Matrix factorizations and families of curves of genus 15

Frank-Olaf Schreyer

Universität des Saarlandes
E-Mail: schreyer@math.uni-sb.de.

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Introduction

The moduli spaces $\mathcal{M}_g$ of curves of genus g is

- unirational for $g \leq 14$, [Severi, Sernesi, Chang-Ran, Verra],
- of general type for $g = 22$ and $g \geq 24$, [Harris-Mumford, Eisenbud-Harris, Farkas].

The cases in between are not fully understood:

- $\mathcal{M}_{23}$ has positive Kodaira dimension [Farkas],
- $\mathcal{M}_{15}$ is rationally connected [Bruno-Verra] ,
- $\mathcal{M}_{16}$ is uniruled [Chang-Ran, Farkas].

In this talk I report on an attempt to prove the unirationality of $\mathcal{M}_{15}$.

$\widetilde{\mathcal{M}}_{15}$

By Brill-Noether theory,

$$W^r_d(C) = \{L \in \mathrm{Pic}^d C \mid h^0(L) \geq r+1\}$$

has dimension at least

$$\rho = g - (r+1)(g-d+r),$$

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a general curve of genus $15 = 5 \cdot 3$ has a finite number of smooth models of degree 16 in $\mathbb{P}^4$. Let

$$\mathcal{H} \subset \mathrm{Hilb}_{16t+1-15}(\mathbb{P}^4)$$

be the corresponding component of the Hilbert scheme, and let

$$\widetilde{\mathcal{M}}_{15} \subset \{(C, L) \mid C \in \mathcal{M}_{15}, L \in W^4_{16}(C)\} \to \mathcal{M}_{15}$$

be the component of the Hurwitz scheme, which dominates generically finite to one. So $\mathcal{H}//PGL(5)$ is birational to $\widetilde{\mathcal{M}}_{15}$.

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Our main result connects the moduli space $\widetilde{\mathcal{M}}_{15}$ to a moduli space of certain matrix factorizations of cubic threefolds.

Main Results

Theorem

The moduli space $\widetilde{\mathcal{M}}_{15}$ of curves of genus 15 together with a g^4_{16} is birational to a component of the moduli space of matrix factorizations of type

$$\mathcal{O}^{18}(-3) \xrightarrow{\psi} \mathcal{O}^{15}(-1) \oplus \mathcal{O}^{3}(-2) \xrightarrow{\varphi} \mathcal{O}^{18}$$

of cubic forms on $\mathbb{P}^4$.

Theorem

$\widetilde{\mathcal{M}}_{15}$ is uniruled.

Overview

1. Introduction
2. Review of matrix factorizations
3. The structure theorem
4. Constructions
5. Tangent space computations
6. Conclusion

Matrix factorizations [Eisenbud, 1980]

R a regular local ring, $f \in \mathfrak{m}^2$. A *matrix factorization* of f is a pair (φ, ψ) of matrices satisfying

$$\psi \circ \varphi = f \; id_G \quad \text{and} \quad \varphi \circ \psi = f \; id_F .$$

$M = \operatorname{coker} \varphi$ is a maximal Cohen-Macaulay R/f-module.

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$M = \operatorname{coker} \varphi$ is a maximal Cohen-Macaulay R/f-module.

Conversely, if M is a MCM on R/f, then as R-module M has a short resolution

$$0 \longleftarrow M \longleftarrow F \longleftarrow G \longleftarrow 0.$$

and multiplication with f on this complex is null homotopic

$$\begin{array}{ccccccccc} 0 & \leftarrow & M & \leftarrow & F & \overset{\varphi}{\leftarrow} & G & \leftarrow & 0 \\ & & \downarrow^{0} & & \downarrow^{f} & \searrow^{\psi} & \downarrow^{f} & & \\ 0 & \leftarrow & M & \leftarrow & F & \overset{\varphi}{\leftarrow} & G & \leftarrow & 0 \end{array}$$

which yields a matrix factorization (φ, ψ).

2-periodic resolutions

As an R/f-module, M has the infinite 2-periodic resolution

$$0 \leftarrow M \leftarrow \overline{F} \xleftarrow{\varphi} \overline{G} \xleftarrow{\psi} \overline{F} \xleftarrow{\varphi} \overline{G} \xleftarrow{\psi} \dots$$

where $\overline{F} = F \otimes R/f$ and $\overline{G} = G \otimes R/f$. In particular, this sequence is exact, and the dual sequence corresponding to the matrix factorization (ψ^t, φ^t) is exact as well.

