

# PDE and Boundary-Value Problems

## Winter Term 2014/2015

### Lecture 5

Saarland University

13. November 2014

## Purpose of Lesson

- To introduce the powerful method of separation of variables and to show how this method can be used to solve some of diffusion-type problems
- To show how problems with **nonhomogeneous** BCs can be solved by transforming them into others with zero BCs.

# Separation of Variables

Separation of variables is one of the oldest techniques for solving IBVPs. It applies to problems where

- 1 PDE is **linear** and **homogeneous** (not necessarily constant coefficients).
- 2 BCs are also **linear** and **homogeneous**.

Example of admissible BCs

$$\alpha u_x(0, t) + \beta u(0, t) = 0$$

$$\gamma u_x(L, t) + \delta u(L, t) = 0$$

Method of separation of variables dates back to the time of Joseph Fourier (in fact, it's occasionally called *Fourier's method*).



- Jean Baptiste Joseph Fourier  
(1768-1830)

<b>Field:</b>	Mathematician, physicist and historian
<b>Alma mater:</b>	École Normale
<b>Advisor:</b>	Joseph Lagrange
<b>Doctoral students:</b>	Gustav Dirichlet, Claude-Louis Navier
<b>Known for:</b>	Fourier series, Fourier transform, Fourier's law of conduction

# Separation of Variables

It is probably the most widely used method of solution (when applicable).

# Overview of Separation of Variables

Separation of variables looks for simple-type solutions to the PDE of the form

$$u(x, t) = X(x)T(t)$$

where  $X(x)$  is some function of  $x$  and  $T(t)$  is some of  $t$ .

The solutions are simple because any temperature  $u(x, t)$  of this form will retain its basic „shape“ for different values of time  $t$ .

## Overview of Separation of Variables $u(x, t) = X(x)T(t)$

- The general idea is that it is possible to find an infinite number of these solutions to the PDE (which, at the same time, also satisfy the BCs).
- These simple functions

$$u_n(x, t) = X_n(x)T_n(t)$$

(called **fundamental solutions**) are the building blocks of our problem.

- We are looking for the solution  $u(x, t)$  of our IBVP as the resulting sum

$$u(x, t) = \sum_{n=1}^{\infty} A_n X_n(x) T_n(t)$$

which satisfies the initial condition.

## Problem 5-1

To find the function  $u(x, t)$  that satisfies

$$\text{PDE: } u_t = \alpha^2 u_{xx}, \quad 0 < x < 1, \quad 0 < t < \infty$$

$$\text{BCs: } \begin{cases} u(0, t) = 0 \\ u(1, t) = 0 \end{cases}, \quad 0 < t < \infty$$

$$\text{IC: } u(x, 0) = \phi(x), \quad 0 \leq x \leq 1$$



## Step 1 (Finding elementary solutions to the PDE)

- We look for solutions of the form  $u(x, t) = X(x)T(t)$  by substituting  $X(x)T(t)$  into the PDE and solving for  $X(x)T(t)$ . As a result we get

$$\boxed{X(x)T'(t) = \alpha^2 X''(x)T(t).} \quad (5.1)$$

- If we **divide** each side of (5.1) by  $\alpha^2 X(x)T(t)$ , we have

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} \quad (5.2)$$

and obtain what is called **separated variables**, that is, the left-hand side of (5.2) depends only on  $t$  and the right-hand side of (5.2), only on  $x$ .

## Step 1 (Finding elementary solutions to the PDE)

- Since  $x$  and  $t$  are **independent of each other**, each side must be a fixed constant (say  $k$ ); hence, we can write

$$T'(t) - k\alpha^2 T(t) = 0$$

$$X''(x) - kX(x) = 0.$$

- Now we can solve each of these two ODEs, multiply them together to get a solution to the PDE (note that we have changed a second-order PDE to two ODEs).

## Step 1 (Finding elementary solutions to the PDE)

- We change  $k := -\lambda^2$ . Otherwise,  $T(t)$  factor doesn't go to zero as  $t \rightarrow \infty$ . As a result we get

$$\begin{aligned}T'(t) + \lambda^2 \alpha^2 T(t) &= 0 \\X''(x) + \lambda^2 X(x) &= 0.\end{aligned}\tag{5.3}$$

- Both of equations in (5.3) are standard-type ODEs and have solutions

$$\begin{aligned}T(t) &= C_1 e^{-\lambda^2 \alpha^2 t} \quad (C_1 \text{ an arbitrary constant}) \\X(x) &= C_2 \sin(\lambda x) + C_3 \cos(\lambda x) \quad (C_2, C_3 \text{ arbitrary}).\end{aligned}$$

## Step 1 (Finding elementary solutions to the PDE)

- Hence all functions

$$u(x, t) = e^{-\lambda^2 \alpha^2 t} (A \sin(\lambda x) + B \cos(\lambda x))$$

(with  $A$ ,  $B$  and  $\lambda$  arbitrary) satisfies the PDE  $u_t = \alpha^2 u_{xx}$ .

## Step 2 (Finding solutions to the PDE and the BCs)

- The next step is to choose a certain **subset** of solutions

$$e^{-\lambda^2 \alpha^2 t} (A \sin(\lambda x) + B \cos(\lambda x)) \quad (5.4)$$

that satisfy the boundary conditions

$$u(0, t) = 0, \quad u(1, t) = 0 \quad \forall t > 0.$$

- To do this, we substitute solutions (4.4) into BCs, getting

$$u(0, t) = B e^{-\lambda^2 \alpha^2 t} = 0 \quad \Rightarrow \quad B = 0$$

$$u(1, t) = A e^{-\lambda^2 \alpha^2 t} \sin \lambda = 0 \quad \Rightarrow \quad \sin \lambda = 0 \quad \Rightarrow \quad \lambda_n = \pm n\pi.$$

- Case  $A = B = 0$  is not interesting.

