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Some Topics in Computational Algebraic Geometry

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Abstract. Brief comments on selected topics in computational algebraic geometry are given. One of the topics is an experimental investigation of the possible Betti numbers of smooth canonical curves of low genus.
Key Words and Phrases: Computer algebra, Gröbner bases, syzygies, resolution of singularities, monodromy, Brieskorn lattice, Tate resolution, cohomology of coherent sheaves, Beilinson monads, invariants rings, binary forms, Green's conjecture, construction of canonical curves.

1. Introduction

Modern computer algebra systems allow to treat impressive examples in computational algebraic geometry. The basic mathematical tool are Gröbner bases as invented by Gordon (1899), Buchberger (1965), Hironaka (1964) and Grauert (1972). In particular Buchberger's algorithm to compute Gröbner bases is essential. For localization of polynomial rings this algorithm was adapted by Mora (1982). A rough classification of the applications is as follows:

(1) Elementary applications: ideal membership, normal forms, Hilbert function, dimension, degree, elimination, projective closure, tangent cone, syzygies, intersections, (I:J), Hom(M,N).
(2) Modifications of algebraic sets: primary decomposition, normalization, Puiseux expansion, rational parameterization of curves (and surfaces), resolution of singularities.
(3) Homological methods: Ext, Tor, cohomology of coherent sheaves, Tate resolutions, monads, resultants.
(4) Parameter spaces: invariant rings, versal deformations of singularities and modules, special families: existence, uni-rationality.
(5) Enumerative geometry.

2000 *Mathematics Subject Classification.* 13P, 14Q.

(6) D-modules and topology: Bernstein-Sato polynomials, monodromy and Brieskorn lattices of isolated singularities, de Rham cohomology.

In this survey we focus on few topics: resolution of singularities, monodromy, Tate resolutions, invariant theory and constructions of special families. For a more complete survey including a short treatment of basic Gröbner basis theory we refer to our survey Decker, Schreyer (2001).

2. Resolution of singularities

Gabor Bodnar and Josef Schicho (2000) implemented Villamayor's algorithm of resolution of singularities (see Encinas, Villamajor 1998) as part of a CASA package for Maple, mainly for surfaces and threefolds, see `http://www.risc.uni-linz.ac.at`. At the current state the algorithm is implemented for characteristic zero, but future implementation will include characteristic $p > 0$, with the expectation, that the algorithm will work in many but not all cases. Note that the running time in characteristic $p > 0$ might be shorter than for characteristic zero due to the fact that the coefficients in a Gröbner basis computation in characteristic $p > 0$ do not accumulate.

The input are the polynomial equations of an affine scheme Z embedded into a nonsingular affine subvariety X of $\mathbf{A}^n$. The output is a tree of charts of blow-ups, whose final leaves consist of a covering of an embedded resolution $f : Y \to X$ of Z, all put together in an HTML document.

The number of charts, which are used to cover Y even in simple examples can be large. For example the desingularisation of the Whitney umbrella $Z = \{z^2 - xy^2 = 0\} \subset X = \mathbf{A}^3$ gives a tree with 50 nodes and 16 final leaves covering Y.

3. Monodromy and Brieskorn lattices

A `SINGULAR` package to compute the monodromy of an isolated hypersurface singularity has been developed by Mathias Schulze. It uses an algorithm by Brieskorn (1970) to compute a connection matrix of the meromorphic Gauss-Manin connection up to arbitrarily high order, and an algorithm of Gerard and Levelt (1973) to transform it to a simple pole.

The computation of the monodromy of the D_4 surface singularity in `SINGULAR` looks as follows:

```
>LIB "mondromy.lib";
>ring R = 0, (x,y,z),ds;
>poly  f= z^2+y^2*x+x^3;
>matrix M =monodromyB(f);
>print(M);
11/6,0,  0,  0,
0,   3/2,0,  0,
```

```
0,   0,  3/2,0,
0,   0,  0,  7/6
```

The monodromy operator is then $\exp(-2\pi i M)$ in terms of the output matrix M.

