A remark on a conjecture of Paranjape and Ramanan

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Abstract. In dieser Notiz zeigen wir, dass die Räum der globalen Schnitten in äusseren Potenzen eines global erzeugten Vektorbündles auf einer Kurve nicht notwendig von lokal zerlegbaren Schnitten erzeugt wird. Die Beispiele basieren auf dem Studium generischer Syzygienvarietäten. Eine weitere Anwendung dieser Syzygienvarietäten ist ein kurzer Beweis von Mukais Satz, dass jede glatte Kurve vom Geschlecht 7 und Cliffordindex 3 als Durchschnitt der Spinorvarietät $S \subset \mathbb{P}^{15}$ mit einem transversalen $\mathbb{P}^6$ entsteht.

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1. Introduction

Let C be a smooth curve of genus g,

$$\phi_k : C \to \mathbb{P}^{g-1}$$

the canonical map, and

$$N_K := \phi_K^*(T_{\mathbb{P}^{g-1}}(-1))$$

the pullback of the twisted tangent bundle. It is well-known that the syzygies of the canonical image of C are controlled by N_K (e.g., [9], 1.10) and that they correspond to the cokernels of the maps

$$\eta_j : \Lambda^j H^0(C, N_K) \to H^0(C, \Lambda^j N_K).$$

In [15], Thm. 1.3, Paranjape and Ramanan proved that all locally decomposable sections of $H^0(C, \Lambda^j N_K)$ lie in the image of η_j provided that $j \leq \mathrm{Cliff}(C)$, the Clifford index of C (cf. [8] for a definition). Moreover, they formulated

Conjecture 1.1. *(cf. [10]) $H^0(C, \Lambda^j N_K)$ is spanned by locally decomposable sections for all j.*

By [15], Thm. 1.3, the Paranjape-Ramanan conjecture implies Green's conjecture on syzygies of canonical curves [7], Conjecture (5.1).

N_K is a semi-stable (even stable if C is not hyperelliptic) globally generated vector bundle on C. One might ask, more generally than the above conjecture 1, whether

$$(*) \qquad H^0(C, \Lambda^j N) \text{ is spanned by locally decomposable sections}$$

holds for every (stable) globally generated vector bundle N on every curve C. The purpose of this note is to give counterexamples to this more general question. Our examples show that it will be rather difficult to give a criterion for pairs (C, N) for which $(*)$ holds, which include canonical curves (C, N_K) (provided this is possible, i.e., Conjecture 1.1 is true).

Acknowledgement. A first version of this paper was written in1997. At that time using the classical *Macaulay* [1] was much more limit in its scope, than nowadays *Macaulay2* [2]. In particular, computations over $\mathbb{Q}$ were not possible at that time. We decided to cut the explicit computation of the original draft. Instead refer to *http://www.math.uni-sb.de/ag/schreyer/home/computeralgebra.htm* were the reader can find explicit *Macaulay2* code which establish Proposition 4.1 and Theorem 2.6. We thank Klaus Hulek for bringing the question of Paranjape and Raman to our attention. We also thank Gavril Farkas, who encouraged us to publish these result after all, as these syzygies schemes occur frequently.

2. The examples

Example 2.1. If (C, N, j) is an example such that $H^0(C, \Lambda^j N)$ is not generated by locally decomposable sections, the cokernel of

$$\eta_j : \Lambda^j H^0(C, N) \to H^0(C, \Lambda^j N)$$

is nontrivial, and $2 \leq j \leq \operatorname{rank} N - 2$. Also in view of the desired application of Conjecture 1, an example where

$$N = N_L = \phi_L^*(T_{\mathbb{P}^r}(-1))$$

for some very ample line bundle L and

$$\phi_L : C \to \mathbb{P}^r = \mathbb{P}H^0(C, L)$$

the corresponding morphism is more interesting.

Example 2.2. In this situation, N_L is globally generated, and the cokernel of η_j corresponds to the $(r-1-j)^{th}$ linear syzygies among the quadrics in the homogeneous ideal of C (cf. [7], 1.b.4 or 2.1 below). In some sense, $j = r-2$, rank $N_L = h^0(L) - 1 \geq 4$, and a single linear relation among quadrics, is the simplest possible case.

