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ENTIRE FUNCTIONS SHARING SIMPLE a-POINTS WITH THEIR FIRST DERIVATIVE

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Abstract. We show that if a nonconstant complex entire function f and its derivative f' share their simple zeroes and their simple a-points for some nonzero constant a, then $f \equiv f'$. We also discuss how far these conditions can be relaxed or generalized. Finally, we determine all entire functions f such that for 3 distinct complex numbers a_1, a_2, a_3 every simple a_j-point of f is an a_j-point of f'.

1. Introduction

Throughout $f(z)$ or f denotes an entire function, i.e., a function that is holomorphic in the whole complex plane, and $f'(z)$ or f' denotes its derivative. We write $f \equiv g$ to say that the two functions are identical.

Everybody knows that $f \equiv f'$ if and only if $f(z) = Ce^z$ with some complex constant C. For an apparently much weaker condition that has the same implication we recall the following.

Two meromorphic functions f and g are said to share the value $a \in \mathbb{C}$ IM (ignoring multiplicity), or just to share the value a, if f takes the value a at exactly the same points as g. If moreover at any given such point the functions f and g take the value a with the same multiplicity, then f and g are said to share the value a CM (counting multiplicity).

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The general philosophy is that two meromorphic functions that share "too many" values must be equal. In the special situation where g is the derivative of f one usually needs even less values.

Theorem 1.1 (Rubel and Yang [17]). *Let f be a nonconstant entire function and let a and b be complex numbers with $a \neq b$. If f and f' share the values a and b CM, then $f \equiv f'$.*

Actually, in this case one doesn't even need the multiplicities, as was proved two years later.

Theorem 1.2 (Mues and Steinmetz [13, Satz 1]). *Let f be a nonconstant entire function and let a and b be complex numbers with $a \neq b$. If f and f' share the values a and b IM, then $f \equiv f'$.*

See also [18, Theorem 8.3] for a proof in English. Theorem 1.1 and later Theorem 1.2 have been generalized in [9] resp. [11] by relaxing the requirements on the sharing. We content ourselves with one representative example, which we will need later.

Theorem 1.3 (Lü, Xu and Yi [11, Corollary 1.1]). *Let b be a nonzero number and let $f(z)$ be a nonconstant entire function. If $f(z) = 0 \Rightarrow f'(z) = 0$ and $f(z) = b \Rightarrow f'(z) = b$, then one of the following cases must occur:*

(a) $f \equiv f'$,
(b) $f = b\{\frac{1}{4}A^2e^{z/2} + Ae^{z/4} + 1\}$, *where A is a nonzero constant.*

There are yet other ways to generalize the sharing of two values between f and f'. For example the papers [8] and [10] also treat the case of f and f' sharing a two-element-set $\{a, b\}$, i.e., the two shared values might get mixed up.

But we want to investigate a generalization that, to the best of our knowledge, has not been applied yet to an entire function and its derivative.

2. Results

Definition. Let f and g be two meromorphic functions and $a \in \mathbb{C}$. We say that f and g *share their simple a-points* if the points where f takes the value a with multiplicity one are exactly those where g takes the value a with multiplicity one.

In the literature sometimes the notation

$$\overline{E}_{1)}(a, f) = \overline{E}_{1)}(a, g)$$

is used to describe this kind of sharing.

Obviously, this property generalizes sharing a CM, and even sharing a with weight one in the sense of [7], but in general it is neither stronger nor weaker than sharing a IM. Note that sharing simple a-points does *not* imply sharing the value a since we make no requirements at all concerning points where the value a is taken with higher multiplicity.

Very roughly, the philosophy is that the bulk of a-points should be simple, at least if one considers enough values a, and hence the loss of information when giving up control over the multiple values can be compensated.

For example, in contrast to Nevanlinna's famous theorem that two nonconstant meromorphic functions that share 5 values must be equal, one obtains that two nonconstant meromorphic functions that share simple a_j-points for 7 values a_j must be equal. See [5] or [18, Section 3.3.1].

