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Fachrichtung Mathematik
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Computer Algebra   Summer Term 2019

Exercise Sheet 11. Hand in by Tuesday, July 2.

Exercise 1. Let $f = a_d x^d + \ldots + a_0$ and $g = b_d x^e + \ldots + b_0 \in K[x]$ and let

$$f = a_d(x-\alpha_1)\cdots(x-\alpha_d), g = b_e(x-\beta_1)\cdots(x-\beta_e)$$

be factorizations over some extension field $L \supset K$. Prove: The resultant satisfies the formulas

$$\begin{aligned} R(f,g) &= a_d^e\, g(\alpha_1)\cdot\ldots\cdot g(\alpha_d) \\ &= (-1)^{de} b_e^d\, f(\beta_1)\cdot\ldots\cdot f(\beta_e) \\ &= a_d^e b_e^d \prod_{\substack{1\le i\le d \\ 1\le j\le e}} (\alpha_i - \beta_j) \end{aligned}$$

Hint: Multiply the Sylvester matrix with the Vandermonde matrix

$$\begin{pmatrix} \beta_1^{d+e-1} & \ldots & \beta_1^2 & \beta_1 & 1 \\ \vdots & & & \vdots & \\ \beta_e^{d+e-1} & \ldots & \beta_e^2 & \beta_e & 1 \\ \alpha_1^{d+e-1} & \ldots & \alpha_1^2 & \alpha_1 & 1 \\ \vdots & & & \vdots & \\ \alpha_d^{d+e-1} & \ldots & \alpha_d^2 & \alpha_d & 1 \end{pmatrix}$$

Exercise 2. Prove:

$$R(f_1 f_2, g) = R(f_1, g) R(f_2, g)$$

for $f_1, f_2, g \in K[x]$.

Exercise 3. A monomial ideal $J \in K[x_0, \ldots, x_n]$ is called Borel-fixed if for $m \in J$ a monomial and x_i a variable dividing m the monomial $\frac{x_j}{x_i} m \in J$ for every $j < i$. Prove:

The syzygy algorithm computes the minimal free resolution in case of a Borel fixed ideal J.

Exercise 4. An alternative way to present the Hilbert function is via its generating series. The Hilbert series of a finitely generated graded $S = K[x_0, \ldots, x_n]$-module M is defined by

$$H_M(t) = \sum_{d \in \mathbb{Z}} h_M(d) t^d \in \mathbb{Z}[[t]][t^{-1}].$$

Prove: H_M is a rational function, more precisely

$$H_M(t) = \frac{\sum_j \sum_i (-1)^i \beta_{ij} t^j}{(1-t)^{n+1}}.$$