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where $\overline{F} = F \otimes R/f$ and $\overline{G} = G \otimes R/f$.

The resolution of an arbitrary R/f-module N is eventually 2-periodic. If

$$0 \leftarrow N \leftarrow F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow \dots \leftarrow F_c \leftarrow 0$$

is the finite resolution of N as R-module then

$$0 \leftarrow N \leftarrow \overline{F}_0 \leftarrow \overline{F}_1 \leftarrow \overline{F}_2 \oplus \overline{F}_0 \leftarrow \overline{F}_3 \oplus \overline{F}_1 \leftarrow \dots \leftarrow \overline{F}_{ev} \leftarrow \overline{F}_{odd} \leftarrow \dots$$

is a R/f-resolution, where

$$F_{ev} = \bigoplus_{i \equiv 0 \mod 2} F_i \quad \text{and} \quad F_{odd} = \bigoplus_{i \equiv 1 \mod 2} F_i.$$

MCM-approximation

The high syzygy modules over a Cohen-Macaulay ring are MCM.
In case of an hypersurface, $M = \operatorname{coker}(\overline{F}_{odd} \to \overline{F}_{ev})$ is a MCM module. There is a natural surjection from M to N with kernel P,

$$0 \leftarrow N \leftarrow M \leftarrow P \leftarrow 0$$

where P is a module of finite projective dimension

$$\operatorname{pd}_{R/f} P < \infty.$$

The graded case: replace R by $S = k[x_0, \ldots, x_n]$

If $f \in S$ is a homogeneous form f degree d then we have to take the grading into account:

- A matrix factorization is now given by a pair

$$G \xrightarrow{\varphi} F \xrightarrow{\psi} G(d)$$

  of maps between graded free S-modules.
- The i-th term in the (not necessarily minimal) eventually 2-periodic S/f-resolution obtained from an S-resolution $F_\bullet$ is

$$\overline{F}_i \oplus \overline{F}_{i-2}(-d) \oplus \ldots \oplus \overline{F}_0(-id/2)$$

  or

$$\overline{F}_i \oplus \overline{F}_{i-2}(-d) \oplus \ldots \oplus \overline{F}_1(-(i-1)d/2)$$

  in case i is even or odd, respectively.

Vector bundles on hypersurfaces

If $X = V(f) \subset \mathbb{P}^n$ is a smooth hypersurface then an MCM module

$$M = \operatorname{coker} \varphi$$

sheafifies to a vector bundle

$$\mathcal{F} = \widetilde{M}$$

on X with no intermediate cohomology,

$$H^p(X, \mathcal{F}(t)) = 0 \text{ for all } p \text{ with } 0 < p < \dim X.$$

If $\det \varphi = \lambda f^r$ with $\lambda \in K$ a scalar, then

$$\operatorname{rank} \mathcal{F} = r.$$

Section 3. The structure theorem

We begin now with the proof of the main theorem.

Theorem

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$$\mathcal{O}^{18}(-3) \xrightarrow{\psi} \mathcal{O}^{15}(-1) \oplus \mathcal{O}^{3}(-2) \xrightarrow{\varphi} \mathcal{O}^{18}$$

of cubic forms on $\mathbb{P}^4$.

Postulation

For $C \subset \mathbb{P}^4$ be a smooth curve of degree $d = 16$ and genus $g = 15$. We have

- $S_C = S/I_C$, the homogeneous coordinate ring, and
- $H^0_*(\mathcal{O}_C) = \oplus_n H^0(\mathcal{O}_C(n))$, the ring of sections.

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Proposition

As S-modules these rings have free resolution with Betti tables

	0	1	2	3	4
0	1	.	.	.	.
1	.	.	.	.	.
2	.	1	.	.	.
3	.	15	30	18	3

and

	0	1	2	3
0	1	.	.	.
1	.	.	.	.
2	3	16	15	0
3	.	.	0	3

iff C has maximal rank and (C, L) is not a ramification point of $\widetilde{\mathcal{M}}_{15} \to \mathcal{M}_{15}$. In particular a general curve C lies on a unique cubic X.

Syzygies $H^0_*(\mathcal{O}_C)$ of as S_X-module

From now on, $C \subset \mathbb{P}^4$ will always denote a general curve of degree 16 and genus 15.

