PDE and Boundary-Value Problems Winter Term 2014/2015

Lecture 7

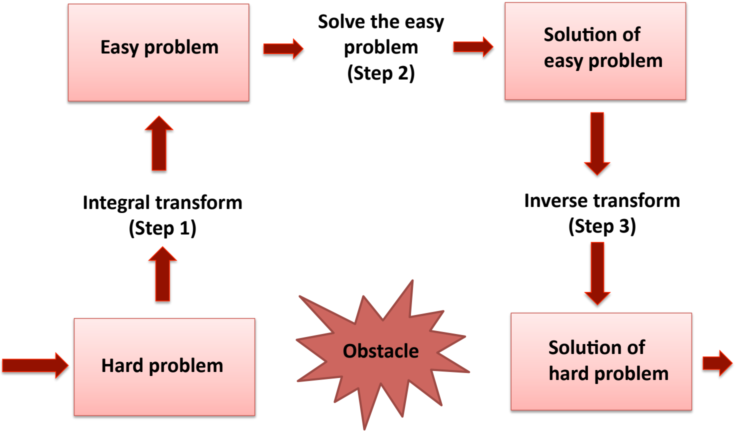
Saarland University

24. November 2014

Purpose of Lesson

- To introduce the idea of integral transforms and show how they transform PDEs in n variables into differential equations in $n-1$ variables.
- To introduce the sine and cosine transforms and use them to solve an infinite-diffusion problem.
- To define the Fourier and inverse Fourier transforms, to illustrate several useful properties of the Fourier transform and to show how these properties can be used to solve PDEs.

General philosophy of transforms



Integral transformation

- An integral transformation is a transformation that assigns to one function $f(t)$ a new function $F(s)$ by means of a formula like

$$f(t) \to F(s) = \int\limits_A^B K(s,t)f(t)dt$$

  Note that we start with a function of t and end with a function of s.

- The function $K(s,t)$ is called the kernel of transformation. It is the major ingredient that distinguishes one transform from another; it is chosen so that the transform has certain properties.

- The limits A and B also change from transformation to transformation.

Integral transformation

- The general philosophy behind integral transformations is that they eliminate partial derivatives with respect to one of the variables; hence, the new equation has one less variable.

Example

If we apply a transform to the PDE

$$u_t = u_{xx}$$

for the purpose of eliminating the time derivative, then we would arrive at an ODE in x.

- The transform and its inverse together form what is called a transform pair.

Sine and Cosine transforms

$$\begin{cases} \mathcal{F}_s[f] = F(\omega) = \dfrac{2}{\pi} \displaystyle\int_0^\infty f(t) \sin(\omega t) dt & \text{(Fourier sine transform)} \\ \mathcal{F}_s^{-1}[F] = f(t) = \displaystyle\int_0^\infty F(\omega) \sin(\omega t) d\omega & \text{(inverse sine transform)} \end{cases}$$

$$\begin{cases} \mathcal{F}_c[f] = F(\omega) = \dfrac{2}{\pi} \displaystyle\int_0^\infty f(t) \cos(\omega t) dt & \text{(Fourier cosine transform)} \\ \mathcal{F}_c^{-1}[F] = f(t) = \displaystyle\int_0^\infty F(\omega) \cos(\omega t) d\omega & \text{(inverse cosine transform)} \end{cases}$$

Sine and Cosine transforms of derivatives

1. $\mathcal{F}_s[f'] = -\omega \mathcal{F}_c[f]$ (proved by integration by parts)
2. $\mathcal{F}_s[f''] = \frac{2}{\pi}\omega f(0) - \omega^2 \mathcal{F}_s[f]$
3. $\mathcal{F}_c[f'] = -\frac{2}{\pi} f(0) + \omega \mathcal{F}_s[f]$
4. $\mathcal{F}_c[f''] = -\frac{2}{\pi}\omega f(0) - \omega^2 \mathcal{F}_c[f]$

Fourier Sine Transform

	$f(x) = \int\limits_0^\infty F(\omega)\sin(\omega x)dx$ $0 < x < \infty$	$F(\omega) = \frac{2}{\pi}\int\limits_0^\infty f(x)\sin(\omega x)d\omega$ $0 < \omega < \infty$
1.	$f(ax)$	$\frac{1}{a}F\left(\frac{\omega}{a}\right)$
2.	e^{-ax}	$\dfrac{2\omega}{\pi(a^2+\omega^2)}$
3.	$x^{-1/2}$	$\sqrt{\dfrac{2}{\pi\omega}}$
4.	$H(a-x)$	$\dfrac{2}{\pi\omega}\left[1-\cos(\omega a)\right]$
5.	x^{-1}	1

Fourier Sine Transform (cont.)

