PDE and Boundary-Value Problems Winter Term 2014/2015

Lecture 16

Saarland University

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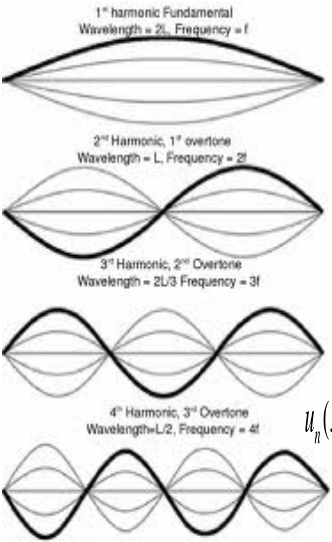
Purpose of Lesson

- To show how the wave equation can describe the vibrations of a drumhead.
- To explain how PDEs that don't involve the time derivative occur in nature. These differential equations have no initial conditions, but only boundary conditions.
- To discuss the most common types of BCs for elliptic-type problems.



$$u_n(x,t) = \sin(n\pi x/L)\left[a_n \sin(n\pi ct/L) + b_n \cos(n\pi ct/L)\right]$$

$$n = 1,2,3,\ldots$$

The Vibrating Drumhead (Wave Equation in Polar Coordinates)

We want to find the vibrations of a circular drumhead with given boundary and initial conditions.

Problem 16-1

To find the function $u(r, \theta, t)$ that satisfies

$$\text{PDE:} \quad u_{tt} = c^2 \left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \right), \quad 0 < r < 1$$

$$\text{BC:} \quad u(1, \theta, t) = 0, \qquad 0 < \theta < 2\pi, \quad 0 < t < \infty$$

$$\text{ICs:} \quad \begin{cases} u(r, \theta, 0) = f(r, \theta) \\ u_t(r, \theta, 0) = g(r, \theta) \end{cases}$$

- Recall that for violin-string problem the solution is a superposition of an infinite number of simple vibrations.
- If we approach the drumhead in a similar manner, we will look for solutions of the form

$$u(r,\theta,t) = U(r,\theta)T(t). \tag{16.1}$$

This gives the shape $U(r,\theta)$ of the vibrations times the oscillatory factor $T(t)$.

Step 1. (Separation of Variables)

- Substituting (16.1) into PDE and BC, we arrive at the equations

$$\begin{cases} U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} + \lambda^2 U = 0 & \text{(Helmholtz equation)} \\ U(1,\theta) = 0 & \\ T'' + \lambda^2 c^2 T = 0 & \text{(Simple harmonic motion)} \end{cases}$$

- We now have to find the shapes $U(r,\theta)$ of the fundamental vibrations

$$\begin{cases} U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} + \lambda^2 U = 0 \\ U(1,\theta) = 0 \end{cases} \qquad (16.2)$$

Step 1. (Separation of Variables)

- (16.2) is the Helmholtz eigenvalue problem (very famous), and our purpose is to seek all λ's (if any) that yield nonzero solutions.

Step 2. (Solving of the Helmholtz Eigenvalue Problem)

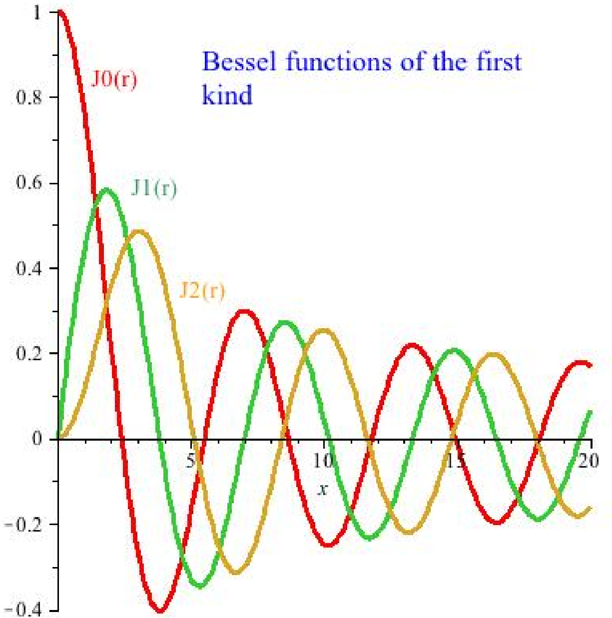
- To solve (16.2) we let $U(r,\theta) = R(r)\Theta(\theta)$ and plug it into (16.2). Doing this, we arrive at

