Algebraic Geometry, Lecture 19

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Overview

Today's topics are

1. The Veronese embeddings
2. The fundamental theorem of elimination
3. Projective morphisms
4. Semi-continuity of the fiber dimensions

The Veronese embeddings

Definition. Let $n, d \geq 1$, $N = \binom{n+d}{n} - 1$ and $m_0 = x_0^d, \ldots, m_N = x_n^d$ be all degree d monomials in $K[x_0, \ldots, x_n]$ in some order. The monomials define a morphism

$$\rho_{n,d} : \mathbb{P}^n \to \mathbb{P}^N$$

which turns out to be an embedding, i.e., an isomorphism to it's image $V_{n,d} \subset \mathbb{P}^N$. $\rho_{n,d}$ is called the **Veronese or d-uple embedding** of $\mathbb{P}^n$.

Example. In case of $n = 1$ the morphism $\rho_{1,d}$ embeds $\mathbb{P}^1 \hookrightarrow \mathbb{P}^d$ as the rational normal curve of degree d in $\mathbb{P}^d$ defined by the 2×2-minors of

$$\begin{pmatrix} x_0 & x_1 & \ldots & x_{d-1} \\ x_1 & x_2 & \ldots & x_d \end{pmatrix}.$$

Example. We discuss

$$\rho_{2,2} : \mathbb{P}^2 \to \mathbb{P}^5, [x : y : z] \mapsto [x^2 : xy : y^2 : xz : yz : z^2]$$

in some details. We use homogeneous coordinates $w_0, \ldots, w_5$ on $\mathbb{P}^5$. Consider the symmetric matrix

$$\Delta = \begin{pmatrix} w_0 & w_1 & w_3 \\ w_1 & w_2 & w_4 \\ w_3 & w_4 & w_5 \end{pmatrix}$$

and let $V \subset \mathbb{P}^5$ be the algebraic set defined by the ideal I of 2×2-minors of Δ. Clearly, these minors vanish on all points of $\rho_{2,2}(\mathbb{P}^2)$. We show that $\rho_{2,2}$ induces an isomorphism of $\mathbb{P}^2$ with V by describing the inverse morphism ψ. $V \subset U_0 \cup U_2 \cup U_5$ because

$$w_1^2, w_3^2, w_4^2 \in I + (w_0, w_2, w_5).$$

On $V \cap U_0$ the inverse map is given by

$$p \mapsto [w_0(p) : w_1(p) : w_3(p)]$$

because $[x^2 : xy : xz] = [x : y : z]$ on $U_x = \{x \neq 0\} \subset \mathbb{P}^2$.

$V = V_{2,2}$ continued

Similarly, the inverse map ψ is given on $V \cap U_2$ and $V \cap U_5$ by the second respectively third row of Δ. The maps coincide on $V \cap U_i \cap U_j$ for $i, j \in \{0, 2, 5\}$ since the 2×2-minors of Δ vanish on V. Thus these pieces glue to a well-defined morphism $\psi : V \to \mathbb{P}^2$ and $\mathbb{P}^2 \cong V$.

Remark. Notice that ψ is a morphism which cannot be defined globally by only one tuple of three homogeneous polynomials of the same degree.

We finish the treatment of this example by proving

Claim. *I is the homogeneous ideal of V.*

Proof. Consider the ring homomorphism

$$\varphi : K[w_0, \ldots, w_5] \to K[x, y, z], w_0 \mapsto x^2, w_1 \mapsto xy, \ldots, w_5 \mapsto z^2.$$

Then $I \subset \ker(\varphi)$. To prove equality we consider a reverse lexicographic order with $w_0, \ldots, w_5$ are ordered such that $w_1, w_3, w_4 > w_0, w_2, w_5$. Then the lead terms of the minors are

$$w_1^2, w_1 w_3, w_3^2, w_1 w_4, w_3 w_4, w_4^2.$$

$V = V_{2,2}$ continued

Thus a remainder of a the division by the minors is at most linear in w_1, w_3 and w_4, and there are precisely

$$\binom{d+2}{2} + 3\binom{d+1}{2} = 2d^2 + 3d + 1 = \binom{2d+2}{2}$$

different monomials of degree d occuring as remainders. Since φ is surjective the homogeneous coordinate ring S_V has precisely that many elements in degree d. Thus $I = \ker(\varphi)$ and the minors form a Gröbner basis.The ideal I is prime because

$$S_V = K[w_0, \dots, w_5]/I \cong K[x^2, xy, y^2, xz, yz, xz] \subset K[x, y, z]$$

is isomorphic to a subring of a domain.