In this paper we try to find similar results corresponding to Theorems 1.1 and 1.2.

Theorem 2.1. *Let $f(z)$ be a nonconstant entire function and $0 \neq a \in \mathbb{C}$. If f and its derivative f' share their simple a-points and their simple zeroes, then $f \equiv f'$.*

Actually, we even prove a slightly stronger statement in the spirit of [9] and [11]. We will be working with the following conditions (in order of decreasing strength):

- f and f' share their simple a-points, i.e.
$$(f = a \text{ and } f' \neq 0) \;\Leftrightarrow\; (f' = a \text{ and } f'' \neq 0);$$
- Every simple a-point of f is a simple a-point of f', i.e.
$$(f = a \text{ and } f' \neq 0) \;\Rightarrow\; (f' = a \text{ and } f'' \neq 0);$$
- Every simple a-point of f is a (not necessarily simple) a-point of f', i.e.
$$(f = a \text{ and } f' \neq 0) \;\Rightarrow\; f' = a.$$
This condition is of course equivalent to
$$f = a \Rightarrow f' \in \{a, 0\}.$$

Theorem 2.2. *Let $f(z)$ be a nonconstant entire function and $0 \neq a \in \mathbb{C}$. If f and f' share their simple zeroes and if every simple a-point of f is a (not necessarily simple) a-point of f', then $f \equiv f'$.*

The proof is given in Section 4.

Example 1. From [9, Theorem 2] we take the function

$$f(z) = Ce^{\frac{b}{b-a}z} + a$$

with nonzero constants C, a, $b(\neq a)$. It shares the value b CM with f' and omits the value a. This shows that in Theorem 2.2 we cannot simply replace sharing simple zeroes by sharing simple b-points. This is perhaps not overly surprising. As the easy Lemma 3.1 in the next section shows, sharing simple zeroes with f' has much stronger implications on f than sharing simple b-points for some $b \neq 0$.

Example 2. Let

$$f(z) = \frac{a}{2}(\sin(2z) + 1);$$

then $f'(z) = a\cos(2z)$. All a-points of f and of f' and all zeroes of f have multiplicity 2. Thus the condition that f and f' share their simple a-points and that every simple zero of f is a simple zero of f' does not imply $f \equiv f'$.

The more interesting question is whether in Theorem 2.1 we can replace sharing the simple zeroes by sharing the simple b-points for some nonzero b different from a. In general the answer again is negative.

Example 3. Let $0 \neq a \in \mathbb{C}$. The entire function $f(z) = a\sin z$ and its derivative $f'(z) = a\cos z$ share their simple a-points and their simple $-a$-points, for the trivial reason that all their a-points and $-a$-points have multiplicity 2.

However, somehow this counterexample seems to hinge on the fact that the second value is the negative of the first. It is still conceivable that if f and f' share their simple a-points and their simple b-points the sufficient condition that forces $f \equiv f'$ is simply $a + b \neq 0$, not $ab = 0$.

For example, in the somewhat similar context of f and f' sharing a two-element-set $\{a, b\}$ CM, the only case for which non-obvious functions f exist is $a + b = 0$ (see [8, Theorem 3] and [10]).

So we ask the following

Question. Let a, b be two distinct nonzero complex numbers with $a + b \neq 0$. If a nonconstant entire function f and its derivative f' share their simple a-points and their simple b-points, does this imply $f \equiv f'$?

At the moment we don't know the answer and we do not even have a clear feeling whether it will be positive or negative. As a small consolation we prove another result, which in case of a positive answer to this question would follow as an immediate corollary.

Theorem 2.3. *Let a_1, a_2, a_3 be three distinct complex numbers and let f be a nonconstant entire function. If f and f' share their simple a_j-points for $j = 1, 2, 3$, then $f \equiv f'$.*

Again, we prove a stronger result.

Theorem 2.4. *Let a_1, a_2, a_3 be three distinct complex numbers. Nonconstant entire functions f with $f \not\equiv f'$ and*

$$f = a_j \Rightarrow f' \in \{a_j, 0\}$$

for $j = 1, 2, 3$ exist if and only if $a_j = \zeta^j a_3$ with ζ being a third root of unity, that is, if $(X - a_1)(X - a_2)(X - a_3)$ is of the form $X^3 - \delta$.