4. Tate resolution

Bernstein, Gel'fand, Gel'fand (1978) established an equivalence between the derived category of coherent sheaves on $\mathbb{P}(W)$ and the stable module category of finitely generated graded modules over the graded exterior algebra $E = \Lambda V$, where $V = W^*$ are dual vector spaces over the ground field K. The heart of the construction associates to a graded $S = \mathrm{Sym}(W)$ module $M = \sum_d M_d$ the infinite linear complex

$$\mathbf{R}(M): \qquad \ldots \to \mathrm{Hom}_K(E, M_d) \to \mathrm{Hom}_K(E, M_{d+1}) \to \ldots$$

and vice versa. In Eisenbud, Fløystad, Schreyer (2001) we review this construction starting from $\mathbf{R}(M)$. We obtain novel methods to compute cohomology of sheaves and to compute the Beilinson monad of a sheaf explicitely.

$\mathbf{R}(M_{\geq r})$ becomes exact precisely for $r > reg(M)$. Thus adjoining a free resolution of $\ker(R^r(M) \to R^{r+1}(M))$, we may extend $\mathbf{R}(M_{\geq r})$ to a doubly infinite exact complex of graded free E-modules

$$\mathbf{T}(\tilde{M}) \qquad \to T^e \to \ldots \to T^{r-1} \to T^r = R^r(M) \to T^{r+1} = R^{r+1}(M) \to \ldots$$

which depends only on the sheaf $\mathcal{F} = \tilde{M}$.

Theorem 4.1 (Eisenbud, Fløystad, Schreyer 2001). *For a coherent sheaf $\mathcal{F} = \tilde{M}$ on $\mathbb{P}(W) = \mathbb{P}^n$ we have*

$$T^e(\mathcal{F}) = \sum_{i=0}^{n} \mathrm{Hom}_K(E, \mathrm{H}^i\mathcal{F}(e-i))$$

where we regard $\mathrm{H}^i\mathcal{F}(e-i)$ as a vector space in degree $e-i$.

Thus syzygies over the exterior algebra allow to compute cohomology groups: Starting from the multiplication map

$$M_d \otimes W \to M_{d+1}$$

for sufficiently high d, we obtain one of the differentials of $\mathbf{R}(M)$ and a Gröbner basis calculation over the exterior algebra gives us any desired finite piece of $\mathbf{T}(\mathcal{F})$.

If we compare this with previous methods to compute cohomology, e.g.

$$\mathrm{H}^i_*(\mathcal{F}) \cong \mathrm{Ext}_S^{n-i}(\Gamma_*(\mathcal{F}), S(-n-1))^*,$$

then we see that to compute e.g. H^1 we do not have to compute the complete free resolution of $\Gamma_*(\mathcal{F})$ but only some steps in the Tate resolution, which seems to be of more appropriate complexity.

The differentials of $\mathbf{T}(\mathcal{F})$ are related to the Beilinson monad of $\mathcal{F}$, c.f. Beilinson (1978). Let Ω be the additive functor, which maps the free E module $\omega_E(i) = \mathrm{Hom}_K(E, K(i))$ to the sheaf $\Omega^i(i)$, the twisted regular i-forms, and which maps morphisms via the identification

$$\mathrm{Hom}_E(\omega_E(i), \omega_E(j)) \cong \Lambda^{i-j} V \cong \mathrm{Hom}_{\mathbb{P}(W)}(\Omega^i(i), \Omega^j(j)).$$

Then

Theorem 4.2 (Eisenbud, Fløystad, Schreyer 2001). *$\Omega(\mathbf{T}(\mathcal{F}))$ is the Beilinson monad for $\mathcal{F}$.*

Thus the differentials of $\mathbf{T}(\mathcal{F})$ give us the differentials of the Beilinson monad. The differential of Beilinson monad were previously very difficult to compute explicitly.