	$f(x) = \int\limits_0^\infty F(\omega)\sin(\omega x)dx$ $0 < x < \infty$	$F(\omega) = \frac{2}{\pi}\int\limits_0^\infty f(x)\sin(\omega x)d\omega$ $0 < \omega < \infty$
6.	$\dfrac{x}{x^2+a^2}$	$e^{-a\omega}$
7.	$\dfrac{x}{x^4+4}$	$\frac{1}{2}e^{-\omega}\sin(\omega)$
8.	$\tan^{-1}\left(\frac{a}{x}\right)$	$\dfrac{1-e^{-a\omega}}{\omega}$
9.	$-x^2 f(x)$	$\dfrac{2}{\pi}F''(\omega)$

Fourier Sine Transform (cont.)

	$f(x) = \int\limits_0^\infty F(\omega)\sin(\omega x)dx$ $0 < x < \infty$	$F(\omega) = \frac{2}{\pi}\int\limits_0^\infty f(x)\sin(\omega x)d\omega$ $0 < \omega < \infty$
10.	$\operatorname{erfc}\left(\dfrac{x}{2\sqrt{a}}\right)$	$\dfrac{2}{\pi}\left[\dfrac{1 - e^{-a\omega^2}}{\omega}\right], \quad a > 0$

Here

$$H(a-x) = \begin{cases} 1, & x \leqslant a \\ 0, & x > a \end{cases} \quad \text{(Reflected Heaviside function)},$$

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int\limits_x^\infty e^{-t^2}dt \quad \text{(complimentary-error function)}.$$

Fourier Cosine Transform

	$f(x) = \int\limits_0^\infty F(\omega)\cos(\omega x)dx$ $0 < x < \infty$	$F(\omega) = \frac{2}{\pi}\int\limits_0^\infty f(x)\cos(\omega x)d\omega$ $0 < \omega < \infty$
1.	$f(ax)$	$\frac{1}{a}F\left(\frac{\omega}{a}\right)$
2.	e^{-ax}	$\dfrac{2a}{\pi(a^2+\omega^2)}$
3.	$x^{-1/2}$	$\sqrt{\dfrac{2}{\pi\omega}}$
4.	$H(a-x)$	$\dfrac{2}{\pi\omega}\sin(a\omega)$
5.	$\delta(x)$	$2/\pi$

Fourier Cosine Transform (cont.)

	$f(x) = \int_0^\infty F(\omega)\cos(\omega x)dx$ $0 < x < \infty$	$F(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos(\omega x)d\omega$ $0 < \omega < \infty$
6.	e^{-ax^2}	$\frac{1}{\sqrt{\pi a}}e^{-\omega^2/(4a)}$
7.	$\frac{\sin(ax)}{x}$	$H(a-\omega)$
8.	$\frac{a}{x^2+a^2}$	$e^{-a\omega}$
9.	$-x^2 f(x)$	$\frac{2}{\pi}F''(\omega)$

Solution of an Infinite-Diffusion Problem via the Sine Transform

We now show how the sine transform can solve an important IBVP (the infinite diffusion problem).

Problem 7-1

To find the function $u(x,t)$ that satisfies

$$\begin{array}{rll} \text{PDE:} & u_t = \alpha^2 u_{xx}, & 0 < x < \infty, \quad 0 < t < \infty \\ \text{BC:} & u(0,t) = A, & 0 < t < \infty \\ \text{IC:} & u(x,0) = 0, & 0 \leqslant x \leqslant \infty \end{array}$$

To solve this, we carry out the following steps.