Lemma 2.3. *(cf. [17], 4.3) If $\ell_1 q_1 + \cdots + \ell_n q_n = 0$ is a linear syzygy among quadrics $q_i \in k[x_0 \ldots, x_m]$ with linearly independent linear forms $\ell_1, \ldots, \ell_n$ then there is a skew-symmetric $n \times n$-matrix $A = (a_{ij})$ of linear forms a_{ij} such that $(q_1, \ldots, q_n) = (\ell_1, \ldots, \ell_n)(a_{ij})$.*

For $n \geq 3$, we consider the following varieties: Let $R_n := \mathbb{Z}[x_1, \ldots, x_n; a_{ij}, 1 \leq i < j \leq n]$ be the polynomial ring in

$$N := \binom{n+1}{2}$$

variables, $A = (a_{ij})$ the generic $n \times n$ skew symmetric matrix, i.e., $a_{ij} = -a_{ji}$ and if $i > j$, and $a_{ii} = 0$, $p := \mathrm{Pfaff}(A)$ the Pfaffian of A if n is even, and $(q_1, \dots, q_n) := (x_1, \dots, x_n)(a_{ij})$ the generic set of n quadrics with a syzygy. We define $X_n \subset \mathbb{P}^{N-1} = \mathrm{Proj}(R_n)$ as the variety defined by $(q_1, \dots, q_n)$ or $(q_1, \dots, q_n, p)$ if n is odd or even, respectively.

Proposition 2.4. *For $3 \le n \le 6$, the variety X_n is arithmetically Cohen-Macaulay of codimension $n-1$ with syzygies:*

$$n = 3: \quad \begin{array}{ccc} 1 & - & - \\ - & 3 & 2 \end{array} \qquad\qquad n = 4: \quad \begin{array}{cccc} 1 & - & - & - \\ - & 5 & 5 & - \\ - & - & - & 1 \end{array}$$

$$n = 5: \quad \begin{array}{ccccc} 1 & - & - & - & - \\ - & 5 & 1 & - & - \\ - & - & 11 & 10 & 1 \\ - & - & - & - & 1 \end{array} \qquad n = 6: \quad \begin{array}{cccccc} 1 & - & - & - & - & - \\ - & 6 & 1 & - & - & - \\ - & 1 & 21 & 21 & 1 & - \\ - & - & - & 1 & 6 & - \\ - & - & - & - & - & 1 \end{array}$$

(cf. [1],[2] for the notation of syzygies).

Proof. It suffices to prove the statement about the syzygies, the other assertions follow then from the Hilbert functions and the Auslander-Buchsbaum-Serre formula. For a fixed small prime p, the syzygies of X_n mod p can be computed by *Macaulay2* [1],[2]. The result will be as stated. From this and the semi-continuity of syzygy numbers, it follows that the syzygies are as stated generically over $\mathrm{Spec}\,\mathbb{Z}$, in particular, the assertion is true over $\mathbb{Q}$. (Note that in case $n = 6$, the syzygy among the the quadrics does not cancel against the Pfaffian by construction. Since X_6 is Gorenstein, the resolution is symmetric. So, also the 1-dimensional pieces of the higher syzygies do not cancel.)

For a proof of this result without a computer and valid for abitrary characteristic and arbitrary $n \ge 3$, we refer to [11] and [12]. □

Remark 2.5. X_3 is $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$, X_4 is isomorphic to $\mathbb{G}(2,5)$. $X_5 \subset \mathbb{P}^{14}$ is isomorphic to the projection of the 10-dimensional spinor variety $S \subset \mathbb{P}^{15}$ from a point, a fact which we will utilize to give an elementary proof of [14], Thm. 2, for genus 7 later on, valid in all characteristics. The variety X_6 was studied in [6], 4.4, from a somewhat different viewpoint. The singular locus of X_5 is isomorphic to the Grassmannian $G(2,5) \subset \mathbb{P}^9 = \{x_1 = x_2 = x_3 = x_4 = x_5 = 0\} \subset \mathbb{P}^{14}$, so X_5 is singular in codimension 4. A similar argument shows that X_6 is singular in codimension 7.