## Step 2 (Finding solutions to the PDE and the BCs)

- We have now finished the second step; we have found an infinite number of functions

$$u_n(x, t) = A_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x) \quad n = 1, 2, 3, \dots$$

each one satisfying the PDE and BCs.

- These functions  $u_n$  are the building blocks (**fundamental solutions**) of the problem, and our desired solution will be a certain sum of these simple functions; the specific sum will depend on the initial conditions.

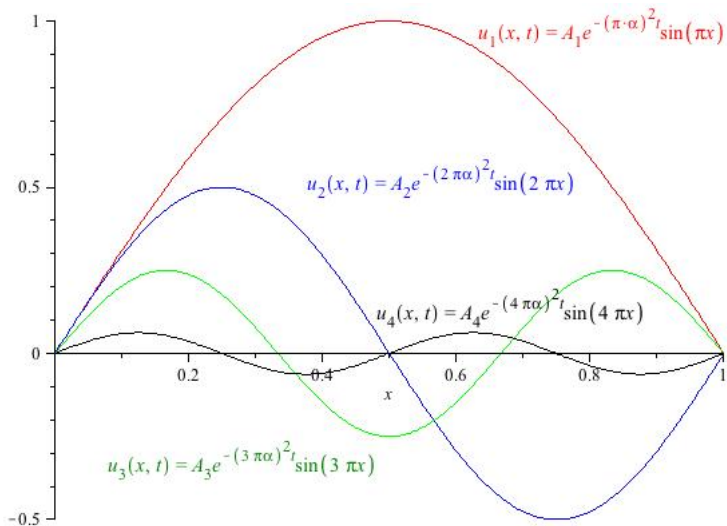


Figure 1: Fundamental solutions

### Step 3 (Finding solutions to the PDE, BCs, and the IC)

- The last step is to add the fundamental solutions

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x) \quad (5.5)$$

in such a way (pick the coefficients  $A_n$ ) that the initial condition  $u(x, 0) = \phi(x)$  is satisfied.

- Substituting (5.5) into the IC gives

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x).$$



## Step 3 (Finding solutions to the PDE, BCs, and the IC)

### Question:

Is it possible to expand the initial temperature  $\phi(x)$  as the sum of the elementary function as follows:

$$A_1 \sin(\pi x) + A_2 \sin(2\pi x) + A_3 \sin(3\pi x) + \dots?$$

### Answer:

The answer to this question is **YES** provided  $\phi(x)$  is **continuous**.

Hence, the question now becomes how to find the coefficients  $A_n$ .

## Remark

The functions

$$\sin(n\pi x), \quad n = 1, 2, \dots$$

are **orthogonal** to each other in the sense

$$\int_0^1 \sin(m\pi x) \sin(n\pi x) dx = \begin{cases} 0, & m \neq n \\ 1/2, & m = n \end{cases}$$

This property can be illustrated by looking at the graph of these functions.

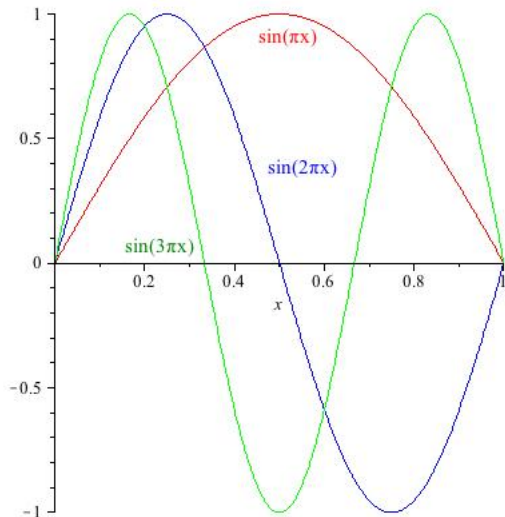


Figure 2: Orthogonal sequence of functions

### Step 3 (Finding solutions to the PDE, BCs, and the IC)

We are now in position to solve for the coefficients in the expression

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x). \quad (5.6)$$

- We **multiply** each side of (5.6) by  $\sin(m\pi x)$  ( $m$  is an arbitrary integer) and **integrate** from zero to one. As a result we get

$$\int_0^1 \phi(x) \sin(m\pi x) dx = A_m \int_0^1 \sin^2(m\pi x) dx = \frac{1}{2} A_m$$
$$\Rightarrow A_m = 2 \int_0^1 \phi(x) \sin(m\pi x) dx.$$

### Step 3 (Finding solutions to the PDE, BCs, and the IC)

We are done; the solution is

$$u(x, t) = 2 \sum_{n=1}^{\infty} \left[ \int_0^1 \phi(s) \sin(n\pi s) ds \right] e^{-(n\pi\alpha)^2 t} \sin(n\pi x).$$

## Remark

The terms in the solution

$$u(x, t) = A_1 e^{-(\pi\alpha)^2 t} \sin(\pi x) + A_2 e^{-(2\pi\alpha)^2 t} \sin(2\pi x) + \dots$$

become small very fast due to the factor

$$e^{-(n\pi\alpha)^2 t}.$$

Hence, for long time periods, the solution is approximately equal to the first term

$$u(x, t) \cong A_1 e^{-(\pi\alpha)^2 t} \sin(\pi x).$$