Algebraic Geometry, Lecture 10

Frank-Olaf Schreyer

Saarland University, Perugia 2021

Overview

Today's topics are rational functions.

1. Rational function field
2. Local ring of a variety at a point
3. Dominant rational maps and birational maps
4. Algebraic and transcendental elements in field extensions
5. Dimension of an algebraic set
6. Appendix: Transcendence degree

Zariski topology on an algebraic set

Let $A \subset \mathbb{A}^n$ be an algebraic set and let $K[A] = K[x_1, \ldots, x_n]/\,\mathrm{I}(A)$ be its coordinate ring.

Definition. The **Zariski topology** on A is the topology induced an A from the Zariski topology of $\mathbb{A}^n$.

Thus the closed subsets of A are the algebraic subsets $B \subset A$. These are in $1-1$ correspondence with radical ideals $J \supset \mathrm{I}(A)$ respectively with radical ideals $\overline{J} = J/\,\mathrm{I}(A) \subset K[A]$:

$$\{\text{algebraic subsets of } A\} \xleftrightarrow{1-1} \{\text{radical ideals of } K[A]\}$$

with

$$B \mapsto \mathrm{I}_A(B) = \{\overline{f} \in K[A] \mid \overline{f}(p) = 0\ \forall p \in B\}$$

and

$$V_A(\overline{J}) = \{p \in A \mid \overline{f}(p) = 0\ \forall \overline{f} \in \overline{J}\} \leftarrow\!\shortmid \overline{J}.$$

In particular we have

$$V_A(\overline{J}) = \emptyset \iff \overline{J} = (1).$$

The rational function field

From now on in today's lecture A denotes an **irreducible algebraic set**. Thus $K[A]$ is an integral domain. We will also drop the overline from $\overline{f}$ in the notation of elements and ideals of $K[A]$.

Definition. The **field of rational functions** on A is the quotient field

$$K(A) = Q(K[A]) = \{f = \frac{g}{h} \mid g, h \in K[A], h \neq 0 \in K[A]\}.$$

We want to interpret $f \in K(A)$ as a partially defined function

$$f : A \dashrightarrow K.$$

Clearly, if $f = g/h$ and $p \in A$ is a point with $h(p) \neq 0$, then $f(p) = g(p)/h(p)$ makes sense. However f has many representatives as fraction. Thus from $h(p) = 0$ we cannot conclude that f is not defined in p.

A non-factorial coordinate ring

Example. Consider $A = V(wx - yz)$. The coordinate ring $K[A] = K[w,x,y,z]/(wx - yz)$ is not factorial. The rational function

$$f = \frac{w}{z} = \frac{y}{x} \in K(A)$$

is defined for all points $p \notin V_A(x, z)$.

However localization of $K[A]$ for A an irreducible algebraic set is simpler than in general.

Proposition. *Let $U \subset K[A]$ be a multiplicative set with $0 \notin U$. Then two fractions*

$$\frac{g_1}{h_1}, \frac{g_2}{h_2} \in K[A][U^{-1}]$$

are equal iff $\frac{g_1}{h_1} = \frac{g_2}{h_2} \in K(A)$.

Proof. $u(h_2 g_1 - h_1 g_2) = 0 \in K[A] \iff h_2 g_1 - h_1 g_2 = 0 \in K[A]$ because $K[A]$ is an integral domain. $\square$

The local ring of a point

Corollary. *Let* $\mathfrak{p}$ *be a prime ideal in* $K[A]$*. Then*

$$K[A]_{\mathfrak{p}} \subset K(A).$$

Definition. Let $p \in A$ be a point and $\mathfrak{m}_p \subset K[A]$ be the corresponding maximal ideal. Then

$$\mathcal{O}_{A,p} = K[A]_{\mathfrak{m}_p}$$

denotes **the local ring of** A **in** p. A rational function $f \in K(A)$ is defined in p iff $f \in \mathcal{O}_{A,p}$.