Theorem 2.6. *a) The curve $C_5 = X_5 \cap \mathbb{P}^5$ for a general $\mathbb{P}^5 \subset \mathbb{P}^{14}$ is a smooth curve of genus 7 embedded by the linear system $L = K(-p)$, $p \in C$ a single point, and there is a linear subspace $\mathbb{P}^5 \subset \mathbb{P}^{14}$ such that for $C_5 = X_5 \cap \mathbb{P}^5$ the above conclusions hold, and $H^0(C_5, \Lambda^3 N_L)$ is not spanned by locally decomposable sections.*

b) The curve $C_6 = X_6 \cap \mathbb{P}^6$ for a general $\mathbb{P}^6 \subset \mathbb{P}^{20}$ is a smooth curve of genus

22 *embedded by a halfcanonical linear system* L*, and there is a subspace* $\mathbb{P}^6 \subset \mathbb{P}^{20}$ *such that for* $C_6 = X_6 \cap \mathbb{P}^6$ *the above conclusions hold and* $H^0(C_6, \Lambda^4 N_L)$ *is not spanned by locally decomposable sections.*

Proof. C_5 and C_6 are smooth for general $\mathbb{P}^5$ and $\mathbb{P}^6$ respectively by Bertini's theorem. That the embedding line bundles are as stated, follows from the syzygies which do not change by cutting down since X_5 and X_6 are arithmetically Cohen-Macaulay. Note that the point $p \in C_5$ is just $V(x_1, \dots, x_5) \cap \mathbb{P}^5$. The assertion about the locally decomposable sections will be proved in section 4. □

Remark 2.7. $C_4 = X_4 \cap \mathbb{P}^4$ does not lead to a counter-example of $(*)$ due to the additional syzygies. Indeed, this is just the elliptic normal curve of degree 5, and $H^0(C_4, \Lambda^j N_L)$ is spanned by locally decomposable sections for all j. Note, however, that the varieties X_3 and X_4 play an important role for the solution of Green's conjecture in case of the second syzygy module in the approaches of [17] and [3]. Also C_5 and C_6 play a somewhat special role for Green's conjecture. C_6 is a curve of Clifford dimension 6, cf. [6], p. 193. C_5 is an important obstacle to any extension of Ehbauer's approach to the next syzygy module.

Remark 2.8. We believe that actually there exists an open subset $U \subset G(\mathbb{P}^5, \mathbb{P}^{14})$ such that for every curve

$$C_5 = X_5 \cap \mathbb{P}^5,\ \mathbb{P}^5 \in U\,,$$

the space $H^0(C_5, \Lambda^3 N_L)$ is not spanned by locally decomposable sections. However, since not spanning is not an open property, one has to have a good knowledge of what the decomposable sections look like to prove this. Set-theoretically, the scheme of locally decomposable sections coincides with the Grassmannian cone in $\Lambda^3 H^0(C_5, N_L) \subset H^0(C_5, \Lambda^3 N_L)$. However, the natural scheme structure comes with embedded components whose behaviour we could not control without an understanding what their geometric explanation is. So this natural question remains open.

3. Properties of N_L

Let L be a base point free line bundle on a curve C,

$$\phi_L :\ C \to \mathbb{P}^r = \mathbb{P}H^0(C, L)$$

the corresponding morphism, and

$$N_L = \phi_L^*(T_{\mathbb{P}^r}(-1)),\ M_L = N_L^*.$$

Thus

$$\Lambda^j N_L \cong \Lambda^{r-j} M_L \otimes L. \tag{1}$$

The syzygies in degree $p+1$ of the $S = \mathrm{Sym}(H^0(C, L))$-module $R_L = \Gamma_*((\phi_L)_*(\mathcal{O}_C))$ can be computed as the homology of the exact sheaf complex

$$\cdots \to \Lambda^{p+1} V \otimes \mathcal{O}_C \to \Lambda^p V \otimes L \to \Lambda^{p-1} V \otimes L^2 \to \cdots \tag{2}$$

