Tate Resolutions on Products of Projective Spaces

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Tate Resolutions

The Tate resolution of a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is a double infinite free complex over an exterior algebra, which encodes the cohomology of $\mathcal{F}$. Applications include

- Beilinson monads
- Chow forms, resultants
- Boij-Söderberg theory
- direct image complexes (local or affine case)

Today, work in progress with David Eisenbud and Daniel Erman

- Extension of this theory to products of projective spaces.
- Application include direct image complexes in the global case: computation of $R\pi_*\mathcal{F}$ for a morphism $\pi : X \to Y$ between projective varieties and $\mathcal{F} \in coh(X)$

Overview

1. Review of Tate resolutions on $\mathbb{P}^n$
2. Construction of the Tate resolution
3. Beilinson monads
4. Exactness property of the Tate resolution
5. Open Questions

Koszul pair

- K a ground field, W an $(n+1)$-dimensional vector space
- $S = Sym\ W = K[x_0, \ldots, x_n]$ coordinate ring of $\mathbb{P}^n$

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so injective=projective over E

BGG-Functors

$M = \oplus_d M_d$ graded S-module

$$\mathbf{R}(M) : \ldots \to Hom_K(E, M_d) \to Hom_K(E, M_{d+1}) \to \ldots$$

with differential

$$\varphi \mapsto \{e \mapsto \sum_{i=0}^{n} x_i \varphi(e_i e)\}$$

$P = \oplus_d P_d$ graded E-module

$$\mathbf{L}(P) : \ldots \to P_1 \otimes S \to P_0 \otimes S \to P_{-1} \otimes S \to \ldots$$

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BGG-Functors

$$\mathbf{R} : \mathit{grmod}(S) \to \mathit{lincplx}(E)$$

and

$$\mathbf{L} : \mathit{grmod}(E) \to \mathit{lincplx}(S)$$

extend to a pair of adjoint functors

$$\mathit{cplx}(S) \overset{\mathbf{L},\mathbf{R}}{\longleftrightarrow} \mathit{cplx}(E)$$

Theorem (Bernstein, Gelfand, Gelfand 1978)

$$D^b(S) \cong D^b(E) \quad \mathit{and} \quad D^b(\mathbb{P}^n) \cong \underline{\mathit{mod}}\ E$$

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Theorem (Eisenbud, Fløystad, S., 2003)

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Cox ring

- $\mathbb{P} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t} = \mathbb{P}(W_1) \times \cdots \times \mathbb{P}(W_t)$
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- $c = (c_1, \ldots, c_t)$ a (multi)-degree, $|c| = \sum_i c_i$ denotes the total degree.

What should be the shape of the Tate resolution?

Example (Künneth case)

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For arbitrary $\mathcal{F} \in coh(\mathbb{P})$ we should have

$$\mathbf{T}^d(\mathcal{F}) = \sum_{0 \le i \le n} \sum_{\substack{a \in \mathbb{Z}^t \\ |a| = d}} H^{|i|}(\mathbb{P}, \mathcal{F}(a-i)) \otimes_K \omega_E(i-a)$$

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no longer finitely generated, but free!

An example on $\mathbb{P}^1 \times \mathbb{P}^1$

Consider $\omega_E \to \omega_E(-2,0) \oplus \omega_E^4(-1,-1) \oplus \omega_E(0,-2)$ defined by the matrix

$$m = (e_0 e_1, e_0 f_0, e_1 f_0, e_0 f_1, e_1 f_1, f_0 f_1)^t$$

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$$\mathbf{L}(\text{image } m) \to M \to 0$$

is the minimal free resolution of the module of global sections $M = \sum_{(a,b)\in\mathbb{Z}^2} H^0(\mathcal{F}(a,b))$ of a rank 3 vector bundle $\mathcal{F}$ with cohomology as indicated on the next slide.

