PDE and Boundary-Value Problems
Winter Term 2014/2015

Lecture 15

Saarland University

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Purpose of Lesson

- To show how BVPs, IBVPs, and other types of physical models can be written in dimensionless form. In this form, we replace the original variables of the problem by new dimensionless ones (they have no units).
- To solve the IBVP for the wave equation in three dimensions and show how this solution satisfies Huygen's principle.
- Using the method of descent to solve the IVP for the wave equation in two dimensions.
- To show that the two-dimensional solution doesn't satisfy Huygen's principle.
- To introduce two new integral transforms (finite sine and cosine transforms) and to show how to solve BVPs (particularly nonhomogeneous ones) by means of these transforms.

Dimensionless Problems

- The basic idea behind dimensional analysis is that by introducing new (dimensionless) variables in a problem, the problem becomes purely mathematical and contains none of the physical constants that originally characterized it.
- In this way, many different equations in physics, biology, engineering and chemistry that contain special nuances via physical parameters are all transformed into the same simple form.

Converting a Diffusion Problem to Dimensionless Form

Problem 15-1

To find the function $u(x, t)$ that satisfies

$$\text{PDE:} \qquad u_t = \alpha^2 u_{xx}, \qquad 0 < x < L, \quad 0 < t < \infty$$

$$\text{BCs:} \qquad \begin{cases} u(0,t) = T_1 \\ u(L,t) = T_2 \end{cases} \qquad 0 < t < \infty$$

$$\text{IC:} \quad u(x,0) = \sin(\pi x/L) \quad 0 \leqslant x \leqslant L$$

Converting a Diffusion Problem to Dimensionless Form

Our goal is to change problem 15-1 to a new equivalent formulation that has the properties:

1. No physical parameters (like α) in the new equation
2. New IC and BCs are simpler.

To do this, we will introduce three new dimensionless variables U, ξ, and τ that take the place of u, x, and t, respectively

$$\begin{array}{llll} u & \longrightarrow & U & \text{(dimensionless temperature)} \\ x & \longrightarrow & \xi & \text{(dimensionless length)} \\ t & \longrightarrow & \tau & \text{(dimensionless time)} \end{array}$$

We carry out these transformations one at a time for simplicity.

Step 1. (Transforming The Dependent Variable $\to U$)

- We define $U(x,t)$ by

$$U(x,t) = \frac{u(x,t) - T_1}{T_2 - T_1}$$

- It's clear that this new temperature $U(x,t)$ has no units, since we are dividing $^\circ C$ by $^\circ C$.

Step 1. (Transforming The Dependent Variable $\to U$)

- The original problem 15-1 has now been transformed into

Problem 15-1a

$$\text{PDE:} \quad U_t = \alpha^2 U_{xx}, \qquad 0 < x < L, \quad 0 < t < \infty$$

$$\text{BCs:} \quad \begin{cases} U(0,t) = 0 \\ U(L,t) = 1 \end{cases} \qquad 0 < t < \infty$$

$$\text{IC:} \quad U(x,0) = \frac{\sin(\pi x/L) - T_1}{T_2 - T_1} \qquad 0 \leqslant x \leqslant L$$

Step 2. (Transformnig the Space Variable $x \to \xi$)

- Since $0 \leqslant x \leqslant L$, we pick

$$\xi = x/L.$$

- The next problem (in U, ξ, and t) is

Problem 15-1b

$$\text{PDE:} \quad U_t = (\alpha/L)^2 U_{\xi\xi}, \qquad 0 < \xi < 1, \quad 0 < t < \infty$$

$$\text{BCs:} \quad \begin{cases} U(0,t) = 0 \\ U(1,t) = 1 \end{cases} \qquad 0 < t < \infty$$

$$\text{IC:} \quad U(\xi, 0) = \frac{\sin(\pi\xi) - T_1}{T_2 - T_1} \qquad 0 \leqslant \xi \leqslant 1$$

Step 3. (Transforming the Time Variable $t \to \tau$)