on global sections where $V = H^0(C, L)$. Breaking (2) into short exact sequences

$$0 \to \Lambda^{p-j} M_L \otimes L^{j+1} \to \Lambda^{p-j} V \otimes L^{j+1} \to \Lambda^{p-j-1} M_L \otimes L^{j+2} \to 0 \tag{3}$$

gives

$$\begin{aligned} \mathrm{Tor}_p^S(R_L, k)_{p+1} &= K_{p,1} \\ &= \mathrm{cokern}(\Lambda^{p+1} V \to H^0(C, \Lambda^p M_L \otimes L)) \\ &= \Lambda^{r+1} V \otimes \mathrm{cokern}(\Lambda^{r-p} V^* \to H^0(\Lambda^{r-p} N_L)) \\ &\cong \mathrm{cokern}(\Lambda^{r-p} H^0(C, N_L) \to H^0(C, \Lambda^{r-p} N_L)) \end{aligned}$$

(cf. [7] or the nice exposition [13], 1.3).

Under the isomorphism (1), locally decomposable sections of $\Lambda^{r-p} N_L$ and $\Lambda^p M_L \otimes L$ correspond to each other. Here, a section $s \in H^0(C, \Lambda^p M_L \otimes L)$ is locally decomposable if for every point $p \in C$ there exists an open neighbourhood U and section $s_1, \ldots, s_p \in \Gamma(U, M_L)$, $t \in \Gamma(U, L)$ such that

$$s|_U = s_1 \wedge \cdots \wedge s_p \otimes t.$$

The following proposition has independently been proved by D. Butler in an unpublished paper.

Proposition 3.1. *Let L be a base point free line bundle on a non-hyperelliptic curve C of genus g, N_L as above. If $\mathrm{Cliff}(L) \le \mathrm{Cliff}(C)$ and $\deg L \ne 2g$ then N_L is stable.*

Proof. For $\deg L \ge 2g + 1$, this is proved in [5], Prop. 3.2, for $L = K$ in [15], 3.5. We follow their argument closely: First, we observe some general facts about quotient bundles of N_L: Let F be a subbundle of N_L then we have the exact sequence

$$0 \to F \to N_L \to G \to 0 \tag{4}$$

with $G = N_L/F$. From the restricted Euler sequence

$$0 \to L^{-1} \to H^0(C, L)^* \otimes \mathcal{O}_C \to N_L \to 0, \tag{5}$$

we see that $N_L^* = M_L$ does not have any nonzero global sections, and because of the dual of (1), the same is true for G^*. Since N_L is globally generated, so is G, and since one can choose rank $G + 1$ global sections to generate G, we have a surjective map

$$0 \to (\det G)^{-1} \to \mathcal{O}_C^{\oplus(\mathrm{rank} G+1)} \to G \to 0$$

whose kernel is isomorphic to $(\det G)^{-1}$. For any quotient bundle G of N_L, this gives the following inequality

$$h^0(C, \det G) - 1 \ge \mathrm{rank} G \tag{6}$$

If L is a special line bundle, i.e., $h^1(C, L) \ge 1$, then by Clifford's theorem $0 \le \deg L - 2(h^0(C, L) - 1) = d - 2r$ and because C is not hyperelliptic, equality only holds for $L \cong \mathcal{O}_C$ (in this case there is nothing to prove) or $L \cong K$ where we have

$$0 = d - 2r < \mathrm{Cliff}(C), \text{ i.e., } \mu(N_K) = \frac{d}{r} = 2.$$

For all other special line bundles L and for all non-special line bundles of $\deg L \leq 2g-1$, we must have

$$0 < d - 2r, \text{ i.e., } \mu(N_L) = \frac{d}{r} > 2.$$

Now, for a non-trivial subbundle $F \subset N_L$, we have two cases:

(i) $h^1(C \det G) \geq 2$, i.e., $\det G \geq 2$ contributes to the Clifford index of C.

