PDE and Boundary-Value Problems
Winter Term 2014/2015

Lecture 18

26. Januar 2015

Purpose of Lesson

- To find particular solutions of the Laplace equation in spherical coordinates. To solve the interior and exterior Dirichlet problems for the Laplace equation in 3D.
- To derive the fundamental solution of the Laplace equation and discuss how to solve with its help the Poisson equation (nonhomogeneous Laplace equation).

Laplace's Equation in Spherical Coordinates (Spherical Harmonics)

An important problem is to find the potential inside or outside a sphere when the potential is given on the boundary. Consider, first, the interior problem:

Problem 18-1

To find the function $u(r,\theta,\phi)$ that satisfies

$$\text{PDE:}\quad \left(r^2 u_r\right)_r + \frac{1}{\sin\phi}\left[\sin\phi\, u_\phi\right]_\phi + \frac{1}{\sin^2\phi} u_{\theta\theta} = 0, \quad 0 < r < 1$$

$$\text{BC:}\quad u(1,\theta,\phi) = g(\theta,\phi), \qquad -\pi \leqslant \theta < \pi, \qquad 0 \leqslant \phi < \pi$$

Remarks

- A typical application of the problem 18-1 would be to find the temperature inside a sphere when the temperature is specified on the boundary.
- Quite often $g(\theta, \phi)$ has a specific form, so that it isn't necessary to solve the problem in its most general form.
- We consider two important cases. One is the case when $g(\theta, \phi)$ is constant, and the other is when it depends only on the angle ϕ (the angle from the north pole).

Special Case 1. ($g(\theta, \phi) = constant$)

- In this case, it is clear that the solution is independent of θ and ϕ, and so Laplace's equation reduces to the ODE

$$(r^2 u_r)_r = 0. \tag{18.1}$$

- The general solution of (18.1) is

$$u(r) = \frac{a}{r} + b$$

- In other words, constants and $\frac{c}{r}$ are the only potentials that depend only on the radial distance from the origin. The potential $\frac{1}{r}$ is called the Newtonian potential.

Special Case 2. ($g(\theta, \phi)$ depends only on ϕ)

In this case, the Dirichlet problem takes the form

Problem 18-1a

To find the function $u(r, \theta, \phi)$ that satisfies

$$\text{PDE:} \quad \left(r^2 u_r\right)_r + \frac{1}{\sin\phi}\left[\sin\phi u_\phi\right]_\phi = 0, \quad 0 < r < 1$$

$$\text{BC:} \quad u(1, \theta, \phi) = g(\phi), \qquad 0 \leqslant \phi < \pi$$

Step 1. (Separation of variables)

- We look for solutions of the form

$$u(r, \phi) = R(r)\Phi(\phi)$$

  and arrive at the two ODEs

$$r^2 R'' + 2rR' - n(n+1)R = 0 \qquad \text{(Euler's equation)}$$

$$\left[\sin\phi\Phi'\right]' + n(n+1)\sin\phi\Phi = 0 \quad \text{(Legendre's equation)}$$

- The separation constant is chosen to be $n(n+1)$ for convenience; later we will see why this choice is made.

Step 2. (Solving the Euler equation)

- We solve Euler's equation by substituting $R(r) = r^\alpha$ in the equation and solving for α. Doing this, we get two values

$$\alpha = \begin{cases} n \\ -(n+1) \end{cases}$$

- Hence, Euler's equation has the general solution

$$R(r) = ar^n + br^{-(n+1)}$$

Step 3. (Solving the Legendre equation)

- Making the substitution $x = \cos\phi$ we get the new Legendre equation

$$(1-x^2)\frac{d^2\Phi}{dx^2} - 2x\frac{d\Phi}{dx} + n(n+1)\Phi = 0, \quad -1 \leqslant x \leqslant 1.$$

  The idea here is to solve for $\Phi(x)$ and then substitute $x = \cos\phi$ in the solution.

- Legendre's equation is a linear second-order ODE with variable coefficients. One of the difficulties in this equation is that the coefficient $(1-x^2)$ is zero at the ends of the interval $[-1, 1]$. Equations like this are called singular differential equations and are often solved by the method of Frobenius.

Step 3. (cont.)

- The only bounded solutions of Legendre's equation occur when $n = 0, 1, 2, \dots$ and these solutions are polynomials $P_n(x)$, $-1 \leqslant x \leqslant 1$ (Legendre polynomials)

$$n = 0 \quad P_0(x) = 1$$

$$n = 1 \quad P_1(x) = x$$

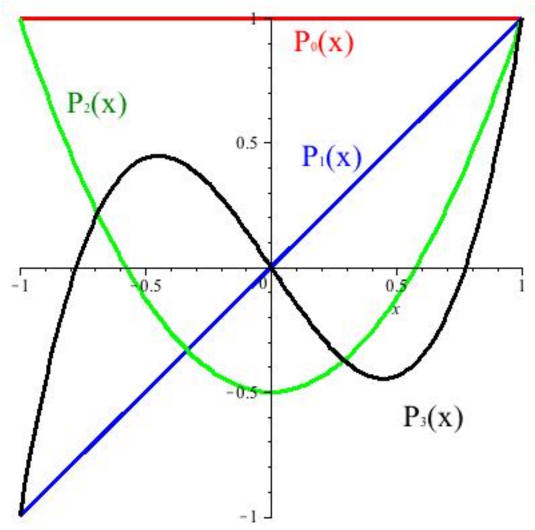
$$n = 2 \quad P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$n = 3 \quad P_3(x) = \frac{1}{2}(5x^2 - 3x)$$

$$\vdots \qquad \vdots \qquad \qquad \vdots$$

$$n \quad P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2 - 1)^n\right] \quad \text{(Rodrigues' formula)}$$