Everywhere defined rational functions

Theorem. *Let A be an irreducible algebraic set. Then*

$$K[A] = \bigcap_{p \in A} \mathcal{O}_{A,p} \subset K(A).$$

Proof. Let $f \in K(A)$. Consider the ideal of denominators of f:

$$\begin{aligned} I_f &= \{h \in K[A] \mid hf \in K[A]\} \\ &= \{h \in K[A] \mid f = \frac{g}{h}\} \cup \{0\}. \end{aligned}$$

Remark. That the set in the second line is an ideal might be a little bit surprising. It says: if h_1 and h_2 are denominators of f and $h_1 + h_2 \neq 0$, then $h_1 + h_2$ is also a denominator of f. Indeed

$$f = \frac{g_1}{h_1} = \frac{g_2}{h_2} \Rightarrow f = \frac{g_1 + g_2}{h_1 + h_2}.$$

Everywhere defined rational functions, continued

Now f is defined at p iff $f \in \mathcal{O}_{A,p}$ iff $p \in A \setminus V(I_f)$, since the elements of $\mathcal{O}_{A,p} = K[A]_{\mathfrak{m}_p}$ are fractions with denominator $h \notin \mathfrak{m}_p \Leftrightarrow h(p) \neq 0$.
If f is everywhere defined, then $V_A(I_f) = \emptyset$ and the Nullstellensatz implies $1 \in I_f$. Hence $f \in K[A]$. $\square$

Definition. Let $f \in K(A)$ be a rational function. Then its **domain of definition of** f is the Zariski open set

$$\mathrm{dom}(f) = A \setminus V_A(I_f) \text{ where } I_f = \{h \in K[A] \mid hf \in K[A]\}.$$

This is a Zariski dense open subset of A on which f defines a K-valued function

$$A \supset \mathrm{dom}(f) \xrightarrow{f} K, \ a \mapsto f(a).$$

Non-empty Zariski open sets are dense

Remark. The fact that $\mathrm{dom}(f)$ is Zariski dense is less spectacular than it might seem on first glance. Actually every non-empty Zariki open subset of A is Zariski-dense:

Proposition. *Let D_1, D_2 be Zariski open subsets of an irreducible algebraic set A. Then*

$$D_1 \cap D_2 = \emptyset \iff D_1 = \emptyset \textit{ or } D_2 = \emptyset.$$

Proof. Let $A_j = A \setminus D_j$ for $j = 1, 2$ be the corresponding closed sets. Then

$$D_1 \cap D_2 = \emptyset \iff A_1 \cup A_2 = A \implies A = A_1 \text{ or } A = A_2$$

because A is irreducible. Thus $D_1 = \emptyset$ or $D_2 = \emptyset$. $\square$

Rational map

Definition. Let $A \subset \mathbb{A}^n$ and $B \subset \mathbb{A}^m$ be irreducible algebraic sets. A **rational map** $\varphi : A \dashrightarrow B$ is given by an m-tuple of rational functions $f_1, \ldots, f_m \in K(A)$ such that

$$\varphi(p) = (f_1(p), \ldots, f_m(p)) \in B \text{ for all } p \in \bigcap_{j=1}^{m} \mathrm{dom}(f_j).$$

Note that the **domain of definition of** φ defined by $\mathrm{dom}(\varphi) = \bigcap_{j=1}^{m} \mathrm{dom}(f_j)$ is not empty by the previous proposition.

Example.

$$\mathbb{A}^1 \dashrightarrow V(x^2 + y^2 - 1), t \mapsto (\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1})$$

is a rational map. Indeed

$$(\frac{2t}{t^2+1})^2 + (\frac{t^2-1}{t^2+1})^2 - 1 = \frac{(2t)^2 + (t^2-1)^2 - (t^2+1)^2}{(t^2+1)^2} = 0.$$

Dominant rational map

Two rational maps $\varphi : A \dashrightarrow B$ and $\psi : B \dashrightarrow C$ might be not composable because it is possible that the image of φ, i.e., $\varphi(\mathrm{dom}(\varphi))$ lies entirely in the complement of $\mathrm{dom}(\psi)$. This does not happen if $\varphi(\mathrm{dom}(\varphi))$ is dense in B.