- How to introduce a new dimensionless time isn't quite so clear as choosing the first two variables.
- Since our goal is to eliminate the constant $[\alpha/L]^2$ from the PDE, we proceed as follows:
  1. Try a transformation of the form $\tau = ct$, where c is an unknown constant.
  2. Compute $u_t = u_\tau \tau_t = cu_t$.
  3. Substitute this derivative into the PDE to obtain

$$cu_\tau = [\alpha/L]^2 u_{\xi\xi}$$

and, hence, pick $c = [\alpha/L]^2$. This gives us our new time

$$\tau = [\alpha/L]^2 t.$$

Step 3. (Transforming the Time Variable $t \to \tau$)

- Using this transformation on our problem 15-1b, we have the completely dimensionless problem (U, ξ and τ)

Problem 15-1c

$$\text{PDE:} \quad U_\tau = U_{\xi\xi}, \qquad 0 < \xi < 1, \quad 0 < \tau < \infty$$

$$\text{BCs:} \quad \begin{cases} U(0,\tau) = 0 \\ U(1,\tau) = 1 \end{cases} \quad 0 < \tau < \infty$$

$$\text{IC:} \quad U(\xi, 0) = \phi(\xi) \quad 0 \leqslant \xi \leqslant 1$$

where $\phi(\xi) = \dfrac{\sin(\pi\xi) - T_1}{T_2 - T_1}$.

- New dimensionless problem 15-1c has the following properties:
  1. No parameters in the PDE.
  2. Simple BCs.
  3. IC hasn't essentially been changed (still a known function)
- The solution to this dimensionless problem can be found once and for all.
- So, if a scientist transformed the original problem to the dimensionless one and found the answer $U(\xi, \tau)$ in a textbook or research journal, he or she could find the solution $u(x, t)$ to the original problem by computing

$$u(x,t) = T_1 + (T_2 - T_1)U(x/L, (\alpha/L)^2 t).$$

Remarks

- Dimensional analysis is especially important in numerical analysis, since most computer programs are written in a general form and don't solve problems with a great many physical parameters.
- It's not always necessary to transform all the variables into dimensionless form; sometimes only one or two have to be transformed.

The Wave Equation in Three Dimensions (Free Space)

- Earlier, we discussed the infinite vibrating string with ICs and showed how it gave rise to the D'Alembert solution.
- Another application of the one-dimensional wave equation would be in describing plane wave in three dimensions.

We will generalize the D'Alembert solution to three dimensions.

Waves in Three Dimensions

We start by considering waves in three dimensions that have given ICs, that is, we would like to solve the IVP:

Problem 15-2

To find the function $u(x, y, z, t)$ that satisfies

$$\text{PDE:} \quad u_{tt} = c^2 \left(u_{xx} + u_{yy} + u_{zz} \right), \qquad \begin{cases} -\infty < x < \infty \\ -\infty < y < \infty \\ -\infty < z < \infty \end{cases}$$

$$\text{ICs:} \quad \begin{cases} u(x, y, z, 0) = \phi(x, y, z) \\ u_t(x, y, z, 0) = \psi(x, y, z) \end{cases}$$

Waves in Three Dimensions (cont.)

To solve problem 15-2, we first solve the simpler one (set $\phi = 0$)

Problem 15-2a

To find the function $u(x, y, z, t)$ that satisfies

$$\text{PDE:} \quad v_{tt} = c^2 \Delta v, \qquad \begin{cases} -\infty < x < \infty \\ -\infty < y < \infty \\ -\infty < z < \infty \end{cases}$$

$$\text{ICs:} \quad \begin{cases} v(x, y, z, 0) = 0 \\ v_t(x, y, z, 0) = \psi(x, y, z) \end{cases}$$

Waves in Three Dimensions (cont.)