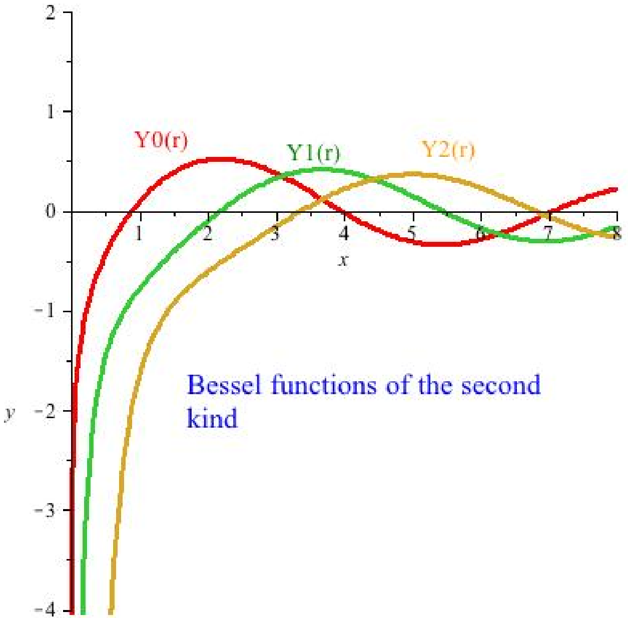
$$\begin{cases} r^2 R'' + rR' + (\lambda^2 r^2 - n^2)R = 0 \\ R(1) = 0 \\ R(0) < \infty \quad \text{(Physical condition)} \\ \Theta'' + n^2\Theta = 0 \end{cases}$$

- Note that we have chosen the new separation constant n^2, $n = 0, 1, 2, \dots$.

Step 2. (Solving of the Helmholtz Eigenvalue Problem (cont.))

- The equation
$$r^2 R'' + rR' + (\lambda^2 r^2 - n^2)R = 0$$
is well known in ODE theory; it is called Bessel's equation and has two linearly independent solutions. They are
$$R_1(r) = AJ_n(\lambda r) \quad n^{th}\text{order Bessel function of the first kind}$$
$$R_2(r) = BY_n(\lambda r) \quad n^{th}\text{order Bessel function of the second kind}$$
- Hence, the general solution to the Helmholtz equation is
$$R(r) = AJ_n(\lambda r) + BY_n(\lambda r)$$


Bessel functions of the first kind
J0(r)
J1(r)
J2(r)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
5
10
15
20
x


2
1
0
-1
-2
-3
-4
y
1
2
3
4
5
6
7
8
x
Y0(r)
Y1(r)
Y2(r)
Bessel functions of the second kind

Step 2. (Solving of the Helmholtz Eigenvalue Problem (cont.))

- Since the functions $Y_n(\lambda r)$ are unbounded at $r = 0$, we set $B = 0$. Therefore

$$R(r) = AJ_n(\lambda r). \tag{16.3}$$

- Substituting the BC $R(1) = 0$ into (16.3), we have

$$J_n(\lambda) = 0.$$

In other words, in order for $R(r)$ to be zero on the boundary of the circle, we must pick the separation constant λ to be one of the roots of $J_n(r) = 0$; that is,

$$\lambda = k_{nm}$$

where k_{nm} is the m-th root of $J_n(r) = 0$.