Definition. A **dominant rational map** is a rational map $\varphi : A \dashrightarrow B$, such that $\varphi(\mathrm{dom}(\varphi))$ is dense in B.

Thus two dominant rational maps $\varphi : A \dashrightarrow B$ and $\psi : B \dashrightarrow C$ can be composed, and the composition $\psi \circ \varphi : A \dashrightarrow C$ is dominant as well.
The category of affine varieties over an algebraically closed field with dominant rational maps as morphisms has the following field theoretic description.

Dominant rational map and field extension

Let

$$\varphi : A \dashrightarrow B \subset \mathbb{A}^m, p \mapsto (f_1(p), \ldots, f_m(p))$$

be a dominant rational map. Then

$$\varphi^* : K(B) \to K(A), F = \frac{G}{H} \mapsto F(f_1, \ldots, f_m) = \frac{G(f_1, \ldots, f_m)}{H(f_1, \ldots, f_m)}$$

is an injective K-algebra map between fields. Note that $H(f_1, \ldots, f_m) \in K(A)$ is not the zero element of $K(A)$ because otherwise $\varphi(\mathrm{dom}(\varphi))$ would be contained in $V_B(H)$ contradicting the assumption that the map is dominant. By the same argument φ^* is injective.

Dominant rational map and field extension

Conversely, if $\phi : K(B) \to K(A)$ is a K-algebra homomorphism between fields and if $\overline{y}_1, \ldots, \overline{y}_m$ denote the coordinate functions on B, then $f_1 = \phi(\overline{y}_1), \ldots, f_m = \phi(\overline{y}_m)$ is a tuple of rational functions which defines a rational map $\varphi : A \dashrightarrow B$. It is dominant because $\phi : K(B) \to K(A)$ is injective, and the composition $K[B] \hookrightarrow K(B) \to K(A)$ is injective as well. Since $\phi(F) = F(f_1, \ldots, f_m)$ we have $\varphi^* = \phi$.

Theorem. *The category of affine varieties over K with dominant rational maps as morphisms and the category of finitely generated field extensions of K with K-algebra injection as morphisms are equivalent via*

$$A \mapsto K(A)$$

and

$$\varphi : A \dashrightarrow B \mapsto \varphi^* : K(B) \hookrightarrow K(A).$$

Proof

Most of the theorem has already been established. It remains to prove that every finitely generated extension field

$$K \subset L$$

arises as $L = K(A)$ for some variety A. Indeed, if

$$L = K(g_1, \ldots, g_n)$$

is generated by elements $g_1, \ldots, g_n$, then the substitution homomorphism

$$K[x_1, \ldots x_n] \to L, x_i \mapsto g_i$$

has a prime ideal J as a kernel because the image as a subring of a field is an integral domain. Then

$$A = V(J) \subset \mathbb{A}^n$$

is an affine variety with $K(A) \cong L$. $\square$

Birational varieties

Remark. The variety A with $L \cong K(A)$ is not uniquely determined. Choosing different generators gives different varieties.
Example. For $A = V(xy-1) \subset \mathbb{A}^2$ we have $L = K(A) = K(\overline{x}, \overline{y})$, and these generators give A back again. Since $\overline{y} = 1/\overline{x}$ we have $K(\overline{x}, \overline{y}) = K(\overline{x})$ and the second choice leads to $B = \mathbb{A}^1$.

Definition. A dominant rational map $\varphi : A \dashrightarrow B$ is called **birational** if there exists a dominant rational map $\psi : B \dashrightarrow A$ such that $\psi \circ \varphi = id_A$ holds, by which we mean that $\psi \circ (\varphi|_D) = id_D$ holds on the (non-empty) open subset $D \subset A$ on which $\psi \circ \varphi$ is defined as a honest map.
By the theorem φ is birational iff $\varphi^* : K(B) \to K(A)$ is an isomorphism. The rational map $\psi : B \dashrightarrow A$ induces the inverse isomorphism $\psi^* = (\varphi^*)^{-1}$, and $\varphi \circ \psi = id_B$ holds automatically as well.

