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Computer Algebra Summer Term 2019

Exercise Sheet 5. Hand in by Tuesday, May 21.

Exercise 1
Let $I \subset K[x_1, \dots, x_n]$ an ideal, $A = V(I)$ the corresponding algebraic set, and let $L(I)$ be the initial ideal with respect to the lexicographic order. Suppose there exist an $N \in \mathbb{N}$ such that

$$\langle x_1, \dots x_k \rangle^N \subset L(I) \subset \langle x_1, \dots x_k \rangle.$$

Prove:

$$\dim A = n - k.$$

If in addition, $L(I)$ is generated by monomials in $K[x_1, \dots, x_k]$ then every component C of A has dimension $\dim C = n - k$.

Exercise 2
Let $A \in \mathbb{Z}^{n\times n}$ be a square matrix with $\det A \neq 0$. Prove: $G = \operatorname{coker} A = \mathbb{Z}^n / \operatorname{image} A$ is a finite abelian group of order

$$|G| = |\det A|.$$

Exercise 3
Let R be a PID, and $A \in R^{n\times n}$. Prove the following third version of the classification theorem: There exist invertible matrices $S, T \in GL(n, R)$ such that SAT^{-1} is a diagonal matrix with entries $d_1, \dots, d_n$ such that $d_i | d_{i+1}$.
(The d_i are called the elementary divisors of A.)

Exercise 4
Let R be a ring and M an R-module, e.g. $M = R$. A subset $S \subset R$ is called multiplicative, if

(1) $1 \in S$, and
(2) $u, v \in S \implies uv \in S$

holds.

Let $S \subset R$ be multiplicative. Prove that

$$(m_1, u_1) \sim (m_2, u_2) \iff \exists v \in S \text{ such that } v(u_2 m_1 - u_1 m_2) = 0 \in M$$

defines an equivalence relation on $M \times S$.

Let $M[S^-1] := (M \times S)/\sim$ denote the set of equivalence classes, and $\frac{m}{u} \in M[S^{-1}]$ the class represented by (m, u). Prove that the formulas

$$\frac{m_1}{u_1} + \frac{m_2}{u_2} := \frac{u_2 m_1 + u_1 m_2}{u_1 u_2}$$

and

$$\frac{r}{u_1} \cdot \frac{m}{u_2} := \frac{rm}{u_1 u_2}$$

gives $R[S^{-1}]$ the structure of a ring and $M[S^{-1}]$ the structure of an $R[S^{-1}]$-module.