Problem 15-2a can be solved by the Fourier transform and has the solution

$$\boxed{v(x,y,z,t) = t\overline{\psi}}, \tag{15.1}$$

where $\overline{\psi}$ is the average of the initial disturbance ψ over the sphere of radius ct centered at (x,y,z); that is,

$$\overline{\psi} = \frac{1}{4\pi c^2 t^2} \int_0^{\pi} \int_0^{2\pi} \psi(x + ct\sin\varphi\cos\theta, y + ct\sin\varphi\sin\theta,$$

$$z + ct\cos\theta)(ct)^2 \sin\varphi d\theta d\varphi.$$

Waves in Three Dimensions (cont.)

- The interpretation of (15.1) is that the initial disturbance ψ radiates outward spherically (velocity c) at each point, so that after so many seconds, the point (x, y, z) will be influenced by those initial disturbances on a sphere (of radius ct) around that point.
- The actual value of the solution (15.1) would most likely have to be computed numerically on a computer for most initial disturbances.

Waves in Three Dimensions (cont.)

Now, we consider the other half of problem 15-2; that is,

Problem 15-2b

To find the function $w(x, y, z, t)$ that satisfies

$$\text{PDE:} \quad w_{tt} = c^2 \Delta w, \qquad\qquad (x, y, z) \in \mathbb{R}^3$$

$$\text{ICs:} \quad \begin{cases} w(x, y, z, 0) = \phi(x, y, z) \\ w_t(x, y, z, 0) = 0 \end{cases}$$

Waves in Three Dimensions (cont.)

We can easily solve problem 15-2b: a famous theorem developed by Stokes says all we have to do to solve this problem is change the ICs to $w = 0$, $w_t = \phi$, and then differentiate this solution with respect to time. In other words, we solve

Problem 15-2c

To find the function $\tilde{w}(x, y, z, t)$ that satisfies

$$\text{PDE:} \quad \tilde{w}_{tt} = c^2 \Delta \tilde{w}, \qquad (x, y, z) \in \mathbb{R}^3$$

$$\text{ICs:} \quad \begin{cases} \tilde{w}(x, y, z, 0) = 0 \\ \tilde{w}_t(x, y, z, 0) = \phi(x, y, z) \end{cases}$$

Waves in Three Dimensions (cont.)

- We get $\tilde{w} = t\overline{\phi}$ and then differentiate with respect to time. This gives us the solution to problem 15-2c

$$\boxed{w = \frac{\partial}{\partial t}\left[t\overline{\phi}\right]}. \tag{15.2}$$

- Combining (15.1) and (15.2) we have the solution to our problem 15-2. It's just

$$\boxed{u(x, y, z, t) = t\overline{\psi} + \frac{\partial}{\partial t}\left[t\overline{\phi}\right]}, \tag{15.3}$$

  where $\overline{\phi}$ and $\overline{\psi}$ are the averages of the functions ϕ and ψ over the sphere of radius ct centered at (x, y, z).

Remarks

- (15.3) is known as Poisson's formula for the free-wave equation in three dimensions. It is the generalization of the D'Alembert formula.
- The most important aspect of the Poisson formula is the fact that the two integrals in $\overline{\phi}$ and $\overline{\psi}$ are integrated over the surface of a sphere.
- When time is $t = t_1$, the solution u at (x, y, z) depends only on the initial disturbances ϕ and ψ on a sphere of radius ct_1 around (x, y, z).

Huygen's principle

The wave disturbance originating from the initial-disturbance region has a sharp trailing edge.

Remark

We know from the D'Alembert solution that the initial disturbance

$$u(x,0) = \phi(x)$$
$$u_t(x,0) = \psi(x)$$

in one dimension does not have a sharp trailing edge (since the D'Alembert solution integrates ψ from $(x - ct)$ to $(x + ct)$.

Two-Dimensional Wave Equation

Consider the two-dimensional problem

Problem 15-3

To find the function $u(x, y, t)$ that satisfies

$$\text{PDE:} \quad u_{tt} = c^2 \left(u_{xx} + u_{yy} \right), \qquad (x, y) \in \mathbb{R}^2$$

$$\text{ICs:} \quad \begin{cases} u(x, y, 0) = \phi(x, y) \\ u_t(x, y, 0) = \psi(x, y) \end{cases}$$

Two-Dimensional Wave Equation (cont.)