(ii) $h^1(C, \det G) \leq 1$

(i) Here we have

$$\begin{aligned} d-2 \underset{(< \ if \ L=K)}{\leq} \operatorname{Cliff}(C) \leq \operatorname{Cliff}(\det G) & \\ &= \deg G - 2(h^0(C, \det G) - 1) \\ &\overset{(6)}{\leq} \deg G - 2 \operatorname{rank} G \\ &= d - \deg F - 2r + 2 \operatorname{rank} F \end{aligned}$$

and therefore

$$\frac{\deg F}{\operatorname{rank} F} \underset{(< \ if \ L=K)}{\leq} 2\,.$$

(ii) Here we have by Riemann-Roch for $\det G$:

$$\begin{aligned} d - \deg F = \deg G &= h^0(C, \det G) - h^1(C, \det G) - 1 + g \\ &\overset{(6)+(ii)}{\geq} \operatorname{rank} G - 1 + g \\ &= r - 1 + g - \operatorname{rank} F. \end{aligned}$$

Together with Riemann-Roch for L, this gives

$$1 \underset{(> \text{ if } L=K)}{\geq} 1 - h^1(C, L) = d - r + 1 - g \geq \deg F - \operatorname{rank} F\,,$$

and therefore

$$\frac{\deg F}{\operatorname{rank} F} \underset{(< \text{ if } L=K)}{\leq} 1 + \frac{1}{\operatorname{rank} F} \leq 2\,.$$

So in both cases, we find

$$\mu(F) = \frac{\deg F}{\operatorname{rank} F} \underset{(< \text{ if } L=K)}{\leq} 2 \underset{(= \text{ if } L=K)}{\leq} \mu(N_L)$$

and N_L is stable. □

Note that N_L is semi-stable for $\deg L = 2g$ but if we take $L = K \otimes F$ for a line bundle F with a global section and $\deg F = 2$ then N_L is not stable because F occurs as a line subbundle of N_L.

Corollary 3.2. (a) *If $C_5 = \mathbb{P}^5 \cap X_5 \subset \mathbb{P}^{14}$, a linear subspace is a smooth curve and $L = \mathcal{O}_C(1)$ then N_L is stable.*

(b) *For $C_6 = \mathbb{P}^6 \cap X_6$, $\mathbb{P}^6 \subset \mathbb{P}^{20}$ a general linear subspace and $L = \mathcal{O}_C(1)$, the vector bundle N_L is stable.*

Proof. (b) By [6], Thm. 3.6, $\mathrm{Cliff}(L) = \mathrm{Cliff}(C) = 21 - 12 = 9$ iff $(I_C)_2$ contains no quadric of rank ≤ 4. This is the case for general $\mathbb{P}^6 \subset \mathbb{P}^{20}$, cf. [6], Thm. 4.4. (a) Since $L = K(-p)$ by 1.7, $\mathrm{Cliff}(L) = 1$ and it suffices to prove $\mathrm{Cliff}(C) = 3$ for all such curves. This follows from our next result. □

Theorem 3.3. *Let C be a smooth curve of genus 7, $p \in C$ a point and $L = K(-p)$. The following are equivalent:*

(1) *L is normally generated, and the homogeneous ideal of the image of C under ϕ_L in $\mathbb{P}^5$ is generated by quadrics.*

(2) $\mathrm{Cliff}(C) = 3$.

(3) *The pair (C, L) is isomorphic to a pair $(C_5, \mathcal{O}_{C_5}(1))$ for $C_5 = X_5 \cap \mathbb{P}^5$ for some linear subspace $\mathbb{P}^5 \subset \mathbb{P}^{14}$.*