Example. Consider the 2×5 matrix

$$\varphi = \begin{pmatrix} e_1e_4 & e_2e_0 & e_3e_1 & e_4e_2 & e_0e_3 \\ e_2e_3 & e_3e_4 & e_4e_0 & e_0e_1 & e_1e_2 \end{pmatrix}$$

over the exterior algebra with generators $e_0, \ldots, e_4$. By direct computation we find the following Betti numbers in the Tate resolution of φ, where φ sits in the indicated spot.

$$\begin{array}{ccccccccccc}
100 & 35 & 4 & . & . & . & . & . & . & . & . \\
. & 2 & 10 & 10 & \mathbf{5} & . & . & . & . & . & . \\
. & . & . & . & . & \mathbf{2} & . & . & . & . & . \\
. & . & . & . & . & . & 5 & 10 & 10 & 2 & . \\
. & . & . & . & . & . & . & . & 4 & 35 & 100
\end{array}$$

The Beilinson functor Ω picks out a finite complex

$$0 \to \oplus^5 \Omega^4(4) \to \oplus^2 \Omega^2(2) \to \oplus^5 \mathcal{O}_{\mathbb{P}^4} \to 0$$

Its homology is the famous Horrocks-Mumford bundle (1973). It is easy to see from these Betti numbers, that it is the Tate resolution of a vector bundle, see Eisenbud, Fløystad, Schreyer (2001) for details.

5. Invariant theory

Let G be a group and $\rho : G \to GL(V)$ a linear representation. The basic problem of invariant theory is to compute for $R = k[V]$ the ring R^G of invariant functions. If G is reductive, then R^G is a finitely generated k-algebra as proved by Hilbert in his first landmark paper (1890). Hilbert himself provided an algorithm to compute generators in his second landmark paper (1893), in which he introduced Noether normalization, the Hilbert-Mumford criterion and the Nullstellensatz, see also Sturmfels (1993) and Decker, de Jong (1999). A variant of Hilbert's original proof of finite generation was turned into an algorithm recently by Derksen (1999)

For finite groups this gives a reasonable good algorithm implemented by Decker and his group into `SINGULAR`. However for algebraic groups none of the algorithms works in practice so far, the reason being that at some step a too expensive elimination computation is required.

A computer implementation of Gordon's method for binary forms including covariants was done by Holger Cröni (2002). It can treat in reasonable time the case of binary septics, the case in which Sylvester's enumerative method predicted too few generators (1878). Later von Gall computed a complete system of invariants for binary septics in 1888. However von Gall got too many, which was finally corrected by Dixmier and Lazard (1986).

The weakest spot of Cröni's program is that the theoretical bounds for the degree of the generators are too large.

Clearly one would hope that the computation of invariant rings with Computer algebra improves upon the state of art a hundred years ago substantially. I think it is time to reconsider this problem from an algorithmic point of view.

6. Constructions

In this last section I would like to comment on computer algebra methods for constructions. For example one might want to prove, that a certain component of the Hilbert scheme is non-empty, and that its general points correspond to smooth varieties, or that the component is uni-rational. Computer algebra for this purpose was very successfully applied by Decker, myself and our students to the study of smooth non-general type surfaces in $\mathbb{P}^4$.

In this survey I will illustrate this method with an investigation of the possible Betti numbers for canonical curves.

Let $C \longrightarrow \mathbb{P}^{g-1}$ be the canonical morphism of a smooth curve of genus g. The syzygies of the canonical ring $R_C = \sum_{n \geq 0} H^0(C, \omega^{\otimes n})$ as $S = Sym(H^0(C, \omega))$ module are conjectured to be closely related to the Brill-Noether theory of C. Since R_C is Gorenstein, it has a self-dual resolution of length $g-2$. Moreover R_C is 3-regular. We summarize the numbers of generators of the modules $F_i = \sum_j S(-j)^{\beta_{ij}}$ in a minimal free resolution

$$0 \leftarrow R_C \leftarrow F_0 \leftarrow F_1 \leftarrow \ldots \leftarrow F_{g-2} \leftarrow 0$$

in a **Betti table**

$$\begin{array}{cccccccccc}
1 & . & \ldots & . & . & \ldots & . & \ldots & . & . \\
. & \beta_{12} & \ldots & * & \beta_{p+1,p+2} & \ldots & \beta_{g-2-p,g-1-p} & \ldots & & \beta_{02} \\
\beta_{02} & \beta_{13} & \ldots & \beta_{p,p+2} & * & \ldots & * & \ldots & \beta_{g-3,g-1} & . \\
. & . & \ldots & . & . & \ldots & . & \ldots & . & 1
\end{array}.$$

Gorenstein gives the symmetry $\beta_{ij} = \beta_{g-2-i,g+1-j}$. The difference $\beta_{i+1,i+2} - \beta_{i,i+2}$ depends only on g and i but not on the curve. The famous conjecture of Green gives a geometric interpretation of the range of nonzero β_{ij}'s.