The m-th Root of $J_n(r) = 0$

		n				
		0	1	2	3	4
	1	2.40	3.83	5.13	6.38	7.59
	2	5.52	7.02	8.42	9.76	11.06
m	3	8.65	10.17	11.62	13.02	14.37
	4	11.79	13.32	14.80	16.22	17.62
	5	14.93	16.47	17.96	19.41	20.83
	⋮	⋮	⋮	⋮	⋮	⋮

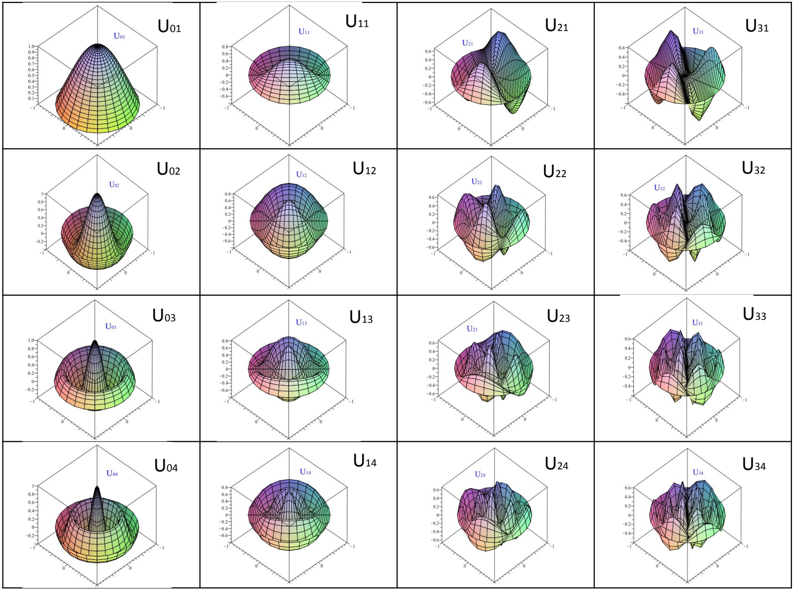
Step 2. (Solving of the Helmholtz Eigenvalue Problem (cont.))

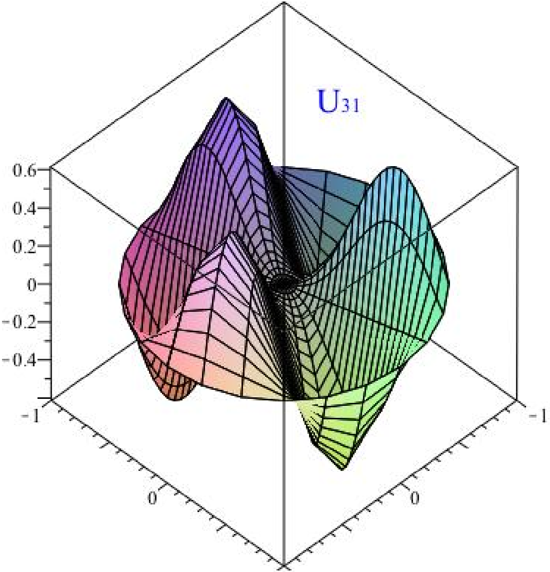
- Thus, the corresponding eigenfunctions $U_{nm}(r, \theta)$ are

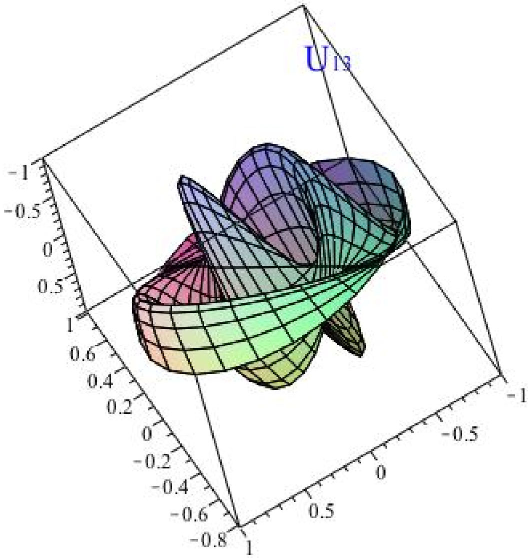
$$\boxed{U_{nm}(r, \theta) = J_n(k_{nm} r) \left[A \sin(n\theta) + B \cos(n\theta) \right]}$$

$$n = 0, 1, 2, \dots \qquad m = 1, 2, 3, \dots$$

- We plot the fundamental vibrations $U_{nm}(r, \theta)$ for the different values of n and m.


