PDE and Boundary-Value Problems
Winter Term 2014/2015

Lecture 6

Saarland University

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Purpose of Lesson

- To show how problems with nonhomogeneous BCs can be solved by transforming them into others with zero BCs.
- To show how more complicated heat-flow problems can be solved by separation of variables.
- Eigenvalue problems, known as Sturm-Liouville problems, are introduced, and some properties of these general problems are discussed.
- To show how to solve the IBVP with nonhomogeneous PDE by the eigenfunction expansion method.

Transforming Nonhomogeneous BCs into Homogeneous Ones

Problem 6-1

Consider heat flow in an insulated rod where two ends are kept at constant temperature k_1 and k_2; that is,

$$\text{PDE:} \quad u_t = \alpha^2 u_{xx}, \qquad 0 < x < L, \quad 0 < t < \infty$$

$$\text{BCs:} \quad \begin{cases} u(0,t) = k_1 \\ u(L,t) = k_2 \end{cases}, \qquad 0 < t < \infty$$

$$\text{IC:} \quad u(x,0) = \phi(x), \qquad 0 \leqslant x \leqslant L.$$

The difficulty here is that since the BCs are not homogeneous, we cannot solve this problem by separation of variables.

Transforming Nonhomogeneous BCs into Homogeneous Ones

- It is obvious that the solution of problem 6-1 will have a steady-state solution (solution when $t = \infty$) that varies linearly (in x) between the boundary temperatures k_1 and k_2.
- It seems reasonable to think of our temperature $u(x, t)$ as the sum of two parts

$$u(x, t) = \text{steady state} + \text{transient},$$

where steady state is eventual solution for large times, while transient is a part of the solution that depends on the IC (and will go to zero). So,

$$u(x, t) = \left[k_1 + \frac{x}{L}(k_2 - k_1)\right] + U(x, t).$$

Transforming Nonhomogeneous BCs into Homogeneous Ones

- Our goal is to find the transient $U(x,t)$. By substituting

$$u(x,t) = \left[k_1 + \frac{x}{L}(k_2 - k_1)\right] + U(x,t)$$

  in the original problem 6-1, we will arrive at a new problem in $U(x,t)$.

- We can solve this new problem for $U(x,t)$ and add it to the steady state to get $u(x,t)$.

Transforming Nonhomogeneous BCs into Homogeneous Ones

Problem 6-1a

$$\text{PDE:} \quad U_t = \alpha^2 U_{xx}, \qquad 0 < x < L,$$

$$\text{BCs:} \quad \begin{cases} U(0,t) = 0 \\ U(L,t) = 0 \end{cases}, \qquad 0 < t < \infty$$

$$\text{IC:} \quad U(x,0) = \overline{\phi}(x), \qquad 0 \leqslant x \leqslant L,$$

where $\overline{\phi}(x) := \phi(x) - \left[k_1 + \frac{x}{L}(k_2 - k_1)\right]$ new IC. But it is known!!!

The problem 6-1a (fortunately) has a homogeneous PDE as well as homogeneous BCs, and so we can solve it by separation of variables.

Question:

What about more realistic-type BCs with time-varying right-hand sides?

Answer:

The ideas are similar to the previous problem 6-1 but a little more complicated.

Transforming Time Varying BCs to Zero BCs

Problem 6-2

Consider the typical problem

$$\text{PDE:} \qquad u_t = \alpha^2 u_{xx}, \qquad 0 < x < L, \quad 0 < t < \infty$$

$$\text{BCs:} \quad \begin{cases} u(0,t) = g_1(t) \\ u_x(L,t) + hu(L,t) = g_2(t) \end{cases}, \qquad 0 < t < \infty$$

$$\text{IC:} \qquad u(x,0) = \phi(x), \qquad 0 \leqslant x \leqslant L.$$

Transforming Time Varying BCs to Zero BCs

- We seek a solution of the form

$$u(x,t) = A(t)\left[1 - \frac{x}{L}\right] + B(t)\frac{x}{L} + U(x,t)$$

  where $A(t)$ and $B(t)$ are chosen so that the steady-state part

$$S(x,t) = A(t)\left[1 - \frac{x}{L}\right] + B(t)\frac{x}{L}$$

  satisfies the BCs of the problem 6-2.

