Algebraic Geometry, Lecture 18

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Overview

Today's topics are

1. Products of projective spaces
2. Morphism
3. Linear projections
4. A dimension bound
5. Rational maps from smooth curves

Products of algebraic sets

For two affine algebraic sets $A \subset \mathbb{A}^n$ and $B \subset \mathbb{A}^m$ the product

$$A \times B \subset \mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$$

is simply the algebraic set defined by

$$(\mathrm{I}(A) \cup \mathrm{I}(B)) \subset K[x_1, \ldots x_n, y_1, \ldots y_m]$$

where $\mathrm{I}(A) \subset K[x_1, \ldots, x_n]$ and $\mathrm{I}(B) \subset K[y_1, \ldots, y_m]$ are the vanishing ideals of A and B respectively.

For projective algebraic sets the definition of a product is not so clear. To start with, it is not a priori clear how to give $\mathbb{P}^n \times \mathbb{P}^m$ the structure of an algebraic set. One uses the **Segre embedding**.

Segre embedding 1

Define

$$\sigma_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N \text{ with } N = (n+1)(m+1) - 1$$

by

$$([a_0 : \ldots : a_m], [b_0 : \ldots : b_n]) \mapsto [a_0 b_0 : \ldots : a_i b_j : \ldots : a_m b_n].$$

This is a well-defined map. For any pair of points at least one component $a_i b_j \neq 0$.
We will use variables $\mathbf{x} = x_0, \ldots, x_n$, $\mathbf{y} = y_0, \ldots, y_m$ and $\mathbf{z} = z_{00}, \ldots, z_{0m}, z_{10}, \ldots, z_{nm}$ for the homogeneous coordinate rings of $\mathbb{P}^n, \mathbb{P}^m$ and $\mathbb{P}^N$. Moreover we call a polynomial

$$f = \sum_{|\alpha|=d, |\beta|=e} f_{\alpha,\beta} x^\alpha y^\beta \in K[\mathbf{x}, \mathbf{y}]$$

bihomogeneous (in x and y) of bidegree (d, e)**.**

Segre embedding 2

Proposition. *Let* $\Sigma_{n,m} \subset \mathbb{P}^N$ *be the projective algebraic set defined by the* 2×2*-minors of the* $(n+1) \times (m+1)$*-matrix* (z_{ij})*. Then*

$$\sigma_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \Sigma_{m,n}$$

is a bijection which induces isomorphisms $U_i \times U_j \cong \Sigma_{n,m} \cap U_{ij}$ *on the standard charts. Moreover* $\Sigma_{n,m} \subset \mathbb{P}^N$ *is irreducible, and the ideal of* 2×2*-minors coincides with the homogeneous ideal of* $\Sigma_{m,m}$*.*

Proof. The minor

$$\det \begin{pmatrix} z_{i_1 j_1} & z_{i_1 j_2} \\ z_{i_2 j_1} & z_{i_2 j_2} \end{pmatrix}$$

vanishes on the image of $\sigma_{n,m}$ because

$$\det \begin{pmatrix} x_{i_1} y_{j_1} & x_{i_1} y_{j_2} \\ x_{i_2} y_{j_1} & x_{i_2} y_{j_2} \end{pmatrix} = 0.$$

Thus the image of $\sigma_{m,n}$ is contained in $\Sigma_{m,n}$.

Segre embedding 3

The point $r = [1 : c_{01} : \ldots : c_{nm}] \in \Sigma_{n,m} \cap U_{00}$ satisfies

$$c_{ij} = c_{i0}c_{0j}.$$

Thus the pair of points

$$(p, q) = ([1 : c_{10} : \ldots, c_{n0}], [1 : c_{01} : \ldots : c_{0m}]) \in U_0 \times U_0 \subset \mathbb{P}^n \times \mathbb{P}^m$$

is the unique preimage point of r and $\Sigma_{n,m} \cap U_{00} \cong U_0 \times U_0$. The same argument in other charts gives that $\sigma_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \Sigma_{n,m}$ is bijective and gives isomorphisms $\Sigma_{n,m} \cap U_{ij} \cong U_i \times U_j$.

To prove that $\Sigma_{m,n}$ is irreducible and that the ideal J of 2×2-minors of (z_{ij}) is its homogeneous ideal, it suffices to prove that J is a prime ideal.

