PDE and Boundary-Value Problems Winter Term 2014/2015

Lecture 4

Saarland University

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Purpose of Lesson

- To discuss the second and the third important types of BCs:
  - temperature of the surrounded medium specified,
  - heat flow across the boundary specified.
- To show how the one-space dimensional heat equation

$$u_t = \alpha^2 u_{xx} + f(x,t)$$

  is derived from the basic principle of *conversation of heat*.
- To show how the rate of heat transfer depends on *thermal conductivity*, *thermal capacity*, and *density*.
- To discuss a few variations of the basic heat equation.

Type 2 BC (Temperature of the surrounding medium specified)

Suppose we have the following experiment that we again break into steps:

1. We consider again our laterally insulated copper rod. Recall that *laterally insulated* means: heat can flow in and out of the rod at the ends, but not across the lateral boundary.

2. We put the left side of the rod in a container that has a changing temperature $g_1(t)$, while the right end is putted in another liquid with temperature $g_2(t)$.

Type 2 BC (Temperature of the surrounding medium specified)

1. We cannot say the boundary temperatures of the rod will be the same as the liquid temperatures $g_1(t)$ and $g_2(t)$.
2. We know (Newton's law of cooling) that whenever the rod temperature at one of the boundaries is less than the respective liquid temperatures, then heat will flow into the rod at a rate proportional to this difference.

Type 2 BC (Temperature of the surrounding medium specified)

Newton's law of cooling states:

$$\begin{cases} \text{Outward flux of heat (at } x = 0) = h\left[u(0,t) - g_1(t)\right] \\ \text{Outward flux of heat (at } x = L) = h\left[u(L,t) - g_2(t)\right]. \end{cases}$$

Here

- Outward flux of the heat $=$ the number of calories crossing the ends of the rod per second.
- h is a heat-exchange coefficient, which is the measure of how many calories flow across the boundary per unit of temperature difference per second per cm.

Type 2 BC (Temperature of the surrounding medium specified)

Remark

Note that the outward flux of heat will be positive at either end provided the temperature of the rod is greater than the surrounding medium.

Otherwise, the outward flux of heat will be negative at either end provided the temperature of the rod is less than the surrounding medium.

Type 2 BC (Temperature of the surrounding medium specified)

In addition to Newton's law we state Fourier's law (proven experimentally)

Outward flux of heat across a boundary is proportional to the inward normal derivative across the boundary

This law says that if the temperature is increasing rapidly in the direction outward from the boundary of a domain, then heat will flow from the surrounding medium into the domain.

Type 2 BC (Temperature of the surrounding medium specified)

In our 1-dimensional problem, Fourier's law takes the form:

$$\begin{cases} \text{Outward flux of heat (at } x = 0) = k\dfrac{\partial u(0,t)}{\partial x} \\ \text{Outward flux of heat (at } x = L) = -k\dfrac{\partial u(L,t)}{\partial x}, \end{cases}$$

where k is the thermal conductivity of the metal, which is a measure of how well the material conducts heat.

Type 2 BC (Temperature of the surrounding medium specified)

Remark

Fourier's law actually holds anywhere inside the rod and not just at the boundary; for example,

$$\boxed{\text{Flux of heat crossing } x_0 \text{ (from left to right)} \; = -k\frac{\partial u(x_0,t)}{\partial x}}$$

Fourier's law says that if $u_x(x_0,t) < 0$, then heat will flow from left to right; if $u_x(x_0,t) > 0$, then the flow of heat through point x_0 will be from right to left.

Heat always flows from high to low temperatures!

Type 2 BC (Temperature of the surrounding medium specified)

Finally, combining two expressions for heat flux, we get our desired BCs in purely mathematical terms; namely,

$$\text{BCs} \quad \begin{cases} \dfrac{\partial u(0,t)}{\partial x} = \dfrac{h}{k}\left[u(0,t) - g_1(t)\right] \\ \dfrac{\partial u(L,t)}{\partial x} = -\dfrac{h}{k}\left[u(L,t) - g_2(t)\right] \end{cases} \qquad 0 < t < \infty$$

Quite often, the constant h/k is simply written as λ.