- The transformed problem in $U(x,t)$ will have homogeneous BCs.

Transforming Time Varying BCs to Zero BCs

- Substituting $S(x,t)$ into BCs we get equations for $A(t)$ and $B(t)$

$$A(t) = g_1(t)$$
$$B(t) = \frac{g_1(t) + Lg_2(t)}{1 + Lh}.$$

- Hence, we have

$$u(x,t) = g_1(t)\left[1 - \frac{x}{L}\right] + \frac{g_1(t) + Lg_2(t)}{1 + Lh}\frac{x}{L} + U(x,t).$$

Transforming Time Varying BCs to Zero BCs

The transformed problem in $U(x,t)$ has a form

Problem 6-2a

PDE: $U_t = \alpha^2 U_{xx} - S_t,$ (nonhomogeneous PDE)

BCs: $\begin{cases} U(0,t) = 0 \\ U_x(L,t) + hU(L,t) = 0 \end{cases},$ (homogeneous BCs)

IC: $U(x,0) = \phi(x) - S(x,0),$ (new IC - but known).

The problem 6-2a has zero BCs (unfortunately, the PDE is nonhomogeneous). We can't solve this problem by separation of variables. But it can be solve by some other methods.

Remark

For BCs of the form

$$\begin{cases} u(0,t) = g_1(t) \\ u(L,t) = g_2(t) \end{cases}$$

the method discussed in the problem 6-2 will give us the transformation

$$u(x,t) = g_1(t) + \frac{x}{L}\left[g_2(t) - g_1(t)\right] + U(x,t).$$

More Complicated Problems and Separation of Variables

We start with a 1-dimensional heat-flow problem where one of the BCs contains derivatives.

Heat-Flow Problem with Derivative BC

- Suppose we have a laterally insulated rod of length 1.
- Consider an apparatus in which we fix the temperature at the left end of the rod at $u(0, t) = 0$ and immerse the right end of the rod in a solution of water fixed at the same temperature of zero (zero refers to some reference temperature).
- Newton's law of cooling says that the BC at $x = 1$ is

$$u_x(1, t) = -hu(1, t).$$

- Suppose now that the initial temperature of the rod is $u(x, 0) = x$, but instantaneously thereafter ($t > 0$), we apply our BCs.

To find the ensuing temperature, we must solve the IBVP

Problem 6-3

To find the function $u(x,t)$ that satisfies

$$\text{PDE:} \qquad u_t = \alpha^2 u_{xx}, \qquad 0 < x < 1, \quad 0 < t < \infty$$

$$\text{BCs:} \quad \begin{cases} u(0,t) = 0 \\ u_x(1,t) + hu(1,t) = 0 \end{cases}, \qquad 0 < t < \infty$$

$$\text{IC:} \qquad u(x,0) = x, \qquad 0 \leqslant x \leqslant 1$$

To apply the separation of variables method, we carry out the following steps:

Step 1 (Finding elementary solutions to the PDE)

We look for solutions of the form $u(x,t) = X(x)T(t)$ by substituting $X(x)T(t)$ into the PDE and solving for $X(x)T(t)$. As a result we get

$$u(x,t) = e^{-\lambda^2\alpha^2 t}\left(A\sin(\lambda x) + B\cos(\lambda x)\right)$$

(with A, B and λ arbitrary).

Step 2 (Finding solutions to the PDE and the BCs)

- The next step is to choose a certain subset of solutions

$$e^{-\lambda^2\alpha^2 t}\left(A\sin\left(\lambda x\right)+B\cos\left(\lambda x\right)\right) \tag{6.1}$$

  that satisfy BCs.