$$\left(\sum_{i=0}^{2} \dim H^i(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{F}(a,b)) \cdot h^i\right)_{-3\leq a,b\leq 3}$$

$$= \begin{pmatrix} 28h & 18h & 8h & 2 & 12 & 22 & 32 \\ 20h & 13h & 6h & 1 & 8 & 15 & 22 \\ 12h & 8h & 4h & 0 & 4 & 8 & 12 \\ 4h & 3h & 2h & h & 0 & 1 & 2 \\ 4h^2 & 2h^2 & 0 & 2h & 4h & 6h & 8h \\ 12h^2 & 7h^2 & 2h^2 & 3h & 8h & 13h & 18h \\ 20h^2 & 12h^2 & 4h^2 & 4h & 12h & 20h & 28h \end{pmatrix} \in \mathbb{Z}[h]^{7\times 7}$$

The injective resolution of $P = \ker m$ has total Betti numbers

	-5	-4	-3	-2	-1	0
-1:	140	84	45	20	6	.
0:	.	.	.	.	.	1

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$$(0 \leftarrow M_{\geq c}(c) \leftarrow) F_0 \leftarrow F_1 \leftarrow \cdots$$

   with $F_k = \oplus_a S^{\beta_{k,a}}(-a)$ satisfies $\beta_{k,a} \neq 0$ only if $k = |a|$,
2. $M_c = H^0(\mathbb{P}, \mathcal{F}(c))$ and $H^p(\mathbb{P}, \mathcal{F}(c)) = 0$ for $p > 0$.

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3. For $\pi^J : \mathbb{P} \to \mathbb{P}^J = \mathbb{P}^{n_{j_1}} \times \cdots \times \mathbb{P}^{n_{j_s}}$ a partial projection
   - 3.1 $\Gamma_{\geq 0}(\pi^J_* \mathcal{F}(c))$ has a linear resolution,
   - 3.2 $R^p \pi^J_* \mathcal{F}(c) = 0$ for $p > 0$.

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Reciprocity still works: **R** and **L** respect the finer grading.
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$$\begin{array}{ccccccc}
\uparrow & & \uparrow & & \uparrow & & \\
\mathbf{R}M_{(1,3)} & \to & \mathbf{R}M_{(2,3)} & \to & \mathbf{R}M_{(3,3)} & \to \\
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Reciprocity still works: **R** and **L** respect the finer grading. $\mathbf{R}(M_{\geq c}(c))$ gives a part of the Tate resolution in a positive quadrant. Say $t = 2$ and $c = (-2, -2)$ sufficiently positive

$$
\begin{tikzcd}
 & & & {} & {} & {} & \\
 & & & \mathbf{R}M_{(1,3)} \arrow[u] \arrow[r] & \mathbf{R}M_{(2,3)} \arrow[u] \arrow[r] & \mathbf{R}M_{(3,3)} \arrow[u] \arrow[r] & {} \\
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Theorem

$c \gg 0$ then

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 & & \uparrow & & \uparrow & & & & & & & \\
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Projective dimension of high truncations

Corollary

M a graded module over the Cox ring S. If $c \gg 0$ then

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Proof.

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Hence $\mathbf{L}P^{(c)} \to M_{\geq c} \to 0$ has length $\dim S - t$. $\square$

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\begin{tikzcd}
 & & & {} & {} & {} & \\
 & & & \mathbf{R}M_{(1,3)} \arrow[u] \arrow[r] & \mathbf{R}M_{(2,3)} \arrow[u] \arrow[r] & \mathbf{R}M_{(3,3)} \arrow[u] \arrow[r] & {} \\
 & & & \mathbf{R}M_{(1,2)} \arrow[u] \arrow[r] & \mathbf{R}M_{(2,2)} \arrow[u] \arrow[r] & \mathbf{R}M_{(3,2)} \arrow[u] \arrow[r] & {} \\
 & & & \textcolor{red}{\mathbf{R}M_{(1,1)}} \arrow[u, red] \arrow[r, red] & \textcolor{red}{\mathbf{R}M_{(2,1)}} \arrow[u, red] \arrow[r, red] & \textcolor{red}{\mathbf{R}M_{(3,1)}} \arrow[u, red] \arrow[r, red] & {} \\
\mathbf{R}M_{(-2,0)} \arrow[r] & \mathbf{R}M_{(-1,0)} \arrow[r] & \mathbf{R}M_{(0,0)} \arrow[ur, red] & & & & \\
 & \mathbf{R}M_{(-1,-1)} \arrow[u] \arrow[r] & \mathbf{R}M_{(0,-1)} \arrow[u] & & & & \\
 & & \mathbf{R}M_{(0,-2)} \arrow[u] & & & &
\end{tikzcd}
$$