U_{01}
U_{11}
U_{21}
U_{31}
U_{02}
U_{12}
U_{22}
U_{32}
U_{03}
U_{13}
U_{23}
U_{33}
U_{04}
U_{14}
U_{24}
U_{34}


U31
0.6
0.4
0.2
0
-0.2
-0.4
-1
0
-1
0


U_{13}
-1
-0.5
0
0.5
1
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
1
0.5
0
-0.5
-1

Step 3. (Solving the Time Equation)

- Each $U_{nm}(r,\theta)$ represents a fundamental vibration of the circular membrane with frequency

$$f_{nm} = k_{nm}\frac{c}{2\pi} \quad \text{cycles / unit time}$$

- We find these frequences by solving the time equation

$$T'' + k_{nm}^2 c^2 T = 0$$

to get

$$\boxed{T_{nm}(t) = A\sin(k_{nm}ct) + B\cos(k_{nm}ct)}.$$

Remarks

- The ratio
$$\frac{f_{nm}}{f_{01}} = \frac{k_{nm}}{k_{01}}$$
is not an integer as it was in the one-dimensional wave equation.
- In other words, higher vibrations for the drumhead are not pure overtones of the basic frequencies.
- The nodal circles (where no vibration takes place) have radii
$$\frac{k_{n1}}{k_{nm}}, \frac{k_{n2}}{k_{nm}}, \dots, \frac{k_{nm}}{k_{nm}} = 1$$

Thus, the solution to our problem 16-1 will be

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(k_{nm}r)\cos(n\theta)\times \times\left[A_{nm}\sin(k_{nm}ct) + B_{nm}\cos(k_{nm}ct)\right].$$

Remarks

- Note that $A\sin(n\theta) + B\cos(n\theta)$ was replaced by $\cos(n\theta)$ by proper choice of the angle θ.
- We have lumped together the constants as A_{nm} and B_{nm}.

Step 4. (Substituting into ICs)

- Rather than going through the complicated process of finding A_{nm} and B_{nm} for the general case, we will find the solution for the situation where u is independent of θ (very common).
- In particular, we consider

$$u(r, \theta, 0) = f(r)$$
$$u_t(r, \theta, 0) = 0$$

- With these assumptions, the solution now becomes

$$\boxed{u(r, t) = \sum_{m=1}^{\infty} A_m J_0(k_{0m} r) \cos(k_{0m} c t)}.$$

Step 4. (Substituting into ICs (cont.))

- Our goal is to find A_m so that

$$f(r) = \sum_{m=1}^{\infty} A_m J_0(k_{0m} r).$$

- Using the orthogonality condition of the Bessel functions

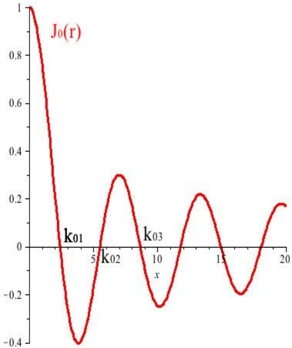
$$\int_0^1 r J_0(k_{0i} r) J_0(k_{0j} r) dr = \begin{cases} 0, & i \neq j \\ \frac{1}{2} J_1^2(k_{0i}), & i = j \end{cases}$$

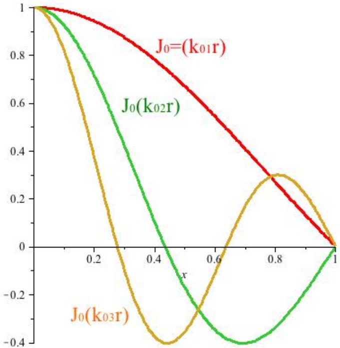
we get

$$A_j = \frac{2}{J_1^2(k_{0j})} \int_0^1 r f(r) J_0(k_{0j} r) dr, \qquad j = 1, 2, \dots$$

Interpretation of $J_0(k_{01}r)$, $J_0(k_{02}r)$, ...