- To do this, we substitute solutions (6.1) into BCs, getting

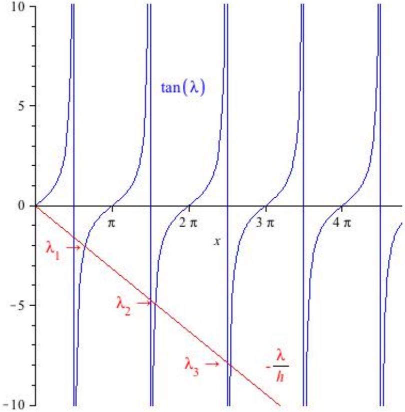
$$Be^{-\lambda^2\alpha^2 t}=0 \quad \Rightarrow \quad B=0$$
$$A\lambda e^{-\lambda^2\alpha^2 t}\cos\lambda+hAe^{-\lambda^2\alpha^2 t}\sin\lambda=0.$$

  Performing alittle algebra on this last equation gives us the condition on λ

$$\tan\left(\lambda\right)=-\lambda/h.$$

Step 2 (Finding solutions to the PDE and the BCs)

In other words, to find λ, we must find the intersections of the curves $\tan(\lambda)$ and $-\lambda/h$.



Step 2 (Finding solutions to the PDE and the BCs)

- These values $\lambda_1, \lambda_2, \ldots$ can be computated numerically for a given h on a computer and are called the eigenvalues of the boundary-value problem

$$\begin{aligned} X'' + \lambda^2 X &= 0 \\ X(0) &= 0 \\ X'(1) + hX(1) &= 0 \end{aligned} \tag{6.2}$$

In other words, they are the values of λ for which there exitst a nonzero solution.

Step 2 (Finding solutions to the PDE and the BCs)

The eigenvalues λ_n of (6.2), which, in this case, are the roots of

$$\tan(\lambda) = -\lambda/h,$$

have been computed (for $h = 1$) numerically, and the first five values are listed in Table 6.1

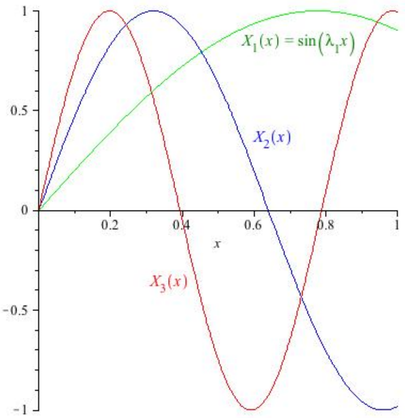
n	λ_n
1	$2,02$
2	$4,91$
3	$7,98$
4	$11,08$
5	$14,20$

Table 6.1: Roots of $\tan(\lambda) = -\lambda$.

Step 2 (Finding solutions to the PDE and the BCs)

- The solutions of (6.2) corresponding to the eigenvalues λ_n are called the eigenfunctions $X_n(x)$, and for problem (6.2), we have

$$X_n(x) = sin(\lambda_n x).$$



Step 2 (Finding solutions to the PDE and the BCs)

- We have now finished the second step; we have found an infinite number of functions (fundamental solutions),

$$u_n(x,t) = e^{-\lambda_n^2 \alpha^2 t} \sin(\lambda_n x)$$

each one satisfying the PDE and BCs.

Step 3 (Finding solutions to the PDE, BCs, and the IC)

- The last step is to add the fundamental solutions

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(\lambda_n \alpha)^2 t} \sin(\lambda_n x) \tag{6.3}$$

  in such a way (pick the coefficients A_n) that the initial condition $u(x,0) = x$ is satisfied.

- Substituting (6.3) into the IC gives

$$x = \sum_{n=1}^{\infty} A_n \sin(\lambda_n x).$$

Step 3 (Finding solutions to the PDE, BCs, and the IC)

We are now in position to solve for the coefficients in the expression

$$x = \sum_{n=1}^{\infty} A_n \sin(\lambda_n x). \tag{6.4}$$

- We multiply each side of (6.4) by $\sin(\lambda_m x)$ (m is an arbitrary integer) and integrate from zero to one. As a result we get

$$\begin{aligned}\int_0^1 x \sin(\lambda_m x) dx &= A_m \int_0^1 sin^2(\lambda_m x) dx \\ &= A_m \left(\frac{\lambda_m - \sin(\lambda_m) \cos(\lambda_m)}{2\lambda_m} \right).\end{aligned}$$

Step 3 (Finding solutions to the PDE, BCs, and the IC)

- Solving for A_n (we'll change the notation to A_n), we get

$$A_n = \frac{2\lambda_n}{\lambda_n - \sin(\lambda_n)\cos(\lambda_n)} \int_0^1 x \sin(\lambda_n x) dx. \tag{6.5}$$

- We are done; our solution is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(\lambda_n \alpha)^2 t} \sin(\lambda_n x)},$$

where the constants A_n are given by (6.5).