Construction

$$
\begin{tikzcd}
 & & & {} & {} & {} & \\
 & & & \mathbf{R}M_{(1,3)} \arrow[u] \arrow[r] & \mathbf{R}M_{(2,3)} \arrow[u] \arrow[r] & \mathbf{R}M_{(3,3)} \arrow[u] \arrow[r] & {} \\
 & & & \mathbf{R}M_{(1,2)} \arrow[u] \arrow[r] & \mathbf{R}M_{(2,2)} \arrow[u] \arrow[r] & \mathbf{R}M_{(3,2)} \arrow[u] \arrow[r] & {} \\
\mathbf{R}M_{(-2,1)} \arrow[r] & \mathbf{R}M_{(-1,1)} \arrow[r] & \mathbf{R}M_{(0,1)} \arrow[ur] & \mathbf{R}M_{(1,1)} \arrow[u] \arrow[r] & \mathbf{R}M_{(2,1)} \arrow[u] \arrow[r] & \mathbf{R}M_{(3,1)} \arrow[u] \arrow[r] & {} \\
\mathbf{R}M_{(-2,0)} \arrow[u] \arrow[r] & \mathbf{R}M_{(-1,0)} \arrow[u] \arrow[r] & \mathbf{R}M_{(0,0)} \arrow[u] \arrow[ur] & & & & \\
 & \mathbf{R}M_{(-1,-1)} \arrow[u] \arrow[r] & \mathbf{R}M_{(0,-1)} \arrow[u] & & & & \\
 & & \mathbf{R}M_{(0,-2)} \arrow[u] & & & &
\end{tikzcd}
$$

Construction

$$
\begin{array}{cccccccccccc}
 & & & & & & \uparrow & & \uparrow & & \uparrow & \\
 & & & & & & \mathbf{R}M_{(1,3)} & \to & \mathbf{R}M_{(2,3)} & \to & \mathbf{R}M_{(3,3)} & \to \\
 & & & & & & \uparrow & & \uparrow & & \uparrow & \\
 & & & & & & \mathbf{R}M_{(1,2)} & \to & \mathbf{R}M_{(2,2)} & \to & \mathbf{R}M_{(3,2)} & \to \\
 & & & & & \nearrow & & & & & & \\
\mathbf{R}M_{(-2,1)} & \to & \mathbf{R}M_{(-1,1)} & \to & \mathbf{R}M_{(0,1)} & & & & & & & \\
\uparrow & & \uparrow & & \uparrow & & & & & & & \\
\mathbf{R}M_{(-2,0)} & \to & \mathbf{R}M_{(-1,0)} & \to & \mathbf{R}M_{(0,0)} & & & & & & & \\
 & & \uparrow & & \uparrow & & & & & & & \\
 & & \mathbf{R}M_{(-1,-1)} & \to & \mathbf{R}M_{(0,-1)} & & & & & & & \\
 & & & & \uparrow & & & & & & & \\
 & & & & \mathbf{R}M_{(0,-2)} & & & & & & &
\end{array}
$$