The eventual 2-periodic resolution of $H^0_*(\mathcal{O}_C)$ as an $S_X = S/f$ has the shape

	0	1	2	3	4	5	6	$\cdots$
0	1	.	.	.	.	.	.	.
1	.	.	1	.	.	.	.	.
2	3	16	15	.	1	.	.	.
3	.	.	3	3+16	15	.	1	.
4	.	.	.	.	3	19	15	$\cdots$
$\vdots$	.	.	.	.	.	.	3	$\cdots$

This is not a minimal resolution.

Syzygies $H^0_*(\mathcal{O}_C)$ of as S_X-module

From now on, $C \subset \mathbb{P}^4$ will always denote a general curve of degree 16 and genus 15.

Proposition

The minimal resolution of $H^0_*(\mathcal{O}_C)$ as an $S_X = S/f$ has the shape

	0	1	2	3	4	5	6	$\cdots$
0	1	.	.	.				
1	.	.	.	.				
2	3	15	15	.	.			
3	.	.	3	18	15	.	.	
4	.	.	.	.	3	18	15	$\cdots$
$\vdots$	.	.	.	.	.	.	3	$\cdots$

From C to a matrix factorization

Corollary

A general curve C determines a matrix factorization of shape

	0	1	2
1	15	.	.
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Define $\mathcal{F}$ via

$$0 \leftarrow \mathcal{F} \leftarrow \mathcal{O}_X^{18}(-3) \stackrel{\varphi}{\leftarrow} \mathcal{O}_X^{15}(-4) \oplus \mathcal{O}_X^{3}(-5)).$$

The composition

$$\mathcal{O}_X^{3}(-2) \leftarrow \mathcal{F} \leftarrow \mathcal{O}_X^{18}(-3)$$

is surjective with a summand $\mathcal{O}_X^3(-3)$ in the kernel, since there are only 5 linear forms on $\mathbb{P}^4$.

From the matrix factorization back to C

Theorem (Structure Theorem)

Given the matrix factorization associated to C then the complex

$$0 \leftarrow \mathcal{O}_X^3(-2) \xleftarrow{\alpha} \mathcal{F} \xleftarrow{\beta} \mathcal{O}_X^3(-3) \leftarrow 0$$

is a monad for the ideal sheaf $\mathcal{J}_{C/X}$ of $C \subset X$, i.e.
α is surjective, β injective and

$$\mathcal{J}_{C/X} \cong \ker \alpha / \operatorname{im} \beta.$$

$\mathcal{F}$ is a rank 7 vector bundle on the cubic X, because

$$\deg \det\begin{pmatrix} 18 & 15 \\ . & 3 \end{pmatrix} = 15 + 3 \cdot 2 = 7 \cdot 3.$$

Proof of the main theorem

Since it is an open condition on matrix factorizations of shape

	0	1	2
1	15	.	.
2	3	18	15
3	.	.	3

to lead to a monad of a smooth curve of genus 15 and degree 16, this completes the proof of the main theorem.

We now could study the moduli space $\mathcal{M}_X(7; c_1\mathcal{F}, c_2\mathcal{F}, c_3\mathcal{F})$ of vector bundles on the cubic threefold X.

Section 4. Constructions

Different approach: construct auxiliary modules N, whose syzygies would lead to a desired matrix factorization.

Possible shape of Betti tables $\beta(N)$ are

	0	1	2	3	4
0	a	.	.	.	.
1	b	c	d	.	.
2	.	.	e	f	h
3	.	.	.	.	i

or

	0	1	2	3	4
0	a	b	.	.	.
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1	.	c	d	e	.
2	.	.	.	f	h

with $(a+d+h, b+e+i, c+f) = (3, 15, 18)$ or $(15, 3, 18)$ for the first case, and $(a+d+h, b+e, c+f) = (18, 15, 3)$ or $(18, 3, 15)$ in the second case.

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or

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2	.	.	.	f	h

A computation shows: There are 39 of the tables in the Boij-Söderberg cone with codim $\beta(N) \geq 3$, in all case we have equality.

Four candiate tables

$\deg \beta(N) = 11$

	0	1	2	3
0	5	9	.	.
1	.	3	13	6
2	.	.	.	.

$\deg \beta(N) = 14$

	0	1	2	3	4
0	2	.	.	.	.
1	1	9	.	.	.
2	.	.	14	9	1

$\deg \beta(N) = 13$

	0	1	2	3
0	2	.	.	.
1	2	15	13	.
2	.	.	1	3

$\deg \beta(N) = 14$

	0	1	2	3
0	6	11	.	.
1	.	2	12	4
2	.	.	.	1

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$$\mathcal{L} = \widetilde{N}$$

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- Since $\mathrm{pd}_S(N) \leq 4$, $N \subset H^0_*(\mathcal{L}) = \oplus_{n \in \mathbb{Z}} H^0(\mathcal{L}(n))$ and local cohomology measures the difference

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The genus g_E and the degree $d_{\mathcal{L}} = \deg \mathcal{L}$ are however not yet determined. Their choice is motivated by a dimension count.