Proof. (1) $\Rightarrow$ (2) is elementary: If $\mathrm{Cliff}(C) \leq 2$, say C is 4-gonal and $|D| = g^1_4$ then the (not necessarily distinct) points $\{p_1 + p_2 + p_3\} \in |D(-p)|$ span a line in $\mathbb{P}^5$. Hence, the homogenous ideal needs cubic generators. If C is trigonal or hyperelliptic then L is even not normally generated.
(2) $\Rightarrow$ (1) is conjectured by [8], Conj. 3.4, since

$$\deg L \geq 2g + 1 + 1 - 2h^1(L) - \mathrm{Cliff}(C) = 14 + 1 + 1 - 2 - 3 = 11$$

is satisfied. Actually, their results [8], Thm. 1, and [13], Prop. 2.4.2, nearly give (1): L is normally generated, and the image of C is scheme theoretically defined by quadrics. To prove the assertion about the homogeneous ideal, we note that by the Hilbert function $h^0(\mathbb{P}^5, I_C(2)) = 5$ and the homogeneous ideal I_C has cubic generators iff there are > 1 linear syzygies among the five quadrics. Now from ≥ 2 syzygies, one can easily derive a contradiction to the fact that C is scheme theoretically cut out by quadrics:

Suppose, there are more syzygies. Let ψ be a 5×2-submatrix of the syzygy matrix with linear entries. If ψ is 1-generic then $\mathrm{cokern}(\psi^{tr} : 5\mathcal{O}(2) \to 2\mathcal{O}(3))$ has support on a rational normal curve of degree 5 in $\mathbb{P}^5$, cf. [4], Thm. 5.1. Hence, $F = \ker(\psi^{tr})$ is locally free of rank 3 away from the rational normal curve. The five quadrics define a section $s \in H^0(\mathbb{P}^5, F)$ whose zero-locus coincides with the intersection C of the quadrics (at least away from the rational normal curve). This gives the contradiction

$$4 = \operatorname{codim} C \leq \operatorname{rank} F = 3.$$

If ψ is not 1-generic then a generalized column of ψ has 5 linearly dependent linear forms $\ell_1, \dots, \ell_5$ as entries. We distinguish the cases

$$n = \dim\langle \ell_1, \dots, \ell_5 \rangle = 1, 2, 3 \text{ or } 4.$$

If $n = 4$ then we may assume by 2.3 that

$$(q_1, \dots, q_4) = (\ell_1, \dots, \ell_4)(a_{ij}),$$

so $\operatorname{rank}(a_{ij}) < 4$ on C. From this we deduce that either $q_5 = \operatorname{Pfaff}(a_{ij})$ or $\operatorname{Pfaff}(a_{ij}) \in (q_1, \dots, q_4)$. In the first case, we obtain $4 = \operatorname{codim} V(q_1, \dots, q_5) \leq \operatorname{codim} X_4 = 3$. In the second case, $\operatorname{codim} V(q_1, \dots, q_4) < \operatorname{codim} X_4 = 3$, hence the contradiction $\operatorname{codim} C < 4$ again. If $n = 3$, either $V(q_1, q_2, q_3)$ is a 3-fold of degree 3 and $C = V(q_1, \dots, q_5)$ has wrong degree 12, or $V(q_1, \dots, q_5)$ has too small codimension again. Finally, the case $n = 2$ leads to reducible quadrics, impossible since C is non-degenerate and integral, and $n = 1$ is absurd, anyway.
(3) $\Rightarrow$ (1) follows from Proposition 2.4 since X_5 is arithmetically Cohen-Macaulay. Finally, (1) $\Rightarrow$ (3) follows from Lemma 2.3: Since $C \subset \mathbb{P}^5$ satisfies (1), the homogeneous ideal is generated by five quadrics with one linear syzygy. By what was proved in (2) $\Rightarrow$ (1) above, the five coefficients $\ell_1, \dots, \ell_5$ have to be linearly independent, hence

$$(q_1, \dots, q_5) = (\ell_1, \dots, \ell_5)(a_{ij})$$

for some skew-symmetric 5×5-matrix (a_{ij}) of linear forms. Writing the ℓ_i's and a_{ij}'s as linear combinations of $x_0, \dots, x_5$, defines the desired $\mathbb{P}^5 \subset \mathbb{P}^{14}$. □

Corollary 3.4. *([14], Thm. 2, $g = 7$) Every smooth curve C of genus 7 and Clifford index 3 is isomorphic to a section $S \cap \mathbb{P}^6$ of the 10-dimensional spinor variety $S \subset \mathbb{P}^{15}$.*