Step 1. (Transformation)

- We transform the *x-variable* via the Fourier sine transform so that we get an ODE in time.

$$\mathcal{F}_s[u] = \frac{2}{\pi}\int\limits_0^\infty u(x,t)\sin(\omega, x)dx =: U(\omega, t) = U(t),$$

$$\mathcal{F}_s[u_t] = \frac{2}{\pi}\int\limits_0^\infty u_t(x,t)\sin(\omega, x)dx$$

$$= \frac{\partial}{\partial t}\left[\frac{2}{\pi}\int\limits_0^\infty u(x,t)\sin(\omega, x)dx\right] = \frac{d}{dt}\mathcal{F}_s[u] = \frac{d}{dt}U(t),$$

$$\mathcal{F}_s[u_{xx}] = \frac{2}{\pi}\omega u(0,t) - \omega^2\mathcal{F}_s[u] = \frac{2A\omega}{\pi} - \omega^2 U(t).$$

Step 1. (Transformation)

- Transformation of the IC provides

$$\mathcal{F}_s[u(x,0)] = U(0) = 0.$$

- Substituting all these expressions into our IBVP, we change the original problem into an initial-value problem

$$\begin{aligned} \text{ODE:} \quad & U'(t) = \alpha^2 \left[-\omega^2 U(t) + \frac{2A\omega}{\pi} \right] \\ \text{IC:} \quad & U(0) = 0. \end{aligned} \tag{7.1}$$

Step 2. (Solving the IVP for ODE)

- To solve (7.1), we could use a variety of elementary techniques from ODEs (integrating factor, homogeneous and particular solution); in any case, the solution is

$$U(t) = \frac{2A}{\pi\omega}\left(1 - e^{-\omega^2\alpha^2 t}\right).$$

Step 3. (Inverse transform)

- The last step is to find the inverse transform of $U(t)$; that is,

$$u(x,t) = \mathcal{F}_s^{-1}[U].$$

We can either evaluate the inverse transform directly from the integral or else resort to the tables. Using the tables we see that

$$\boxed{u(x,t) = A\,\mathrm{erfc}\left(\frac{x}{2\alpha\sqrt{t}}\right)}.$$

Remark

- The exact values of the complimentary-error function

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt$$

  can be found in special tables.
- There is also the so-called error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-t^2} dt$$

- $\operatorname{erf}(x) + \operatorname{erfc}(x) = 1$.
- The integrals in $\operatorname{erf}(x)$ and $\operatorname{erfc}(x)$ cannot be integrated by the usual elementary tricks of calculus.

Exponential Fourier transforms:

$$\mathcal{F}[f] \equiv F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[f(x) e^{-i\xi x} dx \right] \quad \text{(Fourier transform - FT)}$$

$$\mathcal{F}^{-1}[F] \equiv f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[F(\xi) e^{-i\xi x} d\xi \right] \quad \text{(inverse FT)}$$

Exponential Fourier transforms:

Remarks

- The Fourier transform $F(\xi)$ can be a complex function; for example, the Fourier transform of

$$f(x) = \begin{cases} 0, & x \leqslant 0 \\ e^{-x}, & x > 0 \end{cases}$$

is $F(\xi) = \dfrac{1}{\sqrt{2\pi}} \dfrac{1 - i\xi}{1 + \xi^2}$.

- Not all functions have Fourier transforms; in fact, $f(x) = c$, $\sin(x)$, e^x, x^2, do not have Fourier transforms. Only functions that go to zero sufficiently fast as $|x| \to \infty$ have transforms.

Properties of the Fourier Transform:

- Transformation of partial derivatives:

$$\mathcal{F}[u_x] = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} u_x(x,t) e^{-i\xi x} dx = i\xi \mathcal{F}[u]$$

$$\mathcal{F}[u_{xx}] = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} u_{xx}(x,t) e^{-i\xi x} dx = -\xi^2 \mathcal{F}[u]$$

$$\mathcal{F}[u_t] = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} u_t(x,t) e^{-i\xi x} dx = \frac{\partial}{\partial t} \mathcal{F}[u]$$

$$\mathcal{F}[u_{tt}] = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} u_{tt}(x,t) e^{-i\xi x} dx = \frac{\partial^2}{\partial t^2} \mathcal{F}[u]$$