Type 2 BC (Temperature of the surrounding medium specified)

In higher dimensions, we have similar BCs.

Example

For example, if the boundary of a circular disc is interfaced with a moving liquid that has a temperature $g(\theta, t)$, our BCs would be

$$\frac{\partial u(R, \theta, t)}{\partial r} = -\frac{h}{k}\left[u(R, \theta, t) - g(\theta, t)\right].$$

This type of BCs would be called a linear BC (since it is linear in u and u_r) but nonhomogeneous due to the right-hand side $g(\theta, t)$.

Type 3 BC (Flux specified - including the special case of insulated boundaries)

Insulated boundaries are those that do not allow any flow of heat to pass, and, hence, the normal derivative (inward or outward) must be zero on the boundary (since the normal derivative is proportional to the flux).

Example

In the case of the 1-dimensional rod with insulated ends at $x = 0$ and $x = L$, the BCs are

$$\begin{cases} u_x(0, t) = 0 \\ u_x(L, t) = 0 \end{cases} \qquad 0 < t < \infty.$$

Type 3 BC (Flux specified - including the special case of insulated boundaries)

In 2-dimensional domains, an insulated boundary would mean that the normal derivative of the temperature across the boundary is zero.

Example

For example, if the circular disc were insulated on the boundary, then the BC would be

$$u_r(R, \theta, t) = 0 \qquad \forall\, 0 \leqslant \theta < 2\pi \quad \text{and} \quad \forall\, 0 < t < \infty.$$

Type 3 BC (Flux specified - including the special case of insulated boundaries)

On the other hand, if we specify the amount of heat entering across the boundary of our disc, the BC is

$$u_r(R, \theta, t) = f(\theta, t),$$

where $f(\theta, t)$ would represent the amount of heat crossing into the circular disc from am outside heating source.

Typical BCs for 1-dimensional heat flow

Let us consider the following experiment

1. Suppose we have a copper rod 200 cm long that is laterally insulated and has an initial temperature of $0°$C.
2. Suppose the top of the rod ($x = 0$) is insulated, while the bottom ($x = 200$) is immersed in moving water that has a constant temperature of $g(t) = 20°$C.

Typical BCs for 1-dimensional heat flow

The mathematical model for this problem would be the following four equations:

$$\begin{array}{rl}
\text{PDE} & u_t = \alpha^2 u_{xx} \quad 0 < x < 200 \quad 0 < t < \infty \\
\text{BCs} & \left\{ \begin{array}{l} u_x(0,t) = 0 \\ u_x(200,t) = -\dfrac{h}{k}\left[u(200,t) - 20\right] \end{array} \right. \qquad 0 < t < \infty \\
\text{IC} & u(x,0) = 0^\circ C \quad 0 \leqslant x \leqslant 200,
\end{array}$$

where

- $\alpha^2 = 1.16\ cm^2/sec$ is the diffusivity constant for copper;
- $k = 0.93\ cal/cm - sec^\circ$C is the termal conductivity of copper;
- h is heat exchange coefficient.

Typical BCs for 1-dimensional heat flow

Remark

To find h is a hard problem itself. It measures the rate that heat is being exchanged between bottom of the rod and the surrounding water.

We would have to carry out an experiment to determine the value of h.

Derivation of the heat equation

Suppose that we have a 1-dimensional rod of length L for which we make the following assumptions:

1. The rod is made of a single homogeneous conducting material.
2. The rod is laterally insulated (heat flows only in the x-direction).
3. The rod is thin (the temperature at all points of a cross section is constant).

Derivation of the heat equation

If we apply the principle of conversation of heat to the segment $[x, x + \Delta x]$, we can claim

Net change of heat inside $[x, x + \Delta x]$
$=$ total heat generated inside $[x, x + \Delta x]$
$+$ net flux of heat across the boundaries

Derivation of the heat equation

Observe that the total amount of heat (in calories) inside $[x, x + \Delta x]$ at any time t is measured by

$$\boxed{\text{Total heat inside } [x, x + \Delta x] = \int\limits_{x}^{x+\Delta x} c\rho A u(s, t) ds},$$

where

- $c =$ thermal capacity of the rod (measures the ability of the rod to store heat);
- $\rho =$ density of the rod;
- $A =$ cross-section ares of the rod.