Segre embedding 4

Consider the ring homorphism

$$\varphi : K[\mathbf{z}] \to K[\mathbf{x}, \mathbf{y}], z_{ij} \mapsto x_i y_j$$

Clearly, $J \subset \ker \varphi$. To prove equality we consider a reverse lexicographic order $>_{\mathrm{rlex}}$ which refines the following order on the variables

$$\begin{array}{ccccccc} z_{00} & > & z_{01} & > & \dots & > & z_{0m} \\ \vee & & \vee & & & & \vee \\ z_{10} & > & z_{11} & > & \dots & > & z_{1m} \\ \vee & & \vee & & & & \vee \\ \vdots & & \vdots & & & & \vdots \\ \vee & & \vee & & & & \vee \\ z_{n0} & > & z_{n1} & > & \dots & > & z_{nm} \end{array}$$

We have

$$\mathrm{Lt}(\det \begin{pmatrix} z_{i_1 j_1} & z_{i_1 j_2} \\ z_{i_2 j_1} & z_{i_2 j_2} \end{pmatrix}) = -z_{i_2 j_1} z_{i_1 j_2}$$

whenever $i_1 < i_2$ and $j_1 < j_2$.

Segre embedding 5

Thus the remainder of a monomial in $K[\mathbf{z}]$ divided by the 2×2-minors has the form

$$z_{i_1 j_1} z_{i_2 j_2} \cdots z_{i_d j_d} \text{ with } i_1 \leq i_2 \leq \ldots \leq i_d \text{ and } j_1 \leq j_2 \leq \ldots \leq j_d.$$

Since φ induces a bijection between such monomials and bihomogeneous monomials of bidegree (d, d) we conclude that the 2×2-minors form a Gröbner basis of $\ker \varphi$. In particular $J = \ker \varphi$ and this is a prime ideal because $K[\mathbf{z}]/\ker \varphi$ is isomorphic to a subring of the domain $K[\mathbf{x}, \mathbf{y}]$. □

Definition. We give $\mathbb{P}^n \times \mathbb{P}^m$ the structure of a projective variety by identifying $\mathbb{P}^n \times \mathbb{P}^m$ and $\Sigma_{n,m}$.

Example. We identify $\mathbb{P}^1 \times \mathbb{P}^1$ with the quadric

$$\Sigma_{1,1} = V(z_{00} z_{11} - z_{10} z_{01}) \subset \mathbb{P}^3.$$

Hypersurface in $\mathbb{P}^n \times \mathbb{P}^m$ of bidegree (d, e).

Notice that the Zariski topology on $\mathbb{P}^n \times \mathbb{P}^m$ is finer than the product of the Zariski topologies of the factors. For example, if

$$f = \sum_{|\alpha|=d, |\beta|=e} f_{\alpha,\beta} x^\alpha y^\beta \in K[\mathbf{x}, \mathbf{y}]$$

is a bihomogeneous polynomial of bidegree (d, e), then

$$V(f) = \{(a, b) \in \mathbb{P}^n \times \mathbb{P}^m \mid f(a, b) = 0\}$$

is a Zariski closed subset, which for general f is not closed in the product topology. To see that $V(f)$ is an algebraic subset of $\mathbb{P}^n \times \mathbb{P}^m$ we argue as follows: Suppose $d \geq e$. Then multiplying f with monomials $y^\beta \in K[\mathbf{y}]$ of degree $d - e$ we get $\binom{d-e+m}{m}$ polynomials fy^β of bidegree (d, d), each of which is the image of a polynomial in $F_\beta \in K[\mathbf{z}]$ of degree d. $V(f)$ coincides with the zero-loci of $(\{F_\beta \mid \; |\beta| = d - e\}) + \ker \varphi$.

$V(f)$ is called a **hypersurface of bidegree** (d, e) in $\mathbb{P}^n \times \mathbb{P}^m$.