Example 1.

The easiest case is perhaps $d_E = 11$ with Betti table

	0	1	2	3
0	5	9	.	.
1	.	3	13	6
2	.	.	.	0

Altogether we get $g_E + 32$ parameters, and to obtain (at least) 42 motivates the choice $g_E = 10$.

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It is natural to assume that $h^0\mathcal{O}_E(1) = 5$. Riemann-Roch
$\Rightarrow h^1\mathcal{O}_E(1) = g_E - d_E + 4 = g_E - 7$.

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It is natural to assume that $h^0\mathcal{O}_E(1) = 5$. Riemann-Roch $\Rightarrow h^1\mathcal{O}_E(1) = g_E - d_E + 4 = g_E - 7$. Parameter count:

$$\dim\{(E, \mathcal{O}_E(1))\} = 4g_E - 3 - 5 \cdot h^1\mathcal{O}_E(1) = 32 - g_E$$

$$\dim\{X \mid X \supset E\} = 34 - (3d_E + 1 - g_E) = g_E$$

Finally $h^1(\mathcal{L}) = 0$ can be read of the Betti table, so $\mathcal{L}$ is non-special and we obtain further g_E parameters. Altogether we get $g_E + 32$ parameters, and to obtain (at least) 42 motivates the choice $g_E = 10$.

Example 1.

$g_E = 10 \Rightarrow h^1(\mathcal{O}_E(1)) = 3$, so E has a plane model E' of degree $18 - 11 = 7$ with $\delta = \binom{6}{2} - 10 = 5$ double points. So we can choose 5+10 points in $\mathbb{P}^2$,

$$E' \in |7h - \sum_{1=1}^{5} 2p_i - \sum_{j=1}^{10} q_j|,$$

and take $\mathcal{L} = \omega_E(q_1 + q_2 + q_3 - (q_4 + \ldots + q_{10}))$.

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and take $\mathcal{L} = \omega_E(q_1 + q_2 + q_3 - (q_4 + \ldots + q_{10}))$. By checking an example with *Macaulay2* over a finite field we conclude:

Theorem (Family 1)

There exists a 42-dimensional unirational family of tuples

$$(E, \mathcal{O}_E(1), X, \mathcal{L}) \text{ with } (d_E, g_E, d_{\mathcal{L}}) = (11, 10, 14)$$

such that $N = H^0_*(\mathcal{L})$ leads to a matrix factorization of desired shape. The general one gives a smooth curve $C \subset \mathbb{P}^4$ of degree 16 and genus 15.

Example 2.

In case of

	0	1	2	3	4
0	2	.	.	.	.
1	1	9	.	.	.
2	.	.	14	9	1

we have $N \subset H^0_*(\mathcal{L})$ with cokern $K(-1)$. The resolution of N and $H^0_*(\mathcal{L})$ differ by a Koszul complex on 5 linear forms.

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So E has a model in $\mathbb{P}^3$ and to pass from $H^0_*(\mathcal{L})$ to N amounts to the choice a point in a $\mathbb{P}^1$.

Example 2.

The dimension count suggest to take $g_E = 11$. Riemann-Roch $\Rightarrow h^1(\mathcal{O}_E(1)) = 1$, hence

$$\mathcal{O}_E(1) \cong \omega_E(-(p_1 + \ldots + p_6)).$$

Theorem (Family 2)

There exists a 42*-dimensional unirulled family of tuples*

$$(E, \mathcal{O}_E(1), X, \mathcal{L}, N) \text{ with } (d_E, g_E, d_{\mathcal{L}}) = (14, 11, 8)$$

such the general tuple gives a smooth curve $C \subset \mathbb{P}^4$ of degree 16 *and genus* 15*.*

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Take the line bundle $\mathcal{L} = \omega_E(-h)$, where h denotes the hyperplane class of the model $E \subset \mathbb{P}^3$ of degree 12, that is a Chang-Ran curve of genus 11.

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Theorem (Family 2)

There exists a 42-dimensional uniruled family of tuples

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Take the line bundle $\mathcal{L} = \omega_E(-h)$, where h denotes the hyperplane class of the model $E \subset \mathbb{P}^3$ of degree 12, that is a Chang-Ran curve of genus 11.