In the example above $\varphi : V(xy-1) \to \mathbb{A}^1$ is the projection onto the y-axes, while $\psi : \mathbb{A}^1 \dashrightarrow V(xy-1), x \mapsto (x, 1/x)$.

Algebraic and transcendental elements in field extensions

Let $k \subset L$ be a field extension. For $g_1, \ldots, g_n \in L$ we denote by $k(g_1, \ldots, g_n) \subset L$ the smallest subfield of L containing $k \cup \{g_1, \ldots, g_n\}$. In contrast

$$k[g_1, \ldots, g_n] \subset L$$

denotes the smallest subring of L containing $k \cup \{g_1, \ldots, g_n\}$. This is the image under the substitution homomorphism

$$k[x_1, \ldots, x_n] \to L, x_i \mapsto g_i.$$

An element $g \in L$ is called **algebraic over** k if $k[x] \to L, x \mapsto g$ has a nontrivial kernel. In this case the normed generator f of the kernel is called the **minimal polynomial** of g over k, and

$$k[g] \cong k[x]/(f)$$

is a finite-dimensional k-vector space and a field, i.e., $k(g) = k[g]$. If g is not algebraic over k, then g is called **transcendental over** k. In this case $k[g] \cong k[x]$ is an infinite-dimensional k-vector space and not a field: $k[g] \subsetneq k(g)$.

Algebraic independent elements

Elements $g_1, \ldots, g_d \in L$ are called **algebraically independent** over k if

$$k[x_1, \ldots, x_d] \to L, x_i \mapsto g_i$$

has trivial kernel.

A maximal set of algebraic independent elements of L is called a **transcendence basis** for L over k. If $k \subset L$ is finitely generated, then by dropping elements from a generating set one can arrive at a transcendence basis.

Definition. Let $k \subset L$ be a field extension. The **transcendence degree**

$$\operatorname{trdeg}_k L$$

is the number of elements in a transcendence basis of L over k.

Definition of the dimension

Definition. Let A be an irreducible algebraic set. Then the **dimension of** A is

$$\dim A = \operatorname{trdeg}_K K(A).$$

If A is an algebraic set, then we define

$$\dim A = \max\{\dim A_i \mid i = 1, \ldots, r\}$$

where $A = A_1 \cup \ldots \cup A_r$ is the decomposition into irreducible algebraic subsets.

A Gröbner basis criterion

Theorem. *Let* $I \subset K[x_1, \ldots, x_c, y_1, \ldots, y_d]$ *be an ideal and let* $A = V(I) \subset \mathbb{A}^{c+d}$ *be the corresponding algebraic set. Let* $>$ *be a global monomial order. Suppose*

$$\operatorname{rad}(\operatorname{Lt}(I)) = (x_1, \ldots, x_c).$$

Then $\dim A = d$ *and the projection*

$$\pi : A \to \mathbb{A}^d, \quad (a_1, \ldots, a_c, b_1 \ldots, b_d) \mapsto (b_1, \ldots, b_d)$$

onto the last d *components is surjective.*
Moreover, if $\operatorname{Lt}(I)$ *is generated by monomials in the subring* $K[x_1, \ldots, x_c]$*, then every associated prime of* I *defines a variety of dimension* d*. In particular every irreducible component of* A *has dimension* d*.*

One says that I is **unmixed** if the conclusion of the additional hypothesis is satisfied.
Corollary. *I is unmixed if* $\operatorname{Lt}(I)$ *is a primary ideal.*

An example

Consider the ideal I generated by the following polynomials of $K[x_0, \dots x_3]$.
With respect to $>_{rlex}$ the leading terms are the indicated terms, and the calculation shows that the generators are a Gröbner basis.