Algebraic subsets of $\mathbb{P}^n \times \mathbb{P}^m$

Definition. Let $A \subset \mathbb{P}^n \times \mathbb{P}^m$ be a subset. The **bihomogeneous vanishing ideal** of A is

$$\mathrm{I}(A) = (\{f \in K[\mathbf{x}, \mathbf{y}] \text{ bihomogeneous} \mid f(a,b) = 0 \ \forall (a,b) \in A\})$$

and $V(\mathrm{I}(A)) = \overline{A}$ is its Zariski closure. For an algebraic subset $A \subset \mathbb{P}^n \times \mathbb{P}^m$ the bigraded ring $K[\mathbf{x}, \mathbf{y}]/\,\mathrm{I}(A)$ is called the **bihomogeneous coordinate ring** of A.

Remark. For $J \subset K[\mathbf{x}, \mathbf{y}]$ a bihomogenous ideal we have

$$\mathrm{I}(V(J)) = ((\mathrm{rad}(J) : (x_0, \dots, x_n)) : (y_0, \dots, y_m). \quad \square$$

We now are ready to define the product of two arbitrary projective algebraic sets $A \subset \mathbb{P}^n$ and $B \subset \mathbb{P}^m$:

$$A \times B \subset \mathbb{P}^n \times \mathbb{P}^m \subset \mathbb{P}^N$$

is the algebraic set defined by the bihomgeneous polynomials $f_i \in \mathrm{I}(A) \subset K[\mathbf{x}]$ of bidegree $(d_i, 0)$ and $g_j \in \mathrm{I}(B) \subset K[\mathbf{y}]$ of bidegree $(0, e_j)$.

Quasi-projective algebraic sets and regular functions

Definition. A **quasi-affine algebraic set** is an open subset of an affine algebraic set. Similarly we have the notion of a **quasi-projective algebraic set**. Every quasi-affine algebraic set is also quasi-projective because $\mathbb{A}^n = \mathbb{P}^n \setminus V(x_0)$.

The product of two quasi-affine (quasi-projective) algebraic sets $A = A_1 \setminus A_2$ and $B = B_1 \setminus B_2$ is again quasi-affine (quasi-projective).

$$A \times B = A_1 \times B_1 \setminus (A_2 \times B_1 \cup A_1 \times B_2).$$

For $A \subset \mathbb{P}^n$ a quasi-projective algebraic set we define the **ring of regular functions** $\mathcal{O}(A)$ as the ring of functions

$$f : A \to K$$

such that for every point $p \in A$ there exist an open neighbourhood $U \subset A$ and homogeneous polynomials $g, h \in K[x_0, \ldots, x_n]$ of the same degree with $h(p) \neq 0$ for all $p \in U$ such that

$$f(p) = \frac{g(p)}{h(p)}.$$

Morphism

Definition. Let A be a quasi-projective algebraic set.

1. Let $B \subset \mathbb{A}^m$ be a quasi-affine algebraic set. A morphism $\varphi : A \to B$ is a map which is given by an m-tupel of regular functions $f_j \in \mathcal{O}(A)$:
$$\varphi(p) = (f_1(p), \dots, f_m(p)) \, \forall p \in A.$$
2. Let $B \subset \mathbb{P}^m$ be a quasi-projective algebraic set. A map $\varphi : A \to B$ is a morphism if φ is locally given by regular functions, i.e., for each point $p \in A$ there exist an open neighbarhood $U \subset A$ and regular functions $f_0, \dots, f_m \in \mathcal{O}(U)$ such that
$$\varphi(p) = [f_0(p) : \dots : f_m(p)] \, \forall p \in U$$

Clearly, morphisms can be composed.

Definition. A morphism $\varphi : A \to B$ is an isomorphism if there exists a morphism $\psi : B \to A$ such that $\psi \circ \varphi = id_A$ and $\varphi \circ \psi = id_B$.

Examples

1. Let $A \subset \mathbb{P}^n$ be a quasi-projective algebraic set, and let $f_0, \dots, f_m \in K[x_0, \dots, x_n]$ be homogeneous polynomials of the same degree d such that $V(f_0, \dots, f_m) \cap A = \emptyset$. Then

$$\varphi : A \to \mathbb{P}^m, p \mapsto [f_0(p) : \dots : f_m(p)]$$

is a well-defined morphism. Indeed on the open set $U = A \cap (\mathbb{P}^n \setminus V(f_i))$ the map φ is given by the regular functions

$$[\frac{f_0}{f_i} : \dots : \frac{f_m}{f_i}],$$

and these open sets cover A since $V(f_0, \dots, f_m) \cap A = \emptyset$.