Moreover, functions with this property necessarily are of the form

$$f(z) = \frac{4\delta}{27\beta^2} e^{\frac{2}{3}z} + \beta e^{-\frac{1}{3}z}$$

with a nonzero constant β.

Conversely, every function of this form has the stronger property that every simple a_j-point of f is a simple a_j-point of f' for $j = 1, 2, 3$.

The three Examples above and the second case of Theorem 1.3 show that the condition in Theorem 2.4 cannot be reduced to two values a_1, a_2.

3. IDEA OF PROOF AND SOME LEMMAS

The following observation is almost trivial.

Lemma 3.1. *Suppose that f and f' share their simple zeroes. Then*

(a) *f' has no simple zeroes.*
(b) *Every multiple point of f has multiplicity at least 3.*
(c) *Every zero of f has multiplicity at least 3.*

The proofs of the theorems follow an overall strategy that we have seen in several articles from the last ten years on entire (or meromorphic) functions f with certain value sharing properties. This strategy gains its strength from the combination of different methods. To emphasize this we have divided it into four steps.

In Step 1 one constructs from f a family of analytic functions by shifting the argument and then uses the properties of f to show that this family is normal.

In Step 2 one obtains from the normality of that family that f has order at most 1. This is almost automatic. Nevertheless, we want to consider this as a separate step for the following reason: Even if the family in Step 1 is not normal, there might be other ways to show that f has order at most 1.

In Step 3 one constructs an auxiliary function h from f and its derivative(s) that somehow encodes the value sharing property of f. Then one uses Nevanlinna Theory arguments to show that h is constant. This task is greatly facilitated, and sometimes only possible, thanks to the knowledge that f has order at most 1.

In Step 4 one uses the information encoded in $h \equiv const$ to derive the desired properties of f. In the proof of Theorem 2.4 we will to that end emphasize geometric considerations concerning the algebraic curve described by $h \equiv const$.

To start with, i.e. for Step 1, we need the following minor strengthening of the famous Zalcman Lemma.

Lemma 3.2. *Let $\mathcal{F}$ be a family of holomorphic functions on the unit disk. If $\mathcal{F}$ is not normal, then there exist*

(i) *a number $0 < r < 1$,*
(ii) *points z_n, $|z_n| < r$,*
(iii) *functions $f_n \in \mathcal{F}$,*
(iv) *positive numbers $\rho_n \to 0$,*

such that

$$f_n(z_n + \rho_n \xi) =: g_n(\xi) \to g(\xi)$$

uniformly on compact subsets of $\mathbb{C}$, where g is a nonconstant entire function.

Moreover, given a complex number a, if there exists a bound M and a positive integer m such that for every function f in $\mathcal{F}$ and every $z_0 \in \mathbb{C}$ with $f(z_0) = a$ we have $|f^{(k)}(z_0)| \leq M$ for $k = 1, 2, \ldots m$, then every a-point of g has multiplicity at least $m + 1$.

Proof. This is essentially the original version of Zalcman's Lemma [19]. The only thing we have to prove is the last assertion. For fixed n we differentiate $g_n(\xi)$ with respect to ξ and get

$$g_n^{(k)}(\xi) = \rho_n^k f_n^{(k)}(z_n + \rho_n \xi).$$

Now suppose that $g(\xi_0) = a$. Since g is nonconstant, by Hurwitz's theorem there exist ξ_n, $\xi_n \to \xi_0$, such that for sufficiently large n we have $a = g(\xi_0) = g_n(\xi_n)$. Hence our assumptions imply $|g_n^{(k)}(\xi_n)| \leq \rho_n^k M$ for $k = 1, 2, \ldots, m$ and n sufficiently large. Since $g_n^{(k)}(\xi)$ converges locally uniformly to $g^{(k)}(\xi)$, we obtain

$$g^{(k)}(\xi_0) = \lim_{n \to \infty} g_n^{(k)}(\xi_n) = 0$$

for $k = 1, 2, \ldots, m$. $\square$

Remark. For $m = 1$ the same argument is already in [1] (and in other papers).