Finally we note that the Hilbert polynomial of V is

$$p_V(t) = \binom{2t+2}{2} = 4\frac{t^2}{2!} + 3t + 1, \text{ hence } \deg V = 4.$$

□

$V_{n,d}$

Theorem. *The Veronese morphism*

$$\rho_{n,d} : \mathbb{P}^n \to \mathbb{P}^N, [x_0 : \ldots : x_n] \mapsto [x_0^d : x_0^{d-1}x_1 : \ldots : x_n^d]$$

where $N = \binom{n+d}{n} - 1$ *induces an isomorphism onto its image* $V_{n,d}$*, which is a subvariety of* $\mathbb{P}^N$ *of degree* $\deg V_{n,d} = d^n$*.*

Proof. The homogeneous coordinate ring of $\mathbb{P}^N$ has a variable y_α for each $\alpha \in \mathbb{N}^n$ with $|\alpha| = d$, and $\rho_{n,d}$ corresponds to the ring homomorphism

$$\varphi : K[y_\alpha' s] \to K[x_0, \ldots, x_n], y_\alpha \mapsto x^\alpha.$$

We will show that $V_{n,d}$ coincides with the projective variety $V(\ker(\varphi)) \subset \mathbb{P}^N$.

Some equations in $I = \ker(\varphi)$ can be obtained as follows: Consider the $\binom{n+d-1}{n} \times (n+1)$-matrix Δ with rows corresponding to monomials of x^β of degree $d-1$ and columns corresponding to the variables $x_0, \ldots, x_n$ whose entries are $\Delta_{x^\beta, x_j} = y_\alpha$ where $x^\alpha = x^\beta x_j$. The 2×2-minors of Δ are contained in I.

$V_{n,d}$ continued

$V_{n,d}$ is contained in the union of $n+1$ standard charts of $\mathbb{P}^N$:
$V_{n,d} \cap V(y_{(d,0,\ldots,0)}, \ldots, y_{(0,\ldots,0,d)}) = \emptyset$ because

$$y_\alpha^d - y_{(d,0,\ldots,0)}^{\alpha_0} \cdot \ldots \cdot y_{(0,\ldots,0,d)}^{\alpha_n} \in I.$$

Thus

$$V_{n,d} \subset \widetilde{U}_0 \cup \ldots \cup \widetilde{U}_n$$

for $\widetilde{U}_j = \{y_{(0,\ldots,d,\ldots 0)} \neq 0\}$ corresponding to the monomial x_j^d.
$\rho_{n,d}$ induces an isomorphism of U_0 with $V_{n,d} \cap \widetilde{U}_0$: The map

$$p \mapsto [y_{(d,0,\ldots,0)}(p) : y_{(d-1,1,\ldots,0)}(p) : \ldots : y_{(d-1,0,\ldots,1)}(p)]$$

corresponding to the row of $\Delta_{x_0^{d-1}}$ defines the inverse. Similarly

$$V_{n,d} \cap \widetilde{U}_j \cong U_j$$

by the map defined by the row $\Delta_{x_j^{d-1}}$.

$V_{n,d}$ continued

These maps glue to a well-defined inverse morphism

$$\psi : V_{n,d} \to \mathbb{P}^n$$

since the 2×2-minors of Δ vanish on $V_{n,d}$. To compute the degree we compute the Hilbert polynomial. Since $K[y_\alpha\,'s]/I \cong K[x^\alpha\,'s] \subset K[x_0, \ldots, x_n]$ we obtain

$$p_{V_{n,d}}(t) = \binom{dt+n}{n} = d^n \frac{t^n}{n!} + \text{ lower terms }.$$

□

Corollary. *Every quasi-projective algebraic set has a finite open covering by affine algebraic sets.*

Proof of the corollary

Let

$$f = \sum_{\alpha} f_\alpha x^\alpha \in K[x_0, \ldots, x_n]$$

be a homogeneous polynomial of degree d. Consider the open set $U_f = \mathbb{P}^n \setminus V(f)$ and the corresponding hyperplane $H_f = V(\sum_\alpha f_\alpha y_\alpha) \subset \mathbb{P}^N$. Under the Veronese embedding U_f is isomorphic to the Zariski-closed subset of $\mathbb{A}^N$, since

$$U_f \cong V_{n,d} \cap (\mathbb{P}^N \setminus H_f) \subset \mathbb{P}^N \setminus H_f \cong \mathbb{A}^N.$$

Hence U_f is isomorphic to an affine variety.
Let $A = A_1 \setminus A_2$ be a quasi-projective set where $A_2 \subset A_1 \subset \mathbb{P}^n$ are projective algebraic subsets. If $A_2 = V(f_1, \ldots, f_r)$, then

$$A = \bigcup_{j=1}^{r} (A_1 \cap U_{f_j})$$

is an open covering. Since $A_1 \cap U_{f_j}$ is a closed subset of the affine variety U_f it is isomorphic to an affine algebraic set. □