Construction Step 2

Given M and b, sufficiently positive for M, consider free resolutions $T^{(c)}$ of $P^{(c)}$,

$$\begin{tikzcd} T^{(c)} \arrow[rr] \arrow[dr] & & \mathbf{R}M_{\geq c} \\ & P^{(c)} \arrow[ur] & \end{tikzcd}$$

for all $c \geq b$. We have a directed system $\{T^{(c')} \to T^{(c)} | c' \geq c\}$. Define

$$T' = \lim_{\leftarrow} T^{(c)}$$

and finally the **Tate resolution** of $\mathcal{F} = \widetilde{M}$ as the subcomplex of homogeneous elements:

$$\mathbf{T}(\mathcal{F}) = \{f \in T' | f \text{ is homogeneous } \} \subset T'.$$

First Main Theorem

Proposition

The Tate resolution $\mathbf{T}(\mathcal{F})$ *exact. For each multidegree a the space of homogeneous elements* $\mathbf{T}(\mathcal{F})_a$ *of multidegree a is finite dimensional.*

Theorem

The Tate resolution of a coherent sheaf $\mathcal{F}$ *on* $\mathbb{P}$ *has terms*

$$\mathbf{T}(\mathcal{F})^d \cong \sum_{\substack{a \in \mathbb{Z}^t \\ |a|=d}} \sum_{0 \leq i \leq n} H^{|i|}(\mathcal{F}(a-i)) \otimes_K \omega_E(i-a)$$

The derived category $D^b(\mathbb{P})$

$$U_k = \ker(H^0(\mathbb{P}^{n_k}, \mathcal{O}(1)) \otimes \mathcal{O} \to \mathcal{O}(1))$$

tautological rank n_k subbundle on $\mathbb{P}^{n_k}$. Set

$$U^a = \Lambda^{a_1} U_1 \boxtimes \cdots \boxtimes \Lambda^{a_t} U_t$$

Of course, U^a is nonzero if and only if $0 \leq a \leq n$.

Theorem (Beilinson, xyz)

$\{U^a | 0 \leq a \leq n\}$ forms a full strongly exceptional series for the derived category $D^b(\mathbb{P})$, which is right orthogonal to the strongly exceptional series $\{\mathcal{O}(a) | 0 \leq a \leq n\}$ in the sense that

$$H^p \mathrm{RHom}(\mathcal{O}(c), U^a) = H^p(U^a(-c)) = \begin{cases} K & \text{if } a = c \text{ and } p = |a|, \\ 0 & \text{otherwise.} \end{cases}$$

The $\mathbf{U}$-functor

Consider the additive functor on the category of direct sums of finitely generated free graded E-modules defined by

$$\mathbf{U}\colon \omega_E(a) \mapsto U^a$$

on objects. For the morphism given by the multiplication with $e \in E_{b-a} = \otimes_{k=1}^t \Lambda^{a_k-b_k} V_k$ we define $\mathbf{U}$ by the diagram

$$\begin{array}{ccccc}
\omega_E(a) & \mapsto & U^a & \hookrightarrow & \otimes_{k=1}^t \Lambda^{a_k} W_k \otimes_K \mathcal{O}\,, \\
\downarrow{\scriptstyle \times e} & & \downarrow{\scriptstyle \neg e} & & \downarrow{\scriptstyle \neg e} \\
\omega_E(b) & \mapsto & U^b & \hookrightarrow & \otimes_{k=1}^t \Lambda^{b_k} W_k \otimes_K \mathcal{O}
\end{array}$$

where the right hand maps are given by contraction.

Beilinson Monad

Applying **U** to the Tate resolution, we obtain a bounded complex

$$\mathbf{U}(\mathcal{F}) := \mathbf{U}(\mathbf{T}(\mathcal{F})).$$

This is the Beilinson monad for $\mathcal{F}$.