CONJECTURE 6.1 (Green, 1984). *Let C be a smooth curve defined over $\mathbb{C}$. Then $\beta_{p,p+2} \neq 0$ iff C has Clifford index* $\mathrm{Cliff}(C) \leq p$.

The Clifford index of a line bundle L is defined as

$$\operatorname{Cliff}(L) = \deg L - 2(h^0(L) - 1)$$

and the Clifford index of C is

$$\operatorname{Cliff}(C) = \min\{\operatorname{Cliff}(L) \mid L \text{ a line bundle on } C \text{ with } h^0(L), h^1(L) \geq 2\}.$$

The direction from the existence of special line bundles to the existence of exceptional syzygies was established by Green and Lazarsfeld (1984). At present the conjecture is known for curves of genus $g \leq 9$, Mukai (1995). The case $p \leq 2$ is known by M. Noether (1880), Petri (1923), Voisin (1988), Schreyer (1991). Recently Voisin (2001) proved the conjecture for a general k-gonal curves of arbitrary genus except for the case of general curves of odd genus.

Thus it might be time to try to formulate a more precise version of the conjecture, i.e. to answer the question, which Betti tables actually occur for smooth curves. Also the conjecture is known to be false for fields of some finite characteristics. It is interesting to try to explain the exceptional characteristics for various genera. A computational approach to these questions runs along the following lines: Pick curves over finite fields at random, and compute their syzygies. For genus $g \leq 14$ it is possible to pick curves at random, as was shown in Schreyer, Tonoli (2001). Below I summarize, what I think are the possible Betti numbers for small g, and which are the exceptional characteristics. For $g \leq 8$ this was established in Schreyer (1986). For genus 9 we have:

	1	.	.	.	.	.	.	.
general case	.	21	64	70	.	.	.	.
	.	.	.	.	70	64	21	.
	.	.	.	.	.	.	.	1
	1	.	.	.	.	.	.	.
$\exists_1 \quad g^1_5$	.	21	64	70	4	.	.	.
	.	.	.	4	70	64	21	.
	.	.	.	.	.	.	.	1
$\exists \quad g^1_5 \times g^1_5$, more precisely $\exists$ a g^2_8 with 2 triple points, possibly infinitesimal near	1	.	.	.	.	.	.	.
	.	21	64	70	8	.	.	.
	.	.	.	8	70	64	21	.
	.	.	.	.	.	.	.	1
	1	.	.	.	.	.	.	.
$\exists \quad g^2_7$	.	21	64	70	24	.	.	.
	.	.	.	24	70	64	21	.
	.	.	.	.	.	.	.	1
	1	.	.	.	.	.	.	.
$\exists \quad g^1_4$	.	21	64	75	24	5	.	.
	.	.	5	24	75	64	21	.
	.	.	.	.	.	.	.	1

	1	.	.	.	.	.	.	.
$\exists \quad g^1_4 \times g^1_5$	.	21	64	75	48	5	.	.
	.	.	5	48	75	64	21	.
	.	.	.	.	.	.	.	1
	1	.	.	.	.	.	.	.
$\exists \quad g^2_6$ or $g^3_8 \quad (\implies \exists\, g^1_4)$	.	21	64	90	64	20	.	.
	.	.	20	64	90	64	21	.
	.	.	.	.	.	.	.	1
	1	.	.	.	.	.	.	.
$\exists \quad g^1_3$	.	21	70	105	84	35	6	.
	.	6	35	84	105	70	21	.
	.	.	.	.	.	.	.	1
	1	.	.	.	.	.	.	.
$\exists \quad g^1_2$	.	28	112	210	224	140	48	7
	7	48	140	224	210	112	28	.
	.	.	.	.	.	.	.	1

Table: Conjectural Betti numbers for genus 9, characteristic $\neq 3$

It is not known whether this is the correct table for curves of Clifford index 3. For example the table claims that the existence of three g^1_5's implies the existence of a g^2_7.
In characteristic 3 the conjecture fails for the general curve. The following Betti numbers are possible for curves of genus 9 and Clifford index ≥ 3:

	1	.	.	.	.	.	.	.
general case	.	21	64	70	6	.	.	.
	.	.	.	6	70	64	21	.
	.	.	.	.	.	.	.	1
	1	.	.	.	.	.	.	.
	.	21	64	70	8	.	.	.
	.	.	.	8	70	64	21	.
	.	.	.	.	.	.	.	1
	1	.	.	.	.	.	.	.
	.	21	64	70	10	.	.	.
	.	.	.	10	70	64	21	.
	.	.	.	.	.	.	.	1
	1	.	.	.	.	.	.	.
$\exists \quad g^2_7$	.	21	64	70	24	.	.	.
	.	.	.	24	70	64	21	.
	.	.	.	.	.	.	.	1

Table: Conjectural Betti numbers for genus 9 in characteristic 3

For genus 10 over a field of characteristic $\neq 3$ we find the following:

general case	1	.	.	.	.	.	.	.	.
	.	28	105	162	84	.	.	.	.
	.	.	.	.	84	162	105	28	.
	.	.	.	.	.	.	.	.	1
$\exists_1 \quad g^1_5$	1	.	.	.	.	.	.	.	.
	.	28	105	162	89	5	.	.	.
	.	.	.	5	89	162	105	28	.
	.	.	.	.	.	.	.	.	1
$\exists \quad g^1_5 \times g^1_5$, more precisely $\exists\ g^2_8$ with 2 triple points, possibly infinitesimal near	1	.	.	.	.	.	.	.	.
	.	28	105	162	94	10	.	.	.
	.	.	.	10	94	162	105	28	.
	.	.	.	.	.	.	.	.	1
$\exists \quad g^3_9$	1	.	.	.	.	.	.	.	.
	.	28	105	162	104	20	.	.	.
	.	.	.	20	104	162	105	28	.
	.	.	.	.	.	.	.	.	1
$\exists \quad g^2_7$	1	.	.	.	.	.	.	.	.
	.	28	105	162	119	35	.	.	.
	.	.	.	35	119	162	105	28	.
	.	.	.	.	.	.	.	.	1
$\exists \quad g^1_4$	1	.	.	.	.	.	.	.	.
	.	28	105	168	119	35	6	.	.
	.	.	6	35	119	168	105	28	.
	.	.	.	.	.	.	.	.	1
$\exists \quad g^1_4 \times g^1_5$	1	.	.	.	.	.	.	.	.
	.	28	105	168	139	55	6	.	.
	.	.	6	55	139	168	105	28	.
	.	.	.	.	.	.	.	.	1
$\exists \quad g^2_6$	1	.	.	.	.	.	.	.	.
	.	28	105	189	189	105	27	.	.
	.	.	27	105	189	168	105	28	.
	.	.	.	.	.	.	.	.	1
$\exists \quad g^1_3$	1	.	.	.	.	.	.	.	.
	.	28	112	210	224	140	48	7	.
	.	7	48	140	224	210	112	28	.
	.	.	.	.	.	.	.	.	1
$\exists \quad g^1_2$	1	.	.	.	.	.	.	.	.
	.	36	168	378	504	420	216	63	8
	8	63	216	420	504	378	168	36	.
	.	.	.	.	.	.	.	.	1

Table: Conjectural Betti numbers for genus 10 in characteristic $\neq 3$

The general curve of genus 10 over a field of characteristic 3 does not satisfy Green's conjecture.

	1	.	.	.	.	.	.	.	.
general case,	.	28	105	162	85	1	.	.	.
characteristic 3	.	.	.	1	85	162	105	28	.
	.	.	.	.	.	.	.	.	1

In the case of genus 11 Green's conjecture does not hold in characteristic 2 and 3. For other characteristics the following Betti numbers are possible:

	1	.	.	.	.	.	.	.	.	.
general case	.	36	160	315	288	.	.	.	.	.
	.	.	.	.	.	288	315	160	36	.
	.	.	.	.	.	.	.	.	.	1
	1	.	.	.	.	.	.	.	.	.
$k = 1, 2, \ldots, 10,$	.	36	160	315	288	$5k$	.	.	.	.
$12, 20$	.	.	.	.	$5k$	288	315	160	36	.
	.	.	.	.	.	.	.	.	.	1
	1	.	.	.	.	.	.	.	.	.
triple cover of an	.	36	160	315	288	27	.	.	.	.
elliptic curve	.	.	.	.	27	288	315	160	36	.
	.	.	.	.	.	.	.	.	.	1
	1	.	.	.	.	.	.	.	.	.
tri-elliptic with	.	36	160	315	288	$27+5k$	.	.	.	.
further g^1_6 for	.	.	.	.	$27+5k$	288	315	160	36	.
$k = 1, 2, 3.$	.	.	.	.	.	.	.	.	.	1
	1	.	.	.	.	.	.	.	.	.
$\exists \quad g^1_5$	.	36	160	315	294	35	6	.	.	.
	.	.	.	6	35	294	315	160	36	.
	.	.	.	.	.	.	.	.	.	1
	1	.	.	.	.	.	.	.	.	.
$\exists \quad g^1_5 \times g^1_6$	.	36	160	315	294	40	6	.	.	.
	.	.	.	6	40	294	315	160	36	.
	.	.	.	.	.	.	.	.	.	1
	1	.	.	.	.	.	.	.	.	.
$\exists \quad g^1_5$ and tri-	.	36	160	315	294	64	6	.	.	.
elliptic	.	.	.	6	64	294	315	160	36	.
	.	.	.	.	.	.	.	.	.	1

$\exists\ g^2_8$ with a triple point ($\implies \exists\ g^1_5$ and seven g^1_6)	1	.	.	.	.	.	.	.	.	.
	.	36	160	315	294	75	6	.	.	.
	.	.	.	6	75	294	315	160	36	.
	.	.	.	.	.	.	.	.	.	1
$\exists\ g^2_8$ with two triple points ($\implies \exists$ two g^1_5 and ten g^1_6)	1	.	.	.	.	.	.	.	.	.
	.	36	160	315	300	140	12	.	.	.
	.	.	.	12	140	300	315	160	36	.
	.	.	.	.	.	.	.	.	.	1
$\exists\ g^2_7$ ($\implies \exists$ four g^1_5)	1	.	.	.	.	.	.	.	.	.
	.	36	160	315	336	210	48	.	.	.
	.	.	.	48	210	336	315	160	36	.
	.	.	.	.	.	.	.	.	.	1
$\exists\ g^1_4$	1	.	.	.	.	.	.	.	.	.
	.	36	160	322	336	140	48	7	.	.
	.	.	7	48	140	336	322	160	36	.
	.	.	.	.	.	.	.	.	.	1
$\exists\ g^1_4 \times g^1_6$	1	.	.	.	.	.	.	.	.	.
	.	36	160	322	336	210	48	7	.	.
	.	.	7	48	210	336	322	160	36	.
	.	.	.	.	.	.	.	.	.	1
$\exists\ g^2_7$ with a triple point ($\implies \exists\ g^1_4$)	1	.	.	.	.	.	.	.	.	.
	.	36	160	322	392	280	104	7	.	.
	.	.	7	104	280	392	322	160	36	.
	.	.	.	.	.	.	.	.	.	1
$\exists\ g^2_6$ ($\implies$ bi-elliptic)	1	.	.	.	.	.	.	.	.	.
	.	36	160	350	448	350	160	35	.	.
	.	.	35	160	350	448	350	160	36	.
	.	.	.	.	.	.	.	.	.	1
$\exists\ g^1_3$	1	.	.	.	.	.	.	.	.	.
	.	36	168	378	504	420	216	63	8	.
	.	8	63	216	420	504	378	168	36	.
	.	.	.	.	.	.	.	.	.	1
$\exists\ g^1_2$	1	.	.	.	.	.	.	.	.	.
	.	45	240	630	1008	1050	720	315	80	9
	9	80	315	720	1050	1008	630	240	45	.
	.	.	.	.	.	.	.	.	.	1