Morphism from projective algebraic sets

Theorem. *Let A be a projective algebraic set and $\varphi : A \to B$ a morphism to a quasi-projective algebraic set. Then $\varphi(A) \subset B$ is a Zariski-closed subset.*

Corollary. *Let A be a projective variety. Every regular function $f : A \to K$ is constant.*

Proof of the Corollary. f defines a morphism $f : A \to \mathbb{A}^1 \subset \mathbb{P}^1$. The image is closed in $\mathbb{P}^1$ hence different from $\mathbb{A}^1$. Since it is also closed in $\mathbb{A}^1$, it is a finite union of points. Since A is irreducible, it is a single point. □

Remark. In case $K = \mathbb{C}$, this corollary is similar to the maximum principle: If f is a holomorphic function on a compact complex connected manifold A, then $|f|$ attains its maximum, and hence f is constant.

Graph of a morphism

Lemma. *Let $\varphi : A \to B$ a morphism between quasi-projective algebraic sets. Then the graph of φ is a closed subset of $A \times B$.*

Proof. To be a closed subset is a local property. Hence we may replace B by an open affine subset U and A by an open affine subset of $\varphi^{-1}(U)$ since every quasi projective algebraic set has an open affine covering by the corollary to the Theorem on the Veronese embeddings. Thus we may assume the A and B are subsets of $\mathbb{A}^n$ and $\mathbb{A}^m$ respectively and that φ is given by a tuple of polynomial functions $(f_1, \dots, f_m)$. Then the graph of φ is defined by the ideal

$$(y_1 - f_1(x_1, \dots, x_n), \dots, y_m - f_m(x_1, \dots, x_n))$$

on $A \times B$. □

The fundamental theorem of elimination

Passing to the graph reduces the proof of the theorem above to the following:

Theorem. *Let A be a projective algebraic set and B a quasi-projective algebraic set. Then the projection onto the second factor $A \times B \to B$ is a closed map, i.e., maps closed subsets of $A \times B$ to closed subsets of B.*

Proof. We may replace B by an open affine subset. Hence we may assume that $A \subset \mathbb{P}^n$ and $B \subset \mathbb{A}^m$ are closed subsets, and it suffices to prove that the projection $\mathbb{P}^n \times \mathbb{A}^m \to \mathbb{A}^m$ is closed. Any algebraic subset $X \subset \mathbb{P}^n \times \mathbb{A}^m$ is defined by finitely many polynomials $f_1, \ldots, f_r \in K[x_0, \ldots, x_n, y_1, \ldots, y_m]$ where each f_i is homogeneous of some degree d_i in $x_0, \ldots, x_n$. By the projective Nullstellenatz a point $q \in \mathbb{A}^m$ lies in the image of X iff the ideal

$$I(q) := (f_1(\mathbf{x}, q), \ldots, f_r(\mathbf{x}, q)) \subset K[\mathbf{x}]$$

does not contain any of the ideals $(x_0, \ldots, x_n)^d$ for $d \geq 1$.

Proof of the fundamental theorem of elimination

Define

$$Y_d = \{q \in \mathbb{A}^m \mid I(q) \not\supset (x_0, \ldots, x_n)^d\}.$$

Then the image of X is

$$Y = \bigcap_{d=1}^{\infty} Y_d$$

and it suffices to prove that each Y_d is an algebraic subset of $\mathbb{A}^m$. To obtain equations for Y_d we multiply each f_i with all monomials of degree $d - d_i$ in $\mathbf{x}$. Let T_d denote the resulting set of polynomials. Then $q \notin Y_d$ iff each monomial in $K[x_0, \ldots, x_n]_d$ of degree is a linear combination of the polynomials $f(\mathbf{x}, q)$ with $f \in T_d$. Comparing coefficients we obtain a $\binom{d+n}{n} \times \sum_{i=1}^{r} \binom{d-d_i+n}{n}$-matrix M_d with entries in $K[y_1, \ldots, y_m]$ such that $q \in Y_d$ iff rank $M_d(q) < \binom{d+n}{n}$. Thus the $\binom{d+n}{n} \times \binom{d+n}{n}$-minors of M_d define Y_d. □

Computing the elimination ideal in concrete examples

The proof of the theorem does not yield a practical algorithm to compute the image. Here is an approach which works frequently in practise.

Definition. Let I, J be ideals in a ring. Then the **saturation of I with respect to J** is

$$I : J^\infty = \bigcup_{N=1}^{\infty} (I : J^N).$$

To compute the saturation in noetherian rings one can iterate

$$I_{k+1} = I_k : J$$

starting with $I_0 = I$ until $I_{N+1} = I_N$. Then $I_N = I : J^\infty$.

