^1 g(x) \sin(\pi n x) dx$$

$$b_n = 2 \int_0^1 f(x) \sin(\pi n x) dx$$

Remarks.

- Beams are generally fastened in one of three ways

  1. Free (unfastened)
  2. Simply fastened
  3. Rigidly fastened

- Another important vibrating-beam problem is the cantilever-beam problem. The solution to this vibrating beam is not the usual sum of products of sines and cosines, but due to the nonstandard BCs,

$$\begin{aligned} u(0,t) &= 0 & \qquad u_{xx}(1,t) &= 0 \\ u_x(0,t) &= 0 & \qquad u_{xxx}(1,t) &= 0 \end{aligned}$$

  we arrive at the more complicated solution.

Remarks (cont.)

- The solution of the cantilever-beam problem has the form

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x) \left[a_n \sin(\omega_n t) + b_n \cos(\omega_n t) \right],$$

where the eigenfunctions (basic shapes of vibrations) are given by linear combinations of sines, cosines, hyperbolic sines and hyperbolic cosines.