- To solve problem 15-3 we let the initial disturbances ϕ and ψ in the three-dimensional problem depend on only two variables x and y.
- Doing this, the three-dimensional formula

$$u = t\overline{\psi} + \frac{\partial}{\partial t}\left[t\overline{\phi}\right]$$

  for u will describe cylindrical waves and, hence, give us the solution for the two-dimensional problem.
- This technique is called the method of descent.

Two-Dimensional Wave Equation (cont.)

- Carrying out the computations (which are by no means trivial), we get

$$\boxed{\begin{aligned} u(x,y,t) = \frac{1}{2\pi c}\Bigg\{ & \int_0^{2\pi}\int_0^{ct} \frac{\psi(x',y')}{\sqrt{(ct)^2 - r^2}}\, r dr d\theta \\ & + \frac{\partial}{\partial t}\left[\frac{1}{2\pi c}\int_0^{2\pi}\int_0^{ct} \frac{\phi(x',y')}{\sqrt{(ct)^2 - r^2}}\, r dr d\theta\right]\Bigg\} \end{aligned}}, \tag{15.4}$$

where $x' = x + r\cos\theta$ and $y' = y + r\sin\theta$.

Remarks

- In (15.4) the two integrals of the ICs ϕ and ψ are integrated over the interior of a circle (the key word is interior) with center at (x, y) and radius ct.
- If we analyze what this means in a manner similar t the three-dimensional case, we see that initial disturbances give rise to sharp leading waves, but not to sharp trailing waves.
- Thus, Huygen's principle doesn't hold in two dimensions.

The Finite Fourier Transforms (Sine and Cosine Transforms)

Remarks

- Earlier, we learned about the Fourier and Laplace transforms and their applications for problems in free space (no boundaries).
- Now, we show how to solve BVPs (with boundaries) by transforming the bounded variables.

The finite sine and cosine transforms are defined by

$$\begin{cases} S[f] = S_n = \dfrac{2}{L}\displaystyle\int\limits_0^L f(x)\sin(n\pi x/L)dx, \quad \text{(finite sine transform)} \\ \qquad\qquad n = 1, 2, \ldots \\ f(x) = \displaystyle\sum_{n=1}^{\infty} S_n \sin(n\pi x/L) \quad \text{(inverse sine transform)} \end{cases}$$

$$\begin{cases} C[f] = C_n = \dfrac{2}{L}\displaystyle\int\limits_0^L f(x)\cos(n\pi x/L)dx, \quad \text{(finite cosine transform)} \\ \qquad\qquad n = 0, 1, 2, \ldots \\ f(x) = \dfrac{C_0}{2} + \displaystyle\sum_{n=1}^{\infty} C_n \cos(n\pi x/L) \quad \text{(inverse cosine transform)} \end{cases}$$

Properties of the Transforms

- If $u(x,t)$ is a function of two variables, then (note we're transforming the x-variable)

$$S[u] = S_n(t) = \frac{2}{L}\int\limits_0^L u(x,t)\sin\left(n\pi x/L\right)dx$$

$$C[u] = C_n(t) = \frac{2}{L}\int\limits_0^L u(x,t)\cos\left(n\pi x/L\right)dx$$

Properties of the Transforms (cont.)

1. $$S[u_t] = \frac{dS[u]}{dt}$$
2. $$S[u_{tt}] = \frac{d^2 S[u]}{dt^2}$$
3. $$S[u_{xx}] = -[n\pi/L]^2 S[u] + \frac{2n\pi}{L^2}\left[u(0,t) + (-1)^{n+1} u(L,t)\right]$$
4. $$C[u_{xx}] = -[n\pi/L]^2 C[u] - \frac{2}{L}\left[u_x(0,t) + (-1)^{n+1} u_x(L,t)\right]$$

Finite Sine Transform

	$f(x) = \sum\limits_{n=1}^{\infty} S_n \sin(nx)$ $0 \leqslant x \leqslant \pi$	$S_n = \frac{2}{\pi} \int\limits_0^{\pi} f(x) \sin(nx) dx$ $n = 1, 2, \ldots$
1.	$\sin(mx)$	$\begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$
2.	$\sum\limits_{n=1}^{\infty} a_n \sin(nx)$	a_n
3.	$\pi - x$	$\dfrac{2}{n}$
4.	x	$\dfrac{2}{n}(-1)^{n+1}$

Finite Sine Transform (cont.)