Derivation of the heat equation

Therefore,

$$\text{Net change of heat inside } [x, x+\Delta x] = \frac{d}{dt}\left(\text{Total heat inside } [x, x+\Delta x]\right) = \frac{d}{dt}\int\limits_x^{x+\Delta x} c\rho A u(s,t)\,ds$$

Derivation of the heat equation

In addition,

$$\boxed{\begin{array}{l}\text{Net flux of heat across the boundaries} \\ = kA\left[u_x(x+\Delta x, t) - u_x(x,t)\right]\end{array}}$$

It remains only to estimate total heat generated inside $[x, x+\Delta x]$.

- If we assume that our rod has no internal heat source, then

$$\boxed{\text{Total heat generated inside}[x, x+\Delta x] = 0}.$$

- Otherwise, if our rod is supplied with an internal heat source, then

$$\boxed{\text{Total heat generated inside}[x, x+\Delta x] = A \int\limits_{x}^{x+\Delta x} f(s,t)ds}.$$

Derivation of the heat equation

Combining all three equations we can write the conservation of energy equation as

$$\begin{aligned} \frac{d}{dt} \int_{x}^{x+\Delta x} c\rho A u(s,t)ds &= c\rho A \int_{x}^{x+\Delta x} u_t(s,t)ds \\ &= kA\left[u_x(x+\Delta x,t) - u_x(x,t)\right] + A \int_{x}^{x+\Delta x} f(s,t)ds, \end{aligned} \tag{4.1}$$

where

- $k =$ thermal conductivity of the rod (measures the ability to conduct the heat)
- $f(x,t) =$ internal heat source (calories per cm per sec).

Derivation of the heat equation

The problem now is to replace equation (4.1) by one that does not contain integrals.

Recall the Mean Value Theorem from calculus:

Mean Value Theorem

If $f(x)$ is a continuous function on $[a, b]$, then there exists at least one number ξ, $a < \xi < b$ that satisfies

$$\int_a^b f(x)dx = f(\xi)(b - a).$$

Derivation of the heat equation

Applying the Mean Value Theorem to equation (4.1) we arrive for $x < \xi_i < x + \Delta x$ at the following equation:

$$c\rho A u_t(\xi_1, t)\Delta x = kA\left[u_x(x+\Delta x, t) - u_x(x, t)\right] + Af(\xi_2, t)\Delta x$$

or

$$u_t(\xi, t) = \frac{k}{c\rho}\left\{\frac{u_x(x+\Delta x, t) - u_x(x, t)}{\Delta x}\right\} + \frac{1}{c\rho} f(\xi, t).$$

Derivation of the heat equation

Finally, letting $\Delta x \to 0$, we have the desired result

$$\boxed{u_t(x,t) = \alpha^2 u_{xx}(x,t) + F(x,t)},$$

where

- $\alpha^2 = \frac{k}{c\rho}$ (called the diffusivity of the rod);
- $F(x,t) = \frac{1}{c\rho} f(x,t)$ (heat source density).

Derivation of the heat equation

Suppose the rod is not laterally insulated and the heat can flow across the lateral boundary at a rate proportional to the differnce between the temperature $u(x,t)$ and the surrounding medium that we keep at zero.

In this case, the conservation of heat principle will give

$$u_t = \alpha^2 u_{xx} - \beta u + F(x,t),$$

where $\beta =$ rate constant for the lateral heat flow ($\beta > 0$).

Remarks

1. The constant k is the thermal conductivity of the rod and a measure of the heat flow (in calories) that is transmitted per second through a plate 1 cm thick across an area of 1 cm^2 when the temperature difference is $1^\circ C$.

2. If the material of the rod is uniform, then k will not depend on x. For some materials, the value of k depends on the temperature u.

3. The constant c is known as thermal capacity of the substance and measures the amount of energy the substance can store.