Proof. $X_5 \subset \mathbb{P}^{14}$ is isomorphic to the projection of $S \subset \mathbb{P}^{15}$ from a point $p \in S$. In particular, X_5 and S are birational equivalent. Since $C \cong \mathbb{P}^5 \cap X_5$ by 2.5, it follows that $C \cong \mathbb{P}^6 \cap S$ where $\mathbb{P}^6 \subset \mathbb{P}^{15}$ is the cone over $\mathbb{P}^5 \subset \mathbb{P}^{14}$ with vertex p. □

Remark 3.5. This result is valid for arbitrary characteristic of the ground field. The proof of [8], Thm. 1, and [13], Prop. 2.4.2, goes through in arbitrary characteristic, so does Proposition 1.5. Note, however, that contrary to the syzygies of $X_5 \subset \mathbb{P}^{14}$, the syzygies of $S \subset \mathbb{P}^{15}$ depend on the characteristic. In char 2, there is an extra syzygy (cf. [16], p. 108) which shows that Green's conjecture (and also the Paranjape-Ramanan conjecture) is not valid in char 2.

Also, k algebraically closed is not needed in the proof of Corollary 3.4. The existence of a k-rational point suffices.

Corollary 3.6. *The moduli space $M_{7,1}$ of 1-pointed genus 7 curves is unirational.*

Proof. The rational map $\mathbb{G}(6, 15) \to M_{7,1}$, $\mathbb{P} \longmapsto X_5 \cap \mathbb{P}^5$ dominates $M_{7,1}$. □

4. The computation

In this section, we complete the proof of Theorem 2.6 by a computation. We only treat the case C_5, the case C_6 is very similar. Let $C_5 = X_5 \cap \mathbb{P}^5$ be the curve which is determined by the 5×5-matrix

$$A = (a_{ij}) = \begin{pmatrix} 0 & 3x_0 & x_0 - x_2 + x_5 & -x_0 - 2x_2 - x_3 & x_2 - 2x_3 + x_4 \\ -3x_0 & 0 & 2x_1 + x_4 & -x_1 + x_5 & -x_0 + x_3 \\ -x_0 + x_2 - x_5 & -2x_1 - x_4 & 0 & 3x_2 + x_5 & x_2 - x_4 \\ x_0 + 2x_2 + x_3 & x_1 - x_5 & -3x_2 - x_5 & 0 & x_0 - x_3 \\ -x_2 + 2x_3 - x_4 & x_0 - x_3 & -x_2 + x_4 & -x_0 + x_3 & 0 \end{pmatrix}$$

A straightforward computation with *Macaulay2* [1],[2] shows that C_5 is a smooth curve of genus 7 embedded by the linear system $L = K(-p)$, $p = (1:0:0:0:0:0) \in C_5$. The coordinates $(x_0, \ldots, x_5)$ on $\mathbb{P}^5$ are chosen such that $p = p_1 = (1:0:\ldots:0)$, $p_2 = (0:1:0:\ldots:0), \ldots, p_6 = (0:\ldots:0:1)$, $p_7 = (1:\ldots:1)$ and $p_8 = (4:0:0:2:1:2)$ are contained in C_5. With this notation we prove the stronger assertion:

Proposition 4.1. *For $C = C_5 = \mathbb{P}^5 \cap X_5$, every section $s \in H^0(C, \Lambda^3 N_L)$ whose values $s(p_\nu) \in \Lambda^3 N_L \otimes k(p_\nu)$ are decomposable for $\nu = 1, \ldots, 8$ lies in the image of $\Lambda^3 H^0(C, N_L)$ under*

$$\eta_3 : \Lambda^3 H^0(C, N_L) \to H^0(C, \Lambda^3 N_L).$$

Proof. Recall that $V = H^0(C, L)$, $V^* = H^0(C, N_L)$. It is simpler and in view of Section 3 (2) also more natural to to work with

$$\Lambda^3 V \subset H^0(C, \Lambda^2 M_L \otimes L) \subset \Lambda^2 V \otimes V$$

instead of

$$\Lambda^3 V \subset H^0(C, \Lambda^3 N_L) \subset \Lambda^4 V^* \otimes V.$$

A moment's thought gives that the linear syzygy among the quadrics is represented by