	$f(x) = \sum_{n=1}^{\infty} S_n \sin(nx)$ $0 \leqslant x \leqslant \pi$	$S_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$ $n = 1, 2, \dots$
5.	1	$\frac{2}{n\pi}\left[1 - (-1)^n\right]$
6.	$\begin{cases} -x, & x \leqslant a \\ \pi - x, & x > a \end{cases}$	$\frac{2}{n}\cos(na), \quad 0 < a < \pi$
7.	$\begin{cases} (\pi - a)x, & x \leqslant a \\ (\pi - x)a, & x > a \end{cases}$	$\frac{2}{n^2}\sin(na), \quad 0 < a < \pi$

Finite Sine Transform (cont.)

	$f(x) = \sum\limits_{n=1}^{\infty} S_n \sin(nx)$ $0 \leqslant x \leqslant \pi$	$S_n = \frac{2}{\pi} \int\limits_0^{\pi} f(x) \sin(nx) dx$ $n = 1, 2, \ldots$
8.	$\frac{\pi}{2} e^{ax}$	$\frac{n}{n^2 + a^2} \left[1 - (-1)^n e^{a\pi}\right]$
9.	$\frac{\sinh a(\pi - x)}{\sinh a\pi}$	$\frac{2n}{\pi(n^2 + a^2)}$

Finite Cosine Transform

	$f(x) = \frac{C_0}{2} + \sum\limits_{n=1}^{\infty} C_n \cos(nx)$ $0 \leqslant x \leqslant \pi$	$C_n = \frac{2}{\pi} \int\limits_0^{\pi} f(x) \cos(nx) dx$ $n = 0, 1, 2, \ldots$
1.	$\frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos(nx)$	a_n
2.	$f(\pi - x)$	$(-1)^n \frac{2}{\pi} C_n$
3.	1	$\begin{cases} 2, & n = 0 \\ 0, & n = 1, 2, \ldots \end{cases}$
4.	$\cos(mx), \quad m = 1, 2, \ldots$	$\begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$

Finite Cosine Transform (cont.)

	$f(x) = \frac{C_0}{2} + \sum\limits_{n=1}^{\infty} C_n \cos(nx)$ $0 \leqslant x \leqslant \pi$	$C_n = \frac{2}{\pi} \int\limits_0^{\pi} f(x) \cos(nx) dx$ $n = 0, 1, 2, \dots$
5.	x	$\begin{cases} \pi, & n = 0 \\ \frac{2}{\pi n^2}\left[(-1)^n - 1\right], & n = 1, 2, \dots \end{cases}$
6.	x^2	$\begin{cases} 2\pi^2/3, & n = 0 \\ \frac{4}{n^2}(-1)^n, & n = 1, 2, \dots \end{cases}$
7.	$-\log(2\sin(x/2))$	$\begin{cases} 0, & n = 0 \\ \frac{1}{n}, & n = 1, 2, \dots \end{cases}$

Finite Cosine Transform

	$f(x) = \frac{C_0}{2} + \sum\limits_{n=1}^{\infty} C_n \cos(nx)$ $0 \leqslant x \leqslant \pi$	$C_n = \frac{2}{\pi} \int\limits_0^{\pi} f(x) \cos(nx) dx$ $n = 0, 1, 2, \ldots$
8.	$\dfrac{1}{a} e^{ax}$	$\dfrac{2}{\pi} \left[\dfrac{(-1)^n e^{a\pi} - 1}{n^2 + a^2} \right]$
9.	$\begin{cases} 1, & 0 < x < a \\ -1, & a < x < \pi \end{cases}$	$\begin{cases} \dfrac{2}{\pi}(2a - \pi), & n = 0 \\ \dfrac{4}{n\pi} \sin(na), & n = 1, 2, \ldots \end{cases}$