For problem 6-3 the first five constants A_n have been computed and are listed in Table 6.2:

n	A_n
1	$0,24$
2	$0,22$
3	$-0,03$
4	$-0,11$
5	$-0,09$

Table 6.2: Coefficients A_n.

Hence, the first three terms of the IBVP 6-3 are

$$u(x,t) \approx 0,24e^{-4t}\sin(2x) + 0,22e^{-24t}\sin(4,9x) + 0,03e^{-63,3t}\sin(7,98x) + \dots$$

Remark

- The eigenvalue problem (6.2) is a special case of the general problem

$$\text{ODE:} \quad [p(x)Y'(x)]' - q(x)Y(x) + \lambda r(x)Y(x) = 0,$$

$$\text{BCs:} \quad \begin{cases} \alpha_1 Y(0) + \beta_1 Y'(0) = 0 \\ \alpha_2 Y(1) + \beta_2 Y'(1) = 0 \end{cases},$$

known as Sturm-Liouville problem.

Sturm and Liouville proved that under suitable conditions on the functions $p(x)$, $q(x)$ and $r(x)$, the SLP has

- An infinite sequence of eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots \to \infty$$

- Corresponding to each eigenvalue λ_n, there is one nonzero solution $Y_n(x)$.

- If $Y_n(x)$ and $Y_m(x)$ are two different eigenfunctions (corresponding to $\lambda_n \neq \lambda_m$), then they are orthogonal with respect to the weight function $r(x)$ on the interval $[0, 1]$; that is, they satisfy

$$\int_0^1 r(x) Y_n(x) Y_m(x) dx = 0.$$

Solving Nonhomogeneous PDEs (Eigenfunction Expansions)

We have discussed how transform nonhomogeneous BCs into homogeneous ones. Unfortunately, the PDE was left nonhomogeneous by this process and we were left with the problem

Problem 6-4

$$\text{PDE:} \quad u_t = \alpha^2 u_{xx} + f(x,t), \qquad 0 < x < 1, \quad 0 < t < \infty$$

$$\text{BCs:} \quad \begin{cases} \alpha_1 u_x(0,t) + \beta_1 u(0,t) = 0 \\ \alpha_2 u_x(1,t) + \beta_2 u(1,t) = 0 \end{cases}, \qquad 0 < t < \infty$$

$$\text{IC:} \quad u(x,0) = \phi(x), \qquad 0 \leqslant x \leqslant 1$$

Solving Nonhomogeneous PDEs (Eigenfunction Expansions)

We solve problem 6-4 by a method that is analogous to the method of *variation of parameters* in ODEs and is known as the eigenfunction expansion method.

The idea is quite simple. The solution of problem 6-4 with $f(x,t) = 0$ (corresponding homogeneous problem) is given by

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(\lambda_n \alpha)^2 t} X_n(x),$$

where λ_n and $X_n(x)$ are the eigenvalues and eigenfunctions of the Sturm-Liouville problem,

$$\begin{aligned} X'' + \lambda^2 X &= 0 \\ \alpha_1 X'(0) + \beta_1 X(0) &= 0 \\ \alpha_2 X'(1) + \beta_1 X(1) &= 0 \end{aligned}$$

Solving Nonhomogeneous PDEs (Eigenfunction Expansions)

We ask whether the solution of the nonhomogeneous problem 6-4 can be written in the slightly more general form

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x)?$$

To show how this method works, we apply it to a problem more simple as problem 6-4. So, the details aren't as complicated.