$x_1^2 - x_0x_2$	$-x_2$	x_3
$x_1x_2 - x_0x_3$	x_1	$-x_2$
$x_2^2 - x_1x_3$	$-x_0$	x_1

Thus the assumption of the theorem is satisfied for x_1, x_2 and $y_1 = x_0, y_2 = x_3$. Hence I is unmixed, and every component has dimension 2.

Appendix: Transcendence degree

Suppose $L = k(g_1, \dots, g_n)$ and $g_1, \dots, g_d$ is a maximal subset of algebraic independent elements. Then $L = k(g_1, \dots, g_n)$ is an finite dimensional $k(g_1, \dots, g_d)$-vector space. In particular every element $g \in L$ is algebraic over $k(g_1, \dots, g_d)$, i.e., $\{g_1, \dots, g_d, g\}$ are algebraically dependent.

Theorem. *Let $k \subset L$ be a field extension. Any two transcendence bases of L over k have the same cardinality.*

Definition. The common cardinality of all transcendence bases

$$\mathrm{trdeg}_k(L)$$

is called the **transcendence degree** of L over k.

The exchange lemma

We will prove this only in case that $L \supset k$ is finitely generated over k. The proof is similar to the proof that the dimension of a vector space is well-defined.

Lemma. *Let $\{g_1, \ldots, g_d\}$ be a transcendence basis of L over k and let $h \in L$ transcendental over k. Then there exists an index i such that $\{g_1, \ldots, g_{i-1}, h, g_{i+1}, \ldots, g_d\}$ is a transcendence basis as well.*

Proof. Consider an irreducible polynomial $F \in k[x_1, \ldots, x_d, y]$ in the kernel of the map of smallest total degree

$$k[x_1, \ldots, x_d, y] \to L, x_j \mapsto g_j, y \mapsto h.$$

Such an F exists because the kernel is a prime ideal. F involves y because $g_1, \ldots, g_d$ are algebraically independent, and it involves some variable x_i because h is not algebraic over k. After renumbering the x_i's we may assume that $i = d$.

Proof of the exchange Lemma continued

The polynomial $F(g_1, \ldots, g_{d-1}, x_d, h)$ does not vanish, because the leading coefficient F as polynomial in x_d has smaller degree than F, hence does not vanish under the substitution.
Thus g_d is algebraic over $k(g_1, \ldots, g_{d-1}, h)$. Every element of L is algebraic over $k(g_1, \ldots, g_{d-1}, h)$ because

$$k(g_1, \ldots, g_{d-1}, h) \subset k(g_1, \ldots, g_d, h) \subset L$$

is a tower of algebraic field extensions. Finally $g_1, \ldots, g_{d-1}, h$ are algebraic independent because otherwise h would be algebraic over $k(g_1, \ldots, g_{d-1})$. This would imply that also

$$k(g_1, \ldots, g_{d-1}) \subset k(g_1, \ldots, g_{d-1}, h)$$

is an algebraic field extension and g_d would be algebraic over $k(g_1, \ldots, g_{d-1})$, contradicting our assumption. $\square$

Proof of the theorem

We prove by induction on c the following proposition which implies the theorem immediately.

Proposition. *Let $\{g_1, \dots, g_d\}$ be a transcendence basis of L over k, and let $h_1, \dots, h_c \in L$ be elements which are algebraically independent over k. Then after a suitable reordering of $g_1, \dots, g_d$ the set $\{h_1, \dots, h_c, g_{c+1}, \dots, g_d\}$ is a transcendence basis as well. In particular $c \leq d$.*

Proof. The case $c = 1$ is the exchange lemma above after renumbering. By the induction hypothesis we may assume that $\{h_1, \dots, h_{c-1}, g_c, \dots, g_d\}$ is a transcendence basis. By the exchange Lemma we can replace one of these elements by h_c and from the proof we see that this element can be chosen to be different from $h_1, \dots, h_{c-1}$ because $h_1, \dots, h_c$ are algebraically independent. After reordering we may assume that this element is g_c. $\square$