PDE and Boundary-Value Problems Winter Term 2014/2015

Lecture 17

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Purpose of Lesson

- To show how to solve the interior Dirichlet problem for the circle by separation of variables and to discuss also an alternative integral-form of this solution (Poisson integral formula).
- To solve the Dirichlet problem between two circles (annulus)
- To discuss briefly the solution to the exterior Dirichlet problem for the circle

There are many regions of interest where we might solve the Dirichlet problem. Just to name a few, we could have the Dirichlet problem:

- Inside a circle
- In an annulus
- Outside a circle
- Inside a sphere
- Between two spheres
- Between two lines (in two dimensions)
- Between two planes (in three dimensions)

Remarks

- The list of the Dirichlet problems is endless.
- Our intention is to solve a representative sample of Dirichlet problems and learn the general principles.

Interior Dirichlet Problem for a Circle

Problem 17-1

To find the function $u(r, \theta)$ that satisfies

$$\text{PDE:} \quad u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 < r < 1$$

$$\text{BC:} \quad u(1, \theta) = g(\theta), \qquad 0 \leqslant \theta < 2\pi.$$

Remarks

- Problem 17-1 can be interpreted as finding the electrostatic potential inside a circle when the potential is given on the boundary.
- Another application is the soap film model. If we start with a circular wire hoop and distort it so that the distortion is measured by $g(\theta)$ and dip it into a soap solution, a film of soap is formed within the wire. The height of the film is represented by the solution of problem 17-1, provided the displacement $g(\theta)$ is small.

Step 1.

- Substituting $u(r,\theta) = R(r)\Theta(\theta)$ into the Laplace equation we get

$$r^2 R'' + rR' - \lambda^2 R = 0 \quad \text{(Euler's equation)}$$
$$\Theta'' + \lambda^2 \Theta = 0$$

- The separation constant λ must be nonnegative. If the separation constant were negative, the function $\Theta(\theta)$ would not be periodic.
- If $\lambda = 0$, then Euler's equation reduces to

$$r^2 R'' + rR' = 0$$

  and it is easy to see that the general solution is

$$R(r) = a + b \ln r, \qquad \Theta(\theta) = \beta + \gamma\theta.$$

  Since we are looking for bounded and periodic solution, we would throw out the terms $\ln r$ and $\gamma\theta$. So, in this case

$$u(r,\theta) = const.$$

Step 1. (cont.)

- If $\lambda^2 > 0$, then Euler's equation is

$$r^2 R'' + rR' - \lambda^2 R = 0$$

and to solve this, we look for solutions of the form $R(r) = r^\alpha$. The goal is to find two values of α (say α_1 and α_2) so that the general solution will be

$$R(r) = ar^{\alpha_1} + br^{\alpha_2}.$$

Plugging $R(r) = r^\alpha$ into Euler's equation yields $\alpha = \pm\lambda$ and, hence,

$$R(r) = ar^{\lambda} + br^{-\lambda}.$$

- For $\lambda^2 > 0$ the solution of the equation

$$\Theta''(\theta) + \lambda^2\Theta = 0$$

has the form

$$\Theta(\theta) = \beta\cos(\lambda\theta) + \gamma\sin(\lambda\theta).$$

Step 1. (cont.)

- Since Θ should be periodic, we conclude that $\lambda = 0, 1, 2, \dots$. Consequently, all solutions $R(r)\Theta(\theta)$ has the form

$$u_n(r,\theta) = r^n \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right].$$

Thus,

$$u(r,\theta) = \sum_{n=0}^{\infty} r^n \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right]. \qquad (17.1)$$

Step 2.

- Substituting (17.1) into the BC, we conclude

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)d\theta,$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\cos(n\theta)d\theta,$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\sin(n\theta)d\theta.$$

Step 3.

- To summarize, the solution to the interior Dirichlet problem 17-1 is

$$\boxed{u(r,\theta)=\sum_{n=0}^{\infty} r^n \left[a_n \cos(n\theta)+b_n \sin(n\theta)\right]}. \tag{17.2}$$

Observations on the Dirichlet Solution:

- The interpretation of our solution (17.2) is that we should expand boundary function $g(\theta)$ as a Fourier series

$$g(\theta) = \sum_{n=0}^{\infty} [a_n \cos(n\theta) + b_n \sin(n\theta)]$$

and then solve the problem for each sine and cosine in the series.