In particular we see that the regular functions in $\mathcal{O}(U)$ which define φ on U might not exist globally.

2. More specifically, consider the morphism $\rho_d : \mathbb{P}^1 \to \mathbb{P}^d$ defined by

$$[t_0 : t_1] \mapsto [t_0^d : t_0^{d-1}t_1 : \dots : t_1^d]$$

Examples

The image of ρ_d is the so-called **rational normal curve of degree** d. It has the homogeneous ideal generated by the 2×2-minors of the $2 \times d$-matrix

$$\begin{pmatrix} x_0 & x_1 & \dots & x_{d-1} \\ x_1 & x_2 & \dots & x_d \end{pmatrix}$$

Remark. Morphisms $\varphi : A \to B$ between affine algebraic sets are easier to describe because they simply correspond to K-algebra homomorphisms $\varphi^* : K[B] \to K[A]$.
Morphisms $\varphi : A \to B$ between projective algebraic sets have a more complicated description. However they are better behaved:

We will see in one of the next lectures that the image of a projective algebraic set under a morphism is always an algebraic subset of the target.
This was not the case for morphisms between affine algebraic sets.

Example

Consider $A = V(xy - z^2) \subset \mathbb{P}^2$. On the affine chart $U_{z=1}$ we saw that the projection

$$\mathbb{A}^2 \supset V(xy - 1) \to \mathbb{A}^1, (a, b) \mapsto a$$

is not surjective, because the origin o is not in the image.
The map

$$A \setminus \{[0 : 1 : 0]\} \to \mathbb{P}^1, [x : y : z] \mapsto [x : z]$$

extends to a surjective morphism $\pi : A \to \mathbb{P}^1$ because

$$[x : z] = [xy : yz] = [z^2 : yz] = [z : y]$$

holds on $A \setminus V(yz)$. Thus the missing preimage point of $o = [0 : 1] \in \mathbb{A}^1 \subset \mathbb{P}^1$ is the point $p = [0 : 1 : 0]$ on the line $V(z)$ at infinity.

Linear projections

Let $A \subset \mathbb{P}^n$ be a projective variety. Let $\ell_0, \ldots, \ell_r \in K[x_0, \ldots, x_n]$ be $r+1$ linearly independent linear forms such that $L = V(\ell_0, \ldots, \ell_r) \cong \mathbb{P}^{n-r-1}$ does not intersect A. Then

$$\pi_L : A \to \mathbb{P}^r, a \mapsto [\ell_0(a) : \ldots : \ell_r(a)]$$

is called the **linear projection from** L. The condition $A \cap L = \emptyset$ is equivalent to $\operatorname{rad}(\mathrm{I}(A) + (\ell_0, \ldots, \ell_r)) = (x_0, \ldots, x_n)$. If we choose coordinates on $\mathbb{P}^n$ such that $\ell_0 = x_{n-r}, \ldots, \ell_r = x_n$, then $A \cap L = \emptyset$ is equivalent to the condition that there are homogeneous equations $f_i \in \mathrm{I}(A)$ with

$$f_i \equiv x_i^{d_i} \mod (x_{n-r}, \ldots, x_n) \text{ for } i = 0, \ldots, n-r-1.$$

Thus in this case the map

$$\phi : K[x_{n-r}, \ldots, x_n] \to K[A] = K[x_0, \ldots, x_n] / \mathrm{I}(A)$$

induces an integral ring extension $K[A'] \hookrightarrow K[A]$ where $A' = V(\ker(\phi))$.

A dimension bound

Thus in this situation π_L induces a finite and surjective map $A \to A' \subset \mathbb{P}^r$. In particular, $\dim A' = \dim A \leq r$.

Corollary. *Let $A \subset \mathbb{P}^n$ be a projective algebraic set. If there exists a linear subspace $L \subset \mathbb{P}^n$ of dimension $n - r - 1$ with $A \cap L = \emptyset$, then $\dim A \leq r$.* □

Definition. Let $A \subset \mathbb{P}^n$ be a projective algebraic set. We call a linear projection $\pi_L : A \to \mathbb{P}^r$ with $L \cap A = \emptyset$ with $r = \dim A$ a **linear Noether normalization**.