Solving a Nonhomogeneous BVP via the Finite Sine Transform

Consider the nonhomogeneous wave equation

Problem 15-4

To find the function $u(x, t)$ that satisfies

$$\text{PDE:} \quad u_{tt} = u_{xx} + \sin(\pi x), \quad 0 < x < 1, \quad 0 < t < \infty$$

$$\text{BCs:} \quad \begin{cases} u(0, t) = 0 \\ u(1, t) = 0 \end{cases} \quad 0 < t < \infty$$

$$\text{ICs:} \quad \begin{cases} u(x, 0) = 1 \\ u_t(x, 0) = 0 \end{cases} \quad 0 \leqslant x \leqslant 1$$

Step 1. (Determine the transform)

- Since the x-variable ranges from 0 to 1, we use a finite transform.
- We could solve this problem with the Laplace transform by transforming t (it would involve about the same level of difficulty as the finite sine transform).

Step 2. (Carry out the transformation)

- Transforming the PDE and ICs we get the new IVP for $S_n(t) = S[u]$

Problem 15-4a

$$\text{ODE:} \quad \frac{d^2 S_n}{dt^2} + (n\pi)^2 S_n = \begin{cases} 1, & n = 1 \\ 0, & n = 2, 3, \dots \end{cases},$$

$$\text{ICs:} \quad \begin{cases} S_n(0) = \begin{cases} 4/(n\pi), & n = 1, 3, \dots \\ 0, & n = 2, 4, \dots \end{cases} \\ \dfrac{dS_n(0)}{dt} = 0, \quad n = 1, 2, \dots \end{cases}$$

Step 3. (Solving the new IVP)

- Solving the problem 15-4a we get

$$S_1(t) = \left(\frac{4}{\pi} - \frac{1}{\pi^2}\right)\cos(\pi t) + (1/\pi)^2$$

$$S_n(t) = \begin{cases} 0, & n = 2, 4, \ldots \\ \dfrac{4}{n\pi}\cos(n\pi t), & n = 3, 5, 7, \ldots \end{cases}$$

Step 4. (Inverse transform)

- Hence, the solution $u(x, t)$ of the problem is

$$\boxed{\begin{aligned} u(x,t) &= \left(\frac{4}{\pi} - \frac{1}{\pi^2}\right)\cos(\pi t)\sin[\pi x] + (1/\pi)^2 \sin[\pi x] \\ &+ \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n+1}\cos[(2n+1)\pi t]\sin[(2n+1)\pi x] \end{aligned}}$$

Remarks

- In order to apply the finite sine or cosine transform, the BCs at $x = 0$ and $x = L$ must both be of the form

$$\left.\begin{aligned} u(0,t) &= f(t) \\ u(L,t) &= g(t) \end{aligned}\right\} \qquad \text{(use sine transform)}$$

$$\left.\begin{aligned} u_x(0,t) &= f(t) \\ u_x(L,t) &= g(t) \end{aligned}\right\} \qquad \text{(use cosine transform)}$$

In other words, the BCs

$$u(0,t) = f(t) \quad \text{and} \quad u_x(L,t) = g(t)$$

wouldn't work. Also BCs like $u_x(0,t) + hu(0,t) = 0$ don't apply.

Remarks (cont.)

- In order to apply the finite sine and cosine transforms, the equation shouldn't contain first-order derivatives in x (since the sine transform of the first derivative involves the cosine transform and vice versa).
- The finite sine- and cosine-transform method essentially resolves all functions in the original problem (like u_{tt}, u_{xx}, the ICs, BCs) into a Fourier sine and cosine series, solves a sequence of problems (ODE) for the Fourier coefficients, and then adds up the result.