Since each of these terms will give rise to solutions $r^n \sin(n\theta)$ and $r^n \cos(n\theta)$, we can then say (by superposition) that

$$u(r,\theta) = \sum_{n=0}^{\infty} r^n [a_n \cos(n\theta) + b_n \sin(n\theta)] .$$

- The solution of

$$\text{PDE:} \quad \Delta u = 0, \qquad 0 < r < 1$$

$$\text{BC:} \quad u(1,\theta) = 1 + \sin\theta + \tfrac{1}{2}\sin(3\theta) + \cos(4\theta), \quad 0 \leqslant \theta < 2\pi.$$

would be

$$u(r,\theta) = 1 + r\sin\theta + \frac{r^3}{2}\sin(3\theta) + r^4\cos(4\theta).$$

Here, the $g(\theta)$ is already in the form of a Fourier series, with

$a_0 = 1$	$b_1 = 1$
$a_4 = 1$	$b_3 = 0.5$
All other a_n's $= 0$	All other b_n's $= 0$

and so we don't have to to use the formulas for a_n and b_n.

- If the radius of the circle were arbitrary (say R), then the solution would be

$$u(r,\theta) = \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right].$$

- Note that the constant term a_0 in solution (19.2) represents the average of $g(\theta)$

$$a_0 = \frac{1}{2\pi} \int\limits_0^{2\pi} g(\theta) d\theta.$$

- We start with the separation of variables solution

$$u(r,\theta) = \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^n [a_n \cos(n\theta) + b_n \sin(n\theta)]$$

  (we now take an arbitrary radius for the circle) and substitute the coefficients a_n and b_n.

- After a few manipulations involving algebra, calculus, and trigonometry, we have

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)d\theta$$

$$+\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\int_0^{2\pi} g(\alpha)\left[\cos(n\alpha)\cos(n\theta)+\sin(n\alpha)\sin(n\theta)\right]d\alpha$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\left\{1+2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\cos\left[n(\theta-\alpha)\right]\right\}g(\alpha)d\alpha$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\left\{1+\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n\left[e^{in(\theta-\alpha)}+e^{-in(\theta-\alpha)}\right]\right\}g(\alpha)d\alpha$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\left\{1+\frac{re^{i(\theta-\alpha)}}{R-re^{i(\theta-\alpha)}}+\frac{re^{-i(\theta-\alpha)}}{R-re^{-i(\theta-\alpha)}}\right\}g(\alpha)d\alpha$$

$$
\begin{aligned}
u(r,\theta) &= \frac{1}{2\pi}\int\limits_0^{2\pi}\left\{1+\frac{re^{i(\theta-\alpha)}}{R-re^{i(\theta-\alpha)}}+\frac{re^{-i(\theta-\alpha)}}{R-re^{-i(\theta-\alpha)}}\right\}g(\alpha)d\alpha \\
&= \frac{1}{2\pi}\int\limits_0^{2\pi}\left[\frac{R^2-r^2}{R^2-2rR\cos(\theta-\alpha)+r^2}\right]g(\alpha)d\alpha
\end{aligned}
\tag{17.3}
$$

It is the Poisson Integral Formula. So what we have is an alternative form for the solution to the interior Dirichlet problem.

Remarks

- We can interpret the Poisson integral solution (17.3) as finding the potential u at (r, θ) as a weighted average of the boundary potentials $g(\theta)$ weighted by the Poisson kernel

$$\text{Poisson kernel} = \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - \alpha) + r^2}.$$

- For boundary values $g(\alpha)$ close to (r, θ), the Poisson kernel gets large, since the denominator of the Poisson kernel is the square of the distance from (r, θ) to (R, α).

Remarks (cont.)

- Unfortunately, if (r, θ) is extremely close to the boundary $r = R$, then the Poisson kernel gets very large for those values of α that are closest to (r, θ). For this reason, when (r, θ) is close to the boundary, the series solution works better for evaluating the numerical value of the solution.
- If we evaluate the potential at the center of the circle by the Poisson integral, we find

$$u(0,0) = \frac{1}{2\pi} \int_0^{2\pi} g(\alpha) d\alpha.$$

In other words, the potential at the center of the circle is the average of the boundary potentials.

Remarks (cont.)

- We can always solve the BVP (nonhomogeneous PDE)

$$\text{PDE:} \quad \Delta u = f, \quad \text{Inside } D$$

$$\text{BC:} \quad u = 0, \quad \text{On the boundary of } D$$

by

1. Finding any solution V of $\Delta V = f$ (A particular solution).
2. Solving the new BVP

$$\text{PDE:} \quad \Delta W = 0, \quad \text{Inside } D$$

$$\text{BC:} \quad W = V, \quad \text{On the boundary of } D$$

3. Observing that $u = V - W$ is our desired solution.

In other words, we can transfer the nonhomogeneity from the PDE to BC.