Properties of the Fourier Transform:

- The Fourier transform is a linear transformation; that is

$$\mathcal{F}[af + bg] = a\mathcal{F}[f] + b\mathcal{F}[g]$$

- The transform of a product of two functions $f(x) \cdot g(x)$ is not the product of the individual transforms; that is,

$$\mathcal{F}[f(x) \cdot g(x)] \neq \mathcal{F}[f] \cdot \mathcal{F}[g]$$

- Convolution property:

$$\mathcal{F}[f * g] = \mathcal{F}[f] \cdot \mathcal{F}[g],$$

where

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - \xi) g(\xi) d\xi.$$

Properties of the Fourier Transform:

- From the convolution property it follows that

$$f * g = \mathcal{F}^{-1}\left(\mathcal{F}[f] \cdot \mathcal{F}[g]\right).$$

Remark

Hence, to find

$$\mathcal{F}^{-1}\left(\mathcal{F}[f] \cdot \mathcal{F}[g]\right),$$

all we have to do is find the inverse transform of each factor to get f and g and then compute their convolution.

Exponential Fourier transform

	$f(x) = \mathcal{F}^{-1}[F]$	$F(\omega) = \mathcal{F}[f]$
1.	$f^{(n)}(x)$ (n^{th} derivative)	$(i\omega)^n F(\omega)$
2.	$f(ax), \ a > 0$	$\frac{1}{a} F\left(\frac{\omega}{a}\right)$
3.	$f(x-a)$	$e^{-ia\omega} F(\omega)$
4.	$e^{-a^2x^2}$	$\frac{1}{a\sqrt{2}} e^{-\omega^2/(4a^2)}$
5.	$e^{-a\lvert x\rvert}$	$\sqrt{\frac{2}{\pi}} \frac{a}{a^2+\omega^2}$

Exponential Fourier transform (cont.)

	$f(x) = \mathcal{F}^{-1}[F]$	$F(\omega) = \mathcal{F}[f]$
6.	$\begin{cases} 1, & \lvert x\rvert < a \\ 0, & \lvert x\rvert > a \end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(a\omega)}{\omega}$
7.	$\delta(x-a)$	$\frac{1}{\sqrt{2\pi}}e^{-ia\omega}$
8.	$(1+x^2)^{-1}$	$\sqrt{\frac{\pi}{2}}e^{-\lvert\omega\rvert}$
9.	$xe^{-a\lvert x\rvert}, \ a > 0$	$-2\sqrt{\frac{2}{\pi}}\frac{ia\omega}{(\omega^2+a^2)^2}$

Exponential Fourier transform (cont.)

	$f(x) = \mathcal{F}^{-1}[F]$	$F(\omega) = \mathcal{F}[f]$
10.	$H(x+a) - H(x-a)$	$\sqrt{\frac{2}{\pi}}\frac{\sin(a\omega)}{\omega}$
11.	$\frac{a}{x^2+a^2}$	$\sqrt{\frac{\pi}{2}}e^{-a\|\omega\|}$
12.	$\frac{2ax}{(x^2+a^2)^2}$	$-i\sqrt{\frac{\pi}{2}}\omega e^{-a\|\omega\|}$
13.	$\begin{cases} \cos(ax), & \|x\| < \pi/(2a) \\ 0, & \|x\| > \pi/(2a) \end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{a}{a^2-\omega^2}\cos(\pi\omega/(2a))$

Exponential Fourier transform (cont.)

	$f(x) = \mathcal{F}^{-1}[F]$	$F(\omega) = \mathcal{F}[f]$
14.	$\begin{cases} 1 - \lvert x\rvert, & \lvert x\rvert < 1 \\ 0, & \lvert x\rvert > 1 \end{cases}$	$2\sqrt{\frac{2}{\pi}}\left[\frac{\sin(\omega/2)}{\omega}\right]^2$
15.	$\cos(ax)$	$\sqrt{\frac{\pi}{2}}\left[\delta(\omega + a) + \delta(\omega - a)\right]$
16.	$\sin(ax)$	$i\sqrt{\frac{\pi}{2}}\left[\delta(\omega + a) + \delta(\omega - a)\right]$