But over a finite field $\mathbb{F}_q$ there are plenty of points in $E(\mathbb{F}_q)$ which are easy to pick with a probabilistic method.

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such the general tuple gives a smooth curve $C \subset \mathbb{P}^4$ *of degree* 16 *and genus* 15*.*

Section 3. Tangent space computations

All what is needed to conclude from family 1 that $\widetilde{\mathcal{M}}_{15}$ is unirational, is to prove that the map gives an isomorphism on tangent spaces in a random example.
Since the association

$$(N, X) \mapsto (M, X)$$

might not be surjective, this is a nontrivial assertion. So we want to study the natural map

$$Ext^1_{S_X}(N, N)_0 \to Ext^1_{S_X}(M, M)_0.$$

5. Tangent space diagram

The relevant diagram is

$$\begin{array}{ccccccc} Ext^1_{S_X}(M,P) & \to & Ext^1_{S_X}(M,M) & \to & Ext^1_{S_X}(M,N) & \to & Ext^2_{S_X}(M,P) \\ & & & & \uparrow & & \\ & & & & Ext^1_{S_X}(N,N) & & \\ & & & & \uparrow & & \\ & & & & Hom_{S_X}(P,N) & & \end{array}$$

deduced from the MCM approximation

$$0 \leftarrow N \leftarrow M \leftarrow P \leftarrow 0.$$

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\| & & & & \uparrow & & \| \\
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 & & & & \uparrow & & \\
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$\dim Ext^1_{S_X}(M,M)_0 = \dim Ext^1_{S_x}(N,N)_0 = 32$ as expected,
$Hom_{S_X}(P,N)_0 \hookrightarrow Ext^1_{S_x}(N,N)_0$, but

Dimensions of the families

Proposition

For a randomly chosen example,

$$\dim Hom_{S_X}(P, N)_0 = \begin{cases} 3 & \textit{in case of family 1} \\ 0 & \textit{in case of family 2} \end{cases}$$

Hence family 1 leads to a 39-dimensional subvariety of $\widetilde{\mathcal{M}}_{15}$ and family 2 dominates. Inparticular $\widetilde{\mathcal{M}}_{15}$ is unirruled.

Section 6. Conclusion

Altogether I managed to construct 20 families of pairs (X, N) all of dimension at least 42, of which

- 17 families are unirational,
- 3 are (possibly) not, since they required the choice of additional points on the auxiliary curve E.
- The three non-unirational families dominate.
- Most of the unirational families lead to 39-dimensional subvarieties of $\widetilde{\mathcal{M}}_{15}$. One has dimension 40, another one dimension 41.

Could this be just bad luck?

A conjecture

I think no.

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A conjecture

I think no. A good explanation could be

Conjecture

The maximal rationally connected fibration of $\widetilde{\mathcal{M}}_{15}$ has a three dimensional base.

.

A conjecture and a complexity result

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The maximal rationally connected fibration of $\widetilde{\mathcal{M}}_{15}$ *has a three dimensional base.*

Theorem

The probabilistic algorithm, which for a finite field $\mathbb{F}_q$ *selects randomly curves of genus* 15*, has running time* $O((\log q)^3)$*.*

.

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The probabilistic algorithm, which for a finite field $\mathbb{F}_q$ selects randomly curves of genus 15, has running time $O((\log q)^3)$.

- I expect that the algorithm picks points from a subset of $\mathcal{M}_{15}(\mathbb{F}_q)$ of density about 47%. The image of $\widetilde{\mathcal{M}}_{15}(\mathbb{F}_q)$ should have density about 63%. The same should hold for the image of the $\mathbb{F}_q$-rational points of the parameter space in $\widetilde{\mathcal{M}}_{15}(\mathbb{F}_q)$.

.

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- A unirational description of $\mathcal{M}_{15}$ would lead to an algorithm with running time $O((\log q)^2)$.

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- A unirational description of $\mathcal{M}_{15}$ would lead to an algorithm with running time $O((\log q)^2)$.
- For any fixed genus g there exists an algorithm which selects points from a subset of $\mathcal{M}_g(\mathbb{F}_q)$ of positive density in running time $O((\log q)^3)$.

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- For any fixed genus g there exists an algorithm which selects points from a subset of $\mathcal{M}_g(\mathbb{F}_q)$ of positive density in running time $O((\log q)^3)$.

Thank you!