Actually, [1] claims that under certain stronger conditions the function g omits the value a. However, we feel uneasy about Theorem 1 and Theorem 3 in [1] as it seems to us that by practically the same argument one would then be able to prove that a holomorphic family $\mathcal{F}$ with $f(z) \in \{1, -1\} \Rightarrow f'(z) \in \{1, -1\}$ for all $f \in \mathcal{F}$ would be normal, in contradiction to Example 1 in the same paper.

The problem seems to be that on lines 6 and 7 of page 1476 of [1] the value a_l in $g_n'(\xi_n^{(j)}) = \rho_n a_l$ depends on j, and therefore on line 13 one cannot conclude that $g_n'(\xi) - \rho_n a_l$ has k zeroes.

4. Proof of Theorem 2.2

Proof. **Step 1:** We set $f_\omega(z) = f(z+\omega)$ and consider the family of holomorphic functions $\mathcal{F} = \{f_\omega(z) \,:\, \omega \in \mathbb{C}\}$. Note that due to its special form $\mathcal{F}$ is normal on $\mathbb{C}$ if and only if it is normal on the unit disk. Obviously, this family satisfies the conditions of Lemma 3.2 for 0 with $m = 2$ and for a with $m = 1$. We conclude that $\mathcal{F}$ must be normal. Otherwise we could construct a nonconstant entire function g such that all a-points of g have multiplicity at least 2 and all zeroes have multiplicity at least 3. So for the function Θ (the sum of the deficiency and the ramification defect) we would have $\Theta(a, g) \geq \frac{1}{2}$ and $\Theta(0, g) \geq \frac{2}{3}$, in contradiction to the defect relation $\sum_{b\in\mathbb{C}} \Theta(b, g) \leq 1$ for entire functions ([2, Corollary 5.2.4] or [18, Section 1.2.4]).

Step 2: From Step 1 we readily obtain that f has order at most 1. This is a general principle; f is a Yosida function (i.e. its spherical derivative is uniformly bounded on $\mathbb{C}$) if and only if the family $\{f(z+\omega) \,:\, \omega \in \mathbb{C}\}$ is normal on $\mathbb{C}$ [12, p.198], and a holomorphic Yosida function has order at most 1 [12, p.211].

Step 3: Consider the auxiliary function

$$h = \frac{(f')^2(f - f')}{f^2(f - a)}.$$

It is easy to see that the potential poles arising from zeroes of $f - a$ are cancelled either by $f' = a$ or by the zeroes of $(f')^2$. As for the zeroes of f, note that by Lemma 3.1(c) then $f - f'$ has at least a double zero. So h is an entire function.

Using the standard functions from Nevanlinna theory and their basic properties (see e.g. [2] or [18]), from

$$h = \frac{f'}{f} \cdot \frac{f'}{f - a} - \frac{f'}{f} \cdot \frac{f'}{f} \cdot \frac{f'}{f - a}$$

we obtain

$$\begin{aligned} T(r,h) &= m(r,h) \\ &\leq m(r,\frac{f'}{f}) + m(r,\frac{f'}{f-a}) + m(r,\frac{f'}{f}) + m(r,\frac{f'}{f}) + m(r,\frac{f'}{f-a}) + O(1). \end{aligned}$$

From [15] or [6, Theorem 4.1] we see that if f is an entire function of order at most 1, then $m(r,\frac{f'}{f}) = o(\log r)$ (compare [2, Section 3.5]). Hence the above estimate gives $T(r,h) = o(\log r)$, which means that h is constant.

Step 4: If $h \equiv 0$, then $(f')^2(f - f') \equiv 0$, and hence $f \equiv f'$ since f is nonconstant.