Theorem

$\mathbf{U}(\mathcal{F})$ *is a monad for the sheaf* $\mathcal{F}$ *in the sense that*

$$H^p(\mathbf{U}(\mathcal{F})) \cong \begin{cases} \mathcal{F} & \textit{for } p = 0\textit{, and} \\ 0 & \textit{for } p \neq 0. \end{cases}$$

Locally finite E-complexes

Definition

A complex T of graded free E-module with terms

$$T^d = \sum_{a \in \mathbb{Z}^t} B_a^d \otimes \omega_E(-a)$$

with vector spaces B_a^d is **locally finite**, if for each $a \in \mathbb{Z}^t$ the vector space

$$\sum_{d \in \mathbb{Z}} B_a^d$$

is finite dimensional.

Strands, quadrants, regions

T a locally finite complex of graded free E-modules with terms $T^d = \sum_{a\in\mathbb{Z}^t} B_a^d \otimes \omega_E(-a)$. For $c \in \mathbb{Z}^t$ and disjoint subsets $I, J, K \subset \{1, \ldots, t\}$ we call the subquotient complexes $T_c(I, J, K)$ with

$$T_c(I,J,K)^d = \sum_{\substack{a\in\mathbb{Z} \\ a_i<c_i \text{ for } i\in I \\ a_i=c_i \text{ for } i\in J \\ a_i\geq c_i \text{ for } i\in K}} B_a^d \otimes \omega_E(-a)$$

a **proper region complex** of T if $I \cup J \cup K \subsetneq \{1, \ldots, t\}$
$T_c(\emptyset, J, \emptyset)$ with $J \subsetneq \{1, \ldots, t\}$ a **strand** and
$T_c(I, \emptyset, K)$ with $I \cup K = \{1, \ldots, t\}$ a **quadrant** complex.

Corner complex $T_{\ulcorner c}$

$T_{\geq c} = T_c(\emptyset, \emptyset, \{1, \dots, t\})$ and $T_{<c} = T_c(\{1, \dots, t\}, \emptyset, \emptyset)$ denote the first and last quadrant complex and abbreviate

$$T_{c,k} = T_c(\{1, \dots, k\}, \emptyset, \{k+1, \dots, t\})$$

for some of the intermediate quadrant complexes. The **corner complex** $T_{\ulcorner c}$ is the cone over the map

$$T_{<c}[-t] \to T_{\geq c}$$

which we get as composition

$$T_{<c}[-t] = T_{c,t}[-t] \to \dots \to T_{c,k}[-k] \to \dots \to T_{c,0} = T_{\geq c}$$

of the maps in T from one quadrant to the next.

Second Main Theorem

Theorem

T be a locally finite complex of free E-modules. TFAE

1. *Every strand of T is exact.*
2. *Every proper region complex of T is exact.*
3. *Every corner complex $T_{\ulcorner c}$ is exact.*
4. *The corner complexes $T_{\ulcorner c}$ are exact for every sufficiently large c.*
5. *The proper region complexes $T_c(I, \emptyset, \emptyset)$ are exact for every sufficiently large c.*

$\mathbf{T}(\mathcal{F})$ satisfies 5. by construction.

Example on $\mathbb{P}^1 \times \mathbb{P}^1$

$$\begin{pmatrix}
28h & 18h & 8h & 2 & 12 & 22 & 32 \\
20h & 13h & 6h & 1 & 8 & 15 & 22 \\
12h & 8h & 4h & 0 & 4 & 8 & 12 \\
4h & 3h & 2h & h & 0 & 1 & 2 \\
4h^2 & 2h^2 & 0 & 2h & 4h & 6h & 8h \\
12h^2 & 7h^2 & 2h^2 & 3h & 8h & 13h & 18h \\
20h^2 & 12h^2 & 4h^2 & 4h & 12h & 20h & 28h
\end{pmatrix}$$

Total Betti numbers of $T_{\ulcorner 0}$

	-5	-4	-3	-2	-1	0	1	2
-1:	140	84	45	20	6	.		
0:	.	.	.	.	.	1		
1:	.	.	.	.	.	.	.	.
2:	.	.	.	.	.	.	4	15