Corollary. *Let $A \subset \mathbb{P}^n$ be a projective algebraic set of dimension $\dim A = r$. Then every linear subspace L of dimension $\dim L \geq n - r$ intersects A.*

Proof. If $L \cap A = \emptyset$, then $\dim A < r$. □

Theorem. *Let $X, Y \subset \mathbb{P}^n$ be projective algebraic sets. Then*

$$\dim X \cap Y \geq \dim X + \dim Y - n.$$

In particular the intersection of algebraic sets of complementary dimensions is always non-empty.

Proof of the dimension bound

Proof. Consider the projective space $\mathbb{P}^{2n+1}$ with coordinate ring $K[x_0, \dots, x_n, y_0, \dots, y_n]$ and the algebraic set $J(X, Y)$ defined by $\mathrm{I}(X) + \mathrm{I}(Y)$ where $\mathrm{I}(X) \subset K[x_0, \dots, x_n]$ and $\mathrm{I}(Y) \subset K[y_0, \dots, y_n]$ denote the homogeneous ideals in disjoint sets of variables. The algebraic set $J(X, Y)$ is called the **join** of X and Y because it is the union of all lines joining a point of X and with a point of Y,

$$X \subset \mathbb{P}^n \cong V(y_0, \dots, y_n) \subset \mathbb{P}^{2n+1} \supset V(x_0, \dots, x_n) \cong \mathbb{P}^n \supset Y.$$

Clearly,

$$\dim J(X, Y) = \dim X + \dim Y + 1$$

as one can see by combining linear Noether normalizations of X and Y. $X \cap Y = J(X, Y) \cap V(x_0 - y_0, \dots, x_n - y_n)$ is the intersection of $J(X, Y)$ with a linear subspace of dimension n. Thus the intersection $X \cap Y \neq \emptyset$ if

$$n \geq 2n + 1 - (\dim X + \dim Y + 1) \Leftrightarrow \dim X + \dim Y - n \geq 0$$

by the second corollary.

Proof of the dimension bound continued

Suppose $\dim X \cap Y = e > 0$. Let $\ell_0, \ldots, \ell_e \subset K[x_0, \ldots, x_n]$ define a linear Noether normalization of $X \cap Y$. Then

$$J(X, Y) \cap L = \emptyset$$

where $L = V(x_0 - y_0, \ldots, x_n - y_n, \ell_0, \ldots, \ell_e)$ is a linear space of dimension $2n + 1 - (n + 1 + e + 1) = 2n + 1 - (n + 1 + e) - 1$, and

$$\dim J(X, Y) \leq n + 1 + e$$

holds by the first corollary. Thus

$$\dim X \cap Y = e \geq \dim J(X, Y) - n - 1 = \dim X + \dim Y - n.$$

□

Remark. Using Krull's principal ideal theorem one can show for projective **varieties** $X, Y \subset \mathbb{P}^n$ that every component C of $X \cap Y$ has dimension $\dim C \geq \dim X + \dim Y - n$.

Rational maps from smooth curves extend to morphisms

Proposition. *Let C be a smooth irreducible quasi projective curve and $\varphi' : C \dashrightarrow \mathbb{P}^n$ a rational map. Then φ' extends to a morphism*

$$\varphi : C \to \mathbb{P}^n.$$

Proof. Suppose that φ' is given by a tuple $f_0, \dots, f_n$ of rational functions. There are two reasons why $[f_0(p) : \dots : f_n(p)]$ might be not defined in $p \in C$. One of the rational functions might have a pole at p or all rational functions might vanish at p.
Taking $k = \min\{v_p(f_j) \mid j = 0, \dots, n\}$ and $t \in \mathfrak{m}_{C,p} \subset \mathcal{O}_{C,p}$ a generator then we see that $[t^{-k} f_0 : \dots : t^{-k} f_n]$ is defined at $p \in C$ and coincides φ' where t has no zeroes or poles. □

Remark. The proposition is not true for a higher dimension source: The morphism

$$\mathbb{A}^2 \setminus \{o\} \to \mathbb{P}^1, p \mapsto [x(p) : y(p)]$$

has no extension to $\mathbb{A}^2$. Instead the closure of the graph is the blow-up of $o \in \mathbb{A}^2$.