Now consider the case $h \equiv \gamma$ for some nonzero constant γ. Then every a-point of f must be simple; otherwise by Lemma 3.1(b) it would have multiplicity at least 3 and then the term $(f')^2$ would cause a zero of h. So we have $f = a \Rightarrow f' = a$ and by Lemma 3.1(c) also $f = 0 \Rightarrow f' = 0$, that is, we are in the situation of Theorem 1.3. But the second possibility $f = a(\frac{A}{2}e^{\frac{z}{4}}+1)^2$ is ruled out, for example because all zeroes of that function have multiplicity 2, in contradiction to Lemma 3.1(c). □

5. Proof of Theorem 2.4

Proof. We prefer to write a, b, c for a_1, a_2, a_3.

The holomorphic family $\{f_\omega(z) \ : \ \omega \in \mathbb{C}\}$ with $f_\omega(z) = f(z+\omega)$ is normal. If not, as in Step 1 of the proof of Theorem 2.2 we could construct a nonconstant entire function g with 3 totally ramified values (namely a, b and c); this would contradict the defect relations [2, Theorem 5.4.1]. But actually our claim is just a special case of [3, Lemma 4].

Next, exactly the same argument as in Step 2 shows that f has order at most 1, and as in Step 3 we see that

$$h = \frac{(f')^2(f-f')}{(f-a)(f-b)(f-c)}$$
$$= \frac{f'}{f-c}\left(\frac{b}{b-a}\cdot\frac{f'}{f-b} - \frac{a}{b-a}\cdot\frac{f'}{f-a} - \frac{f'}{f-a}\cdot\frac{f'}{f-b}\right)$$

must be constant. If $h \equiv 0$, again we get $f \equiv f'$ since $f' \not\equiv 0$.

Now we discuss the case $h \equiv \gamma$ for a nonzero constant γ. This can probably be done by some case distinctions as in [11], or rather, more complicated ones. But we prefer a more geometric argument that would also work in many more complicated situations.

Consider the holomorphic map

$$z \mapsto (f(z), f'(z)) = (X, Y)$$

from $\mathbb{C}$ to the affine curve

$$A: \quad Y^3 - XY^2 + \gamma(X-a)(X-b)(X-c) = 0.$$

This polynomial is irreducible in $\mathbb{C}[X,Y]$. If not, it would have a factor $Y - uX + v$ with $u \neq 0$; but then by the equation above $X - \frac{v}{u}$ would be a multiple factor of $(X-a)(X-b)(X-c)$.

Now we write $(X-a)(X-b)(X-c)$ as $X^3 + c_2X^2 + c_1X + c_0$ and examine the corresponding projective curve

$$R: \quad Y^3 - XY^2 + \gamma(X^3 + c_2X^2Z + c_1XZ^2 + c_0Z^3) = 0.$$

This is an irreducible cubic curve. So either it is smooth and has genus 1, or it has exactly one singular point and genus 0. In the latter case the smooth model of R is a Riemann sphere. We suppress discussing the somewhat complicated conditions on γ and c_2, c_1, c_0 that distinguish the two cases, as it would not really help us in finishing the proof.

If the genus is 1, the affine curve A is obtained by removing at least one point from a smooth, projective curve (equivalently, from a compact Riemann surface) of positive genus. Hence A is hyperbolic [4, Theorem 27.12], i.e., its universal covering is the unit disk. Hence (essentially by Liouville's Theorem) every holomorphic map from $\mathbb{C}$ to A must be constant. This would mean that f is constant.

Alternatively, as Andreas Sauer has pointed out to me, hyperbolicity of algebraic curves can also be obtained from value distribution theory of meromorphic functions. See [14, Chapter X, §3] and the references given there. As mentioned there, it is already a classical theorem by Picard [16] that if an algebraic curve $F(X,Y) = 0$ is uniformized by two nonconstant meromorphic functions $X(z)$ and $Y(z)$ then the curve necessarily has genus 0 or 1. But if the genus is 1, the functions are elliptic and hence not entire.