Table: Conjectural Betti numbers for genus 11, characteristic $\neq 2, 3$

Some remarks are in place: The case $\beta_{46} = 50$ occurs, when C is a double cover of a plane quartic, or, if C has a birational g^2_8 with nodes in general position. A special position of the nodes might result in an extra g^1_6 and $\beta_{46} = 60$. A curve which is simultaneously tri-elliptic and a double cover of

a plane quartic has $\beta_{46} = 100$. The same number occurs for curves with a g^3_{10}. I am not certain whether the number k in these tables can always be interpreted as the number of extra g^1_6's counted suitably.
I do not present a conjectural table of possible Betti numbers for higher genera. Conjecturally exceptional characteristics are summarized in the following table:

genus g	characteritic p	extra syzygies
≤ 6	none	$--$
7	2	$\beta_{24} = 1$
8	none	$--$
9	3	$\beta_{35} = 6$
10	3	$\beta_{35} = 1$
11	$2, 3$	$\beta_{46} = 28, 10$
12	2	$\beta_{46} = 1$
13	$2, 5$	$\beta_{57} = ??, 120$

Table: Exceptional characteristics and Betti numbers of a general curve

What is the evidence for the correctness of these tables? First the exceptional characteristics in the case of the even genera are really counter examples to Green's conjecture, because in these cases the number of additional syzygies is too small to come from a linear system. However that the generic curve of that genus has extra syzygies is not fully established. We just have a probabilistic argument, as in the case of the odd genera. For $g = 9$ we can use Mukai's theorem (1995), which says that all curves of Clifford index 4 are transversal sections $C = X \cap \mathbb{P}^8$ of the symplectic Grassmanian

$$X = LG(3, 6) \subset \mathbb{P}^{13}$$

of Lagrangian subspaces, and compute the syzygies of X for various small p. The evidence is then based on our believe that exceptions occur only for small p. For larger odd genera we can compute examples for each small p. Our evidence is, that it is unlikely, that we always hit the loci of curves with extra syzygies, if we pick random different examples.

For the table of all **possible Betti numbers** we know for **odd genus** $g = 2k + 1$ by Hirschowitz and Ramanan (1998), that curves with extra syzygies lie in the locus $M_g(g^1_{k+1})$ of curves with a g^1_{k+1} and that $\beta_{k-1,k+1} \geq k$ with equality on an open set of $M_g(g^1_{k+1})$. On the other side every smooth curve of any genus with $\beta_{i,i+2} \neq 0$ for $i \leq 2$ satisfies Green's Conjecture by Max Noether (1880), Petri (1923), Voisin (1988) and Schreyer (1991), and their Betti numbers are computed in (Schreyer, 1986). So for odd genus $g \leq 11$ the Betti numbers are not yet known only in the case of Clifford index 3 and $g = 9$ and Clifford index 3,4 and $g = 11$. Turning to Green's Conjecture the only open question in this range is, whether a curve of genus

$g = 11$ and Clifford index 4 satisfies $\beta_{35} = 0$. However there could be many more cases of possible Betti tables. To get some confidence in its completeness, we can do the following. Take a small characteristic p, say $p = 5$, and construct curves in each stratum at random over $\mathbb{Z}/5$. Then since $\beta_{k-2,k} = \beta_{k-1,k}$ we know that this number jumps up in codimension 1. Hence we roughly expect to see such a phenomenon with a chance of $1 : p$. If this expectation turns out to be true, and no new Betti tables are found, we may have some more confidence. In particular I expect, that $\beta_{46} = 100$ is the maximum possible value for curves of genus $g = 11$ and Clifford index 4. In this stratum I did not find any jumps at all. I checked this over $\mathbb{Z}/5$ running 20 random examples observing no jumping up. The probability, that all twenty curves miss the jump loci in this stratum, is (if the jump loci is nonempty) roughly $0.8^{20} = 0.0115$. So in some sense we can be certain with 98% that this jump loci is really empty.

For **even genus** $g = 2k$ I am less confident. By Voisin's result (2001) we know, that Green's conjecture holds for the general curve and for general q-gonal curves. On the other hand there is no apriori reason why jumps in Betti numbers, say for $\beta_{k,k+1} = \beta_{k-2,k-1}$, occur in small codimension. The loci of curves with $\beta_{k,k+1} \neq 0$ is reducible for $g = 10$. We have the loci

$$M_{10}(g^1_5) \text{ and } M_{10}(\text{half canonical } g^3_9),$$

where $M_g(g^r_d) = \{\text{curves of genus g with a } g^r_d\}$, which have dimension 25 and 21 hence codimension 2 and 6 in M_{10} respectively. None of these strata lies in the closure of the other.

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