Legendre Polynomials $P_n(x)$



Step 4. (Combination)

- We now have that the bounded solutions of

$$r^2 R'' + 2rR' - n(n+1)R = 0 \qquad 0 < r < 1$$
$$\left[\sin\phi\Phi'\right]' + n(n+1)\sin\phi\Phi = 0 \qquad -\pi \leqslant \phi \leqslant \pi$$

are

$$R(r) = ar^n$$
$$\Phi(\phi) = aP_n(\cos\phi)$$

- Therefore,

$$u(r,\phi) = \sum_{n=0}^{\infty} a_n r^n P_n(\cos\phi). \tag{18.2}$$

Step 5. (Substituting into BC)

- Substituting solution (18.2) into the BC gives

$$\sum_{n=0}^{\infty} a_n P_n(\cos\phi) = g(\phi) \tag{18.3}$$

- Observe that the Legendre polynomials are orthogonal on $[-1, 1]$.

Step 5. (cont.)

So, if we multiply each side of (18.3) by $P_m(\cos\phi)\sin\phi$ and integrate ϕ from 0 to π, we get

$$\int\limits_0^\pi g(\phi)P_m(\cos\phi)\sin\phi d\phi = \sum_{n=0}^\infty a_n \int\limits_0^\pi P_n(\cos\phi)P_m(\cos\phi)\sin\phi d\phi$$

$$= \sum_{n=0}^\infty a_n \int\limits_{-1}^1 P_n(x)P_m(x)dx$$

$$= \begin{cases} 0, & n \neq m \\ \dfrac{2a_m}{2m+1}, & m = n \end{cases}$$

Step 5. (cont.)

- Hence

$$a_n = \frac{2n+1}{2} \int_0^{\pi} g(\phi) P_n(\cos\phi) \sin\phi \, d\phi$$

and the solution to our Dirichlet problem 18-1a is

$$u(r, \phi) = \sum_{n=0}^{\infty} a_n r^n P_n(\cos\phi)$$

Remarks

- The solution of the exterior Dirichlet problem

$$\text{PDE:} \quad \Delta u = 0, \qquad 1 < r < \infty$$

$$\text{BC:} \quad u(1, \theta, \phi) = g(\phi), \quad 0 \leqslant \phi < \pi$$

is

$$u(r, \phi) = \sum_{n=0}^{\infty} \frac{b_n}{r^{n+1}} P_n(\cos\phi),$$

where

$$b_n = \frac{2n+1}{2} \int_0^{\pi} g(\phi) P_n(\cos\phi) \sin\phi d\phi.$$

Remarks (cont.)

- For example, the BC $g(\phi) = 3$ would yield for the solution of the exterior problem
$$u(r, \phi) = \frac{3}{r}.$$
Note that in this problem (in 3D!!!), the solution goes to zero, while in two dimensions, the exterior solution with constant BC was itself a constant.

Fundamental Solution

Problem 18-2

To find a function $u(x)$ that satisfies

$$\Delta u = 0 \qquad x \in \mathbb{R}^n$$

- We attempt to find a solution u of Laplace's equation in $\mathbb{R}^n$, having the form

$$u(x) = v(r),$$

where $r = |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.

- It is clear that $\Delta u = 0$ if and only if that

$$v''(r) + \frac{n-1}{r} v'(r) = 0.$$

- If $r > 0$ we have

$$v(r) = \begin{cases} a \ln r + b & (n = 2) \\ \dfrac{a}{r^{n-2}} + b & (n \geqslant 3), \end{cases}$$

where a and b are constants.

The above consideration motivate the following

Definition

The function

$$\Phi(x) := \begin{cases} -\dfrac{1}{2\pi} \ln |x|, & (n = 2) \\ \dfrac{1}{n(n-2)\omega(n)} \dfrac{1}{|x|^{n-2}} & (n \geqslant 3), \end{cases}$$

defined for $x \in \mathbb{R}^n$, $x \neq 0$, is the fundamental solution of Laplace's equation.

- $\omega(n)$ denotes the volume of the unit ball in $\mathbb{R}^n$.
- $|D\Phi(x)| \leqslant \dfrac{C}{|x|^{n-1}}, \quad |D^2\Phi(x)| \leqslant \dfrac{C}{|x|^n}, \quad (x \neq 0)$

Theorem (Solving Poisson's equation)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice contionuous differentiable with compact support, and let u satisfy

$$u(x) = \int\limits_{\mathbb{R}^n} \Phi(x-y) f(y) dy$$

$$= \begin{cases} -\dfrac{1}{2\pi} \displaystyle\int\limits_{\mathbb{R}^2} \ln|x-y| f(y) dy & (n=2) \\ \dfrac{1}{n(n-2)\omega(n)} \displaystyle\int\limits_{\mathbb{R}^n} \dfrac{f(y)}{|x-y|^{n-2}} dy & (n \geqslant 3) \end{cases}$$

Then $u \in C^2(\mathbb{R}^n)$ and $-\Delta u = f \quad \text{in } \mathbb{R}^n$.

Remarks

- We cannot just compute

$$\Delta u(x) = \int\limits_{\mathbb{R}^n} \Delta_x \Phi(x-y) f(y) dy = 0.$$

- Indeed, $D^2\Phi(x-y)$ is not summable near the singularity at $y = x$, and so the differentiation under the integral sign is unjustified (and incorrect).
- We must proceed more carefully in calculating Δu.