Either way, we can assume from now on that the genus of R is 0. Then the function field $\mathbb{C}(f, f')$ is a rational function field $\mathbb{C}(t)$. It is a classical result that f' has the same order (in the sense of Nevanlinna theory) as f and that adding, multiplying and dividing functions does not increase the order. As some textbooks do not mention this, we give the reference [18, Theorem 1.21 and Section 1.3.4]. Since t is a rational expression in f and f', we thus obtain that $t(z)$ is a meromorphic function of order at most 1.

Plugging $Z = 0$ into the homogeneous equation for R, we see that there are at least 2 points outside the affine part. After a Möbius transformation we can assume that t has a zero and a pole at these two points. Then $t(z)$ is an entire function of order at most 1 without zeroes. By Hadamard's factorization theorem [18, Theorem 2.5] we have

$$t(z) = \frac{e^{\alpha z}}{\beta}$$

with nonzero constants α, β.

Of course, f is a rational function of degree 3 in t. If $t(z)$ omits other values than ∞ and 0, it must be constant by Picard's Theorem, so f would be constant. Thus the poles of f are exactly at $t = 0$ and $t = \infty$. Replacing t by $\frac{1}{t}$ if necessary, we can assume that the double pole is at $t = \infty$. Then $f = \frac{b_2t^3+b_1t^2+b_0t+b_{-1}}{t}$ with $b_2b_{-1} \neq 0$. Choosing β suitably, we can assume $b_{-1} = 1$, that is,

$$f = b_2t^2 + b_1t + b_0 + \frac{1}{t},$$

and hence

$$f' = 2\alpha b_2t^2 + \alpha b_1t - \frac{\alpha}{t}.$$

We plug this into the equation for A and compare coefficients for the powers of t. From the coefficients of t^6 and of t^{-3} we obtain $8\alpha^3b_2^3 - 4\alpha^2b_2^3 + \gamma b_2^3 = 0$ and $-\alpha^3 - \alpha^2 + \gamma = 0$, so together

$$\alpha = \frac{1}{3} \quad \text{and} \quad \gamma = \frac{4}{27}.$$

Using this, the coefficient of t^4 forces $c_2 = 0$, and then from the t^{-2}-coefficient $b_0 = 0$ follows. From the t^3-coefficient we get $b_1 = 0$, and with that the t^{-1}-coefficient implies $c_1 = 0$. Finally, the coefficient of t^0 tells us that $b_2 = \frac{-4}{27}c_0 = \frac{4}{27}\delta$. This shows

$$f(z) = \frac{4\delta}{27\beta^2}e^{\frac{3}{2}z} + \beta e^{-\frac{1}{3}z}$$

and proves the main part of the theorem.

Checking the coefficients for the remaining powers of t confirms that such f do indeed satisfy the differential equation

$$(f')^3 - f(f')^2 + \frac{4}{27}(f^3 - \delta) \equiv 0.$$

So, conversely, assume f is of the above form. Then the differential equation shows that if $f = \zeta^j c$ and $f' \neq 0$ then $f' = \zeta^j c$. Moreover, since f also satisfies the differential equation $f'' \equiv \frac{1}{3}f' + \frac{2}{9}f$, such $\zeta^j c$-points of f' are simple. $\square$

6. Proof of Theorem 2.3

Proof. If $f \not\equiv f'$, then by Theorem 2.4 we must have $a_j = \zeta^j c$ and $f = \frac{4c^3}{27}t^2 + \frac{1}{t}$ with $t = \frac{1}{\beta}e^{\frac{1}{3}z}$. Correspondingly, $f' = \frac{8c^3}{81}t^2 - \frac{1}{3t}$ and $f'' = \frac{16c^3}{243}t^2 + \frac{1}{9t}$. In particular, $f' = c$ if and only if $t \in \{\frac{-3}{c}, \frac{3(2\pm\sqrt{6})}{4c}\}$. But for $t = \frac{3(2+\sqrt{6})}{4c}$ we get $f'' \neq 0$ and $f = (\sqrt{6} - \frac{1}{2})c \neq c$. So not every simple c-point of f' is a c-point of f. □

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