Solution by the Eigenfunction Expansion Method

Consider the nonhomogeneous problem

Problem 6-5

$$\text{PDE:} \quad u_t = \alpha^2 u_{xx} + f(x,t), \quad 0 < x < 1, \quad 0 < t < \infty$$

$$\text{BCs:} \quad \begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \end{cases}, \qquad 0 < t < \infty$$

$$\text{IC:} \quad u(x,0) = \phi(x), \qquad 0 \leqslant x \leqslant 1$$

To solve problem 6-5 we divide the procedure into the following steps:

Step 1 (Find $X_n(x)$, that is, the solutions of the associated SLP)

- We find the functions $X_n(x)$ which are the eigenvectors of the associated Sturm-Liouville system

$$\begin{aligned} X'' + \lambda^2 X &= 0 \\ X(0) &= 0 \\ X(1) &= 0. \end{aligned}$$

It is clear that

$$X_n(x) = \sin(n\pi x), \qquad n = 1, 2, \ldots$$

Step 2 (Decomposition of $f(x,t)$)

- We decompose the heat source $f(x,t)$ into simple components

$$f(x,t) = f_1(t)X_1(x) + f_2(t)X_2(x) + \cdots + f_n(t)X_n(x) + \ldots$$

- For problem 6-5, our decomposion of the heat source has the form

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin(n\pi x). \tag{6.6}$$

Step 2 (Decomposion of $f(x,t)$)

- To find the functions $f_n(t)$ we merely multiply each side of (6.6) by $\sin(m\pi x)$ and integrate from zero to one (with respect to x); hence, we have

$$\int_0^1 f(x,t)\sin(m\pi x)dx = \sum_{n=1}^{\infty} f_n(t)\int_0^1 \sin(m\pi x)\sin(n\pi x)dx$$
$$= \frac{1}{2}f_m(t).$$

- Changing m to n we get

$$f_n(t) = 2\int_0^1 f(x,t)\sin(n\pi x)dx.$$

Step 3 (Find the responce $u_n(x,t) = T_n(t)X_n(x)$)

- We try to find our solution as a sum of the individual responses

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(n\pi x);$$

in other words, we seek the functions $T_n(t)$.

Step 3 (Find the responce $u_n(x,t) = T_n(t)X_n(x)$)

- Substituting $u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(n\pi x)$ into the system 6-5 gives us

$$\text{PDE:} \quad \sum_{n=1}^{\infty}\left[T_n'(t) + (n\pi\alpha)^2 T_n(t) - f_n(t)\right]\sin(n\pi x) = 0$$

$$\text{BCs:} \quad \begin{cases} \sum_{n=1}^{\infty} T_n'(t)\sin(0) = 0 & \text{(says nothing; zero=zero)} \\ \sum_{n=1}^{\infty} T_n'(t)\sin(n\pi) = 0 & \text{(says nothing; zero=zero)} \end{cases}$$

$$\text{IC:} \quad \sum_{n=1}^{\infty} T_n(0)\sin(n\pi x) = \phi(x).$$

Step 3 (Find the responce $u_n(x,t) = T_n(t)X_n(x)$)

- From PDE and IC it follows that $T_n(t)$ will satisfy the simple initial value problem

$$T_n' + (n\pi\alpha)^2 T_n = f_n(t)$$

$$T_n(0) = 2\int_0^1 \phi(x)\sin(n\pi x)dx =: a_n$$

- This ODE problem has the solution

$$T_n(t) = a_n e^{-(n\pi\alpha)^2 t} + \int_0^1 e^{-(n\pi\alpha)^2(t-\tau)} f_n(\tau)d\tau.$$

Step 4 (Find the solution $u(x,t)$)

- Hence, the solution of problem 7-2 is

$$u(x,t) = \sum_{n=1}^{\infty} \left[a_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x) \right] + \sum_{n=1}^{\infty} \left[\sin(n\pi x) \int_0^1 e^{-(n\pi\alpha)^2 (t-\tau)} f_n(\tau) d\tau \right].$$

- Here $\sum_{n=1}^{\infty} \left[a_n e^{-(n\pi\alpha)^2 t} \sin(n\pi x) \right]$ is the transient part (because of IC), and $\sum_{n=1}^{\infty} \left[\sin(n\pi x) \int_0^1 e^{-(n\pi\alpha)^2 (t-\tau)} f_n(\tau) d\tau \right]$ is the „steady state“ (because of the right-hand side $f(x,t)$).

Remarks

- The method of eigenfunction expansion is one of the most powerful for solving nonhomogeneous PDEs.
- The eigenfunctions $X_n(x)$ in the expansion change from problem to problem and depend on PDE and BCs.