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Computer Algebra Summer Term 2019

Exercise Sheet 3. Hand in by Tuesday, May 7.

Exercise 1

Let F be an infinite field and let $f \in F[x_1, \ldots, x_n]$ be a non-zero polynomial. Prove:
There exists a point $a \in F^n$ such that $f(a) \neq 0$.

Exercise 2

Let $F = \mathbb{C}$ and let $f \in \mathbb{C}[x_1, .., x_n]$ be a non-zero polynomial. Prove for every $a \in \mathbb{A}^n(\mathbb{C}) = \mathbb{C}^n$ and every $\epsilon > 0$ the ball $B_\epsilon(a)$ with radius ϵ around a intersects the complement $\mathbb{C}^n \setminus V(f)$ of $V(f)$.

Hint: If $f \in F[x_1]$ is polynomial in one variable of degree $\leq d$, and $b_1, \ldots, b_{d+1} \in F$ are pairwise distinct, then $f(b_j) = 0$ for all $j = 1, \ldots, d+1$ holds if and only if f is the zero polynomial.

Exercise 3

Let $I \subset F[x_1, \ldots, x_n]$ be an ideal, let $L(I)$ denote its lead ideal with respect to a global monomial order and let $A = V(I) \subset \mathbb{A}^n = \mathbb{A}^n(\overline{F})$ the corresponding algebraic set. Prove: TFAE:

(1) A is finite.
(2) The set of monomials $\{m \notin L(I)\}$ is finite.
(3) $F[x_1, \ldots, x_n]/I$ is finite-dimensional as an F-vector space.

If this is the case, then $|\{m \notin L(I)\}|$ bounds the number of solutions $|A|$, with equality, if $F = \overline{F}$ and $I = I(A)$.

Exercise 4

Show that the parabola $V(y - x^2) \subset \mathbb{A}^2$ and the hyperbola $V(xy - 1) \subset \mathbb{A}^2$ are not isomorphic.