Example on $\mathbb{P}^1 \times \mathbb{P}^1$

$$\begin{pmatrix}
. & . & . & 2 & 12 & 22 & 32 \\
. & . & . & 1 & 8 & 15 & 22 \\
. & 8h & . & 0 & 4 & 8 & 12 \\
. & . & 2h & h & 0 & 1 & 2 \\
4h^2 & 2h^2 & 0 & . & . & . & . \\
12h^2 & 7h^2 & 2h^2 & . & . & . & . \\
20h^2 & 12h^2 & 4h^2 & . & . & . & .
\end{pmatrix}$$

Total Betti numbers of $T_{\geq 0}$

	-5	-4	-3	-2	-1	0	1	2
-1:	140	84	45	20	6	.		
0:	.	.	.	.	.	1		
1:	.	.	.	.	.	.	.	.
2:	.	.	.	.	.	.	4	15

Direct image complexes

$\mathcal{F}$ coherent sheaf on $\mathbb{P}$, and $T = \mathbf{T}(\mathcal{F})$ it's Tate resolution. For each proper subset $J = \{j_1, \dots, j_s\} \subset \{1, \dots, n\}$ with complement J' we have the projection

$$\pi^J : \mathbb{P} \to \mathbb{P}^{n_{j_1}} \times \cdots \times \mathbb{P}^{n_{j_s}} = \mathbb{P}^J.$$

Corollary

For $c \in \mathbb{Z}^t$ the strand $T_c(\emptyset, J', \emptyset)$ is exact, and after twist and shift

$$T_c(\emptyset, J', \emptyset)(c)[|c|] \cong T_J \otimes_K \omega_{E^{J'}}$$

is a flat extension of an minimal complex T_J of free E^J-modules such that

$$\mathbf{U}_J(T_J) \cong R\pi^J_*(\mathcal{F}(c)) \in D^b(\mathbb{P}^J)$$

Half plane complexes

The Tate resolution $\mathbf{T}(\mathcal{F})$ has many exact subquotient complexes.

Question

What is the geometric meaning of say, the ***half plane complexes***

$$T_c(I, \emptyset, K)$$

for $I \cup K = \{1, \ldots, n\} \setminus \{j\}$ *?*

Double complexes

For simplicity, assume $t = 2$, hence $\mathbb{P} = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$.

$$\mathcal{F} \cong \bigoplus_j \mathcal{F}_j \boxtimes \mathcal{G}_j \quad \Rightarrow \quad \mathbf{T}(\mathcal{F}) \text{ is a double complex.}$$

Question

Is the converse true?

Objects in $D^b(\mathbb{P}^n)$ as image sheaves ?

In the case of an affine space Spec A, David and I proved that any bounded complex

$$0 \to A^{\alpha_0} \to \dots \to A^{\alpha_n} \to 0$$

arises as $R\pi_*\mathcal{F}$ of a vector bundle $\mathcal{F}$ on Spec $A \times \mathbb{P}^n$.

Question

Could it be that any object in $D^b(\mathbb{P}^n)$ arises as $R\pi_*\mathcal{F}$ for a coherent sheaf $\mathcal{F}$ on a product $\mathbb{P}$ for a suitable projection $\pi : \mathbb{P} \to \mathbb{P}^n$ onto a factor?

Tate resolution of elements in $D^b(\mathbb{P})$

Any object in $F \in D^b(\mathbb{P})$ can be represent by a bounded minimal complex

$$0 \to F^k \to F^{k+1} \to \ldots \to F^\ell \to 0$$

with $F^j = \oplus_i U^{a^{ij}}$. So there exist a smallest complex T of free E module such that $\mathbf{U}(T) \cong F$

Question

How to compute the Tate resolution of F, i.e. an exact complex T' of free E-modules such that $\mathbf{U}(T'(c)[|c|]) = F(c)$ for every $c \in \mathbb{Z}^t$?

We have a nice simple Macaulay2 code in case of $\mathbb{P} = \mathbb{P}^n$ of a single factor.