Remarks (cont.)

- We can solve the BVP (nonhomogeneous BC)

$$\text{PDE:} \quad \Delta u = 0, \quad \text{Inside } D$$

$$\text{BC:} \quad u = f, \quad \text{On the boundary of } D$$

by

1. Finding any solution V that satisfies $V = f$ on the boundary of D.
2. Solving the new BVP

$$\text{PDE:} \quad \Delta W = \Delta V, \quad \text{Inside } D$$

$$\text{BC:} \quad W = 0, \quad \text{On the boundary of } D$$

3. Observing that $u = V - W$ is the solution to our problem.

In other words, we can transfer the nonhomogeneity from the BC to the PDE.

The Dirichlet Problem in an Annulus

Problem 17-2

To find the function $u(r,\theta)$ that satisfies

$$\text{PDE:}\quad u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad R_1 < r < R_2$$

$$\text{BCs:}\quad \begin{cases} u(R_1,\theta) = g_1(\theta), \\ u(R_2,\theta) = g_2(\theta) \end{cases} \qquad 0 \leqslant \theta < 2\pi.$$

Step 1. (Separation of Variables)

- Substituting $u(r,\theta) = R(r)\Theta(\theta)$ into the Laplace equation and arguing similar to the interior Dirichlet problem we arrive at our general solution

$$\boxed{\begin{aligned} u(r,\theta) = {} & a_0 + b_0 \ln r \\ & + \sum_{n=1}^{\infty} (a_n r^n + b_n r^{-n}) \cos(n\theta) \\ & + \sum_{n=1}^{\infty} (c_n r^n + d_n r^{-n}) \sin(n\theta) \end{aligned}} \qquad (17.4)$$

Step 2. (Substituting into BCs)

- Substituting the solution (17.4) into the BCs and integrating gives the following equations:

$$\begin{cases} a_0 + b_0 \ln R_1 = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} g_1(s)ds \\ a_0 + b_0 \ln R_2 = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} g_2(s)ds \end{cases} \qquad \text{(Solve for } a_0, b_0\text{)}$$

Step 2. (cont.)

$$\begin{cases} a_n R_1^n + b_n R_1^{-n} = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} g_1(s) \cos(ns) ds \\ a_n R_2^n + b_n R_2^{-n} = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} g_2(s) \cos(ns) ds \end{cases}$$

(Solve for a_n, b_n)

$$\begin{cases} c_n R_1^n + d_n R_1^{-n} = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} g_1(s) \sin(ns) ds \\ c_n R_2^n + d_n R_2^{-n} = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} g_2(s) \sin(ns) ds \end{cases}$$

(Solve for c_n, d_n)

Exterior Dirichlet Problem

Problem 17-3

To find the function $u(r, \theta)$ that satisfies

$$\text{PDE:} \quad u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 1 < r < \infty$$

$$\text{BC:} \quad u(1, \theta) = g(\theta), \qquad 0 \leqslant \theta < 2\pi.$$

Problem 17-3 is solved exactly like the interior Dirichlet problem. The only exception is that now we throw out the solutions that are unbounded as r goes to infinity

$$r^n \cos(n\theta), \quad r^n \sin(n\theta), \quad \ln r$$

Exterior Dirichlet Problem (cont.)

Hence, we are left with the solution

$$\boxed{u(r,\theta) = \sum_{n=0}^{\infty} r^{-n} \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right],}$$

where a_n and b_n are exactly as before

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} g(s)ds,$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} g(s)\cos(ns)ds, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} g(s)\sin(ns)ds$$

Remarks

- The exterior Dirichlet problem for arbitrary radius R

$$\text{PDE:} \quad u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad R < r < \infty$$

$$\text{BC:} \quad u(R,\theta) = g(\theta), \qquad 0 \leqslant \theta < 2\pi.$$

has the solution

$$u(r,\theta) = \sum_{n=0}^{\infty} \left(\frac{r}{R}\right)^{-n} [a_n \cos(n\theta) + b_n \sin(n\theta)]$$

- The only solution of the 2-D Laplace equation that depend only on r are constants and $\ln r$. The potential $\ln r$ is very important and is called the logarithmic potential.