- We start by drawing $J_0(r)$. In order to graph the functions $J_0(k_{01}r)$, $J_0(k_{02}r)$, ..., $J_0(k_{0m}r)$ we rescale the r-axis so that $m-th$ root passes through $r = 1$.





Remark

- For the vibrating membrane with ICs $u = f(r)$, $u_t = 0$, we can interpret the solution as expanding the IC $f(r)$ as a sum of basic building blocks $A_m J_0(k_{0m} r)$ and let each of them vibrate with its own frequency $\cos(k_{0m} ct)$, giving the fundamental vibration

$$A_m J_0(k_{0m} r) \cos(k_{0m} ct).$$

- We then add them up to get vibration resulting from the IC $f(r)$.

Chapter 4. Elliptic-Type Problems

- Until now, the problems we've discussed involved phenomena that changed over space and time. There are, however, many important problems whose outcomes do not change with time, but only with respect to space.
- These problems, for the most part, are described by elliptic boundary-value problems.
- There are two common situations that give rise to PDEs that don't involve time; they are
  1. Steady-state problems.
  2. Problems where we factor out the time component in the solution.

Steady-State Problems

- Steady-state solution is a solution when $t \to \infty$.
- It's obvious if the solution doesn't change in time, then $u_t = 0$. To find the steady-state solution $u(x, \infty)$ (if it exists), we let $u_t = 0$ and solve the corresponding BVP.
- For some problems, a steady-state solution may not exist.

Factoring out the Time Component in Hyperbolic and Parabolic Problems

- In the circular drumhead problem 16-1 we looked for solutions of the form $u(r,\theta,t) = U(r,\theta)T(t)$ which yielded the Helmholtz BVP

$$\begin{aligned} \text{PDE:} \quad & \Delta U + \lambda^2 U = 0 \\ \text{BC:} \quad & U(1,\theta) = 0 \end{aligned}$$

  This situation is common in PDEs where the solution represents a shape factor $U(r,\theta)$ multiplied by a time factor $T(t)$.

- As a matter of fact, we arrive at the same Helmholtz equation by factoring out the time component in the heat equation.

The Three Main Types of BCs in BVPs

There are three types of BCs that are most common for elliptic-type problems.

- 1. BVPs of the First Kind (Dirichlet Problems)
  - The PDE holds over the given region of space, and the solution is specified on the boudary of the region.
  - There are interior and exterior Dirichlet problems.
  - Dirichlet problems are common in electrostatics when we want to find the potential in a region with the potential given on the boundary.

- 2. BVPs of the Second Kind (Neumann Problems)
  - The PDE holds in some region of space, but now the outward normal derivative $\dfrac{\partial u}{\partial n}$ (which is proportional to the inward flux) is specified on the boundary.
  - Neumann problems are common in steady-state heat flow and electrostatistics, where the flux (in heat energy, electrons, and so forth) is given over the boundary.
  - Neumann problems make sense only if the net gain across the boundary is zero. Mathematically, this says that

$$\int_C \frac{\partial u}{\partial n} = 0.$$

    Otherwise the problem has no solution.
  - The Neumann problem differs from other BCs in that solutions are not unique.

- 3. BVPs of the Third Kind (Mixture Problems)
  - These problems correspond to the PDEs being given in some region of space, but now the condition on the boundary is a mixture of the first two kinds

    $$\frac{\partial u}{\partial n} + h(u - g) = 0,$$

    where h is a constant (input to the problem) and g is a given function that can vary over the boundary.
  - A more suggestive form of this BC would be

    $$\frac{\partial u}{\partial n} = -h(u - g)$$

    which says the inward flux across the boundary is proportional to the difference between the temperature u and some specified temperature g.