$$s_0 = \sum_{0 \le i < j \le 5} x_i \wedge x_j \otimes a_{ij} \in \Lambda^2 V \otimes V$$

where $\tilde{A} = (a_{ij})$ is the skew-symmetric 5×5-matrix from the definition of C, extended by a row and column of zeroes, i.e., $a_{0j} = 0 \in V$. Indeed, the Koszul differential

$$d : \Lambda^2 V \otimes V \to V \otimes S_2 V$$

maps

$$s_0 \longmapsto \sum_{i=0}^{5} x_i \otimes q_i \neq 0$$

where $q_0 = 0$, $q_1, \ldots, q_5 \in (I_C)_2$. Hence,

$$[s_0] \in \frac{\ker(H^0(C, \Lambda^2 V \otimes L) \to H^0(C, V \otimes L^2))}{\operatorname{Im}(H^0(C, \Lambda^3 V \otimes \mathcal{O}_C) \to H^0(C, \Lambda^2 V \otimes L))} = \operatorname{Tor}_2^S(R_L, \mathbb{Q})_3$$

gives a non-trivial cohomology class, and s_0 together with the elements

$$s_{i,j,k} = x_i \wedge x_j \otimes x_k - x_i \wedge x_k \otimes x_j + x_j \wedge x_k \otimes x_i \in \Lambda^2 V \otimes V$$

forms a basis of $H^0(C, \Lambda^2 M_L \otimes L)$ because the syzygy module is 1-dimensional. To prove the assertion for C, we have to show that if

$$\begin{aligned} s = s_b = {} & b_0 s_0 + b_1 s_{0,1,2} + b_2 s_{0,1,3} + b_3 s_{0,1,4} + b_4 s_{0,1,5} + b_5 s_{0,2,3} \\ & + b_6 s_{0,2,4} + b_7 s_{0,2,5} + b_8 s_{0,3,4} + b_9 s_{0,3,5} + b_{10} s_{0,4,5} \\ & + b_{11} s_{1,2,3} + b_{12} s_{1,2,4} + b_{13} s_{1,2,5} + b_{14} s_{1,3,4} + b_{15} s_{1,3,5} \\ & + b_{16} s_{1,4,5} + b_{17} s_{2,3,4} + b_{18} s_{2,3,5} + b_{19} s_{2,4,5} + b_{20} s_{3,4,5} \end{aligned}$$

with $b_0, b_1, \ldots, b_{20} \in \mathbb{C}$ is decomposable in p_ν for all $\nu = 1, 2, \ldots, 8$ then b_0 is zero. Now, s_b similarly to s_0 is given by $\sum_{0 \le i < j \le 5} x_i \wedge x_j \otimes b_{ij}$ where $B = (b_{ij})$ is a skew-symmetric 6×6-matrix whose entries b_{ij} are linear in x_i and b_j. The section s_b is decomposable in p_ν iff the matrix $B(p_\nu) = (b_{ij}(p_\nu))$ represents a decomposable skew form, i.e., $\operatorname{rank} B(p_\nu) \le 2$. Note that $\operatorname{rank} B(p_\nu) \le 4$ is clear since

$$(x_0(p_\nu), \ldots, x_5(p_\nu))(b_{ij}(p_\nu)) = b_0(0, q_1(p_\nu), \ldots, q_5(p_\nu)) = 0$$

as $p_\nu \in C$. Hence, $B(p_\nu)$ induces indeed a skew symmetric bilinear form on $N_L \otimes k(p_\nu)$ with values in $L \otimes k(p_\nu)$. Now rank $B(p_\nu) \le 2$ holds if and only if all 4×4-Pfaffians of $B(p_\nu)$ vanish. All these Pfaffians generate an ideal

$$J \subset \mathbb{Q}[b_0, b_1, \ldots, b_{20}],$$

and a straight forward *Macaulay2* computation shows that $b_0^2 \in J$. For details, we refer to *http://www.math.uni-sb.de/ag/schreyer/home/computeralgebra.htm*.

□

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