In the situation of $X \subset \mathbb{P}^n \times \mathbb{A}^m$ defined by $f_1, \ldots, f_r$ as above, we obtain equations of the image $Y \subset \mathbb{A}^m$ by taking the elements of degree 0 in $\mathbf{x}$ of

$$(f_1, \ldots, f_r) : (x_0, \ldots, x_n)^\infty.$$

Projective morphism

The proof above actually gives a stronger result.

Definition. A morphism $\varphi : A \to B$ is called a **projective morphism** if φ is the composition of a closed embedding $\iota : A \to \mathbb{P}^n \times B$ with the projection onto B. Here a morphism $\iota : A \to C$ is called a **closed embedding** if ι induces an isomorphism $A \to \iota(A)$ and $\iota(A)$ is a Zariski-closed subset of C.

Theorem. *A projective morphism is a closed map.*

Proof. Indeed in the proof of the fundamental theorem of elimination we replaced $A \subset \mathbb{P}^n$ with the graph of φ in $A \times B \subset \mathbb{P}^n \times B$. The map $\iota : A \to \mathbb{P}^n \times B$ to the graph of φ in $\mathbb{P}^n \times B$ is a closed embedding. In the remaining part of the proof all we used was that $A \cong \iota(A) \subset \mathbb{P}^n \times B$ is closed. Thus closed subsets of $X \subset A$ are also closed subsets of $\mathbb{P}^n \times B$, and our argument showed that the image Y of X under the projection to B is closed in B. □

Semi-continuity of the fiber dimension

Definition. Let $\varphi : X \to Y$ be a morphism. The **fiber** of φ over a point $q \in Y$ is $X_q := \varphi^{-1}(q) \subset X$. Since morphisms are continuous in the Zariski topology, the preimage of the point p is a Zariski closed subset of X.

If φ is a projective morphism, say φ factors over a closed embedding $\iota : X \to \mathbb{P}^n \times Y$, then the situation is better: The fiber

$$X_p \subset \mathbb{P}^n \times \{p\} \cong \mathbb{P}^n$$

is a projective algebraic set.

Theorem. *Let $\varphi : X \to Y$ be a projective morphism and $r \geq -1$ an integer. Then the set*

$$U_r = \{q \in Y \mid \dim X_q \leq r\}$$

is Zarsiki-open in Y.

In other words, the fiber dimension of special points can be larger than the fiber dimension for more general points.

Proof of the semi-continuity

Proof. The set

$$U_{-1} = \{q \in Y \mid \dim X_q \leq -1\} = \{q \in Y \mid X_q = \emptyset\}$$

is open in Y because it is the complement of the closed subset $\varphi(X) \subset Y$.
Suppose $\dim X_q = r \geq 0$. We assume that φ factors over $\mathbb{P}^n \times Y$.
Consider a linear space $L \subset \mathbb{P}^n$ of dimension $n - r - 1$ with $X_q \cap L = \emptyset$ and

$$Z = X \cap (L \times Y) \subset \mathbb{P}^n \times Y.$$

Fibers X_q with $\dim X_q > r$ intersect Z. Thus $U = Y \setminus \varphi(Z)$ is an open neighbourhood of $p \in U_r$. □

Dimension of general fibers

In case of a surjective projective morphism between varieties the result can be strengthened.

Theorem. *Let $\varphi : X \to Y$ be a surjective projective morphism between varieties. Then*

$$\dim X_q \geq \dim X - \dim Y,$$

and equality holds for $q \in U$ of a non-empty open subset U of Y.
Proof. We may assume that Y is affine and that $X \subset \mathbb{P}^n \times Y$ is a closed subset. Consider the function fields

$$K(Y) \subset K(X).$$

We have

$$\operatorname{trdeg}_K K(X) = \operatorname{trdeg}_{K(Y)} K(X) + \operatorname{trdeg}_K K(Y).$$

Let $I \subset K[Y][x_0, \ldots, x_n]$ be the ideal of $X \subset \mathbb{P}^n \times Y$. Consider the ideal $J \subset K(Y)[x_0, \ldots, x_n]$ generated by I. J corresponds to a variety $V(J)$ defined over the function $K(Y)$ of dimension

$$\dim V(J) = \operatorname{trdeg}_{K(Y)} K(X) = \dim X - \dim Y.$$

The proof continued

We compute a normalized Gröbner basis of J, i.e., one where the leading coefficients of all Gröbner basis elements are 1. In doing so we have to divide by finitely many polynomial functions of $K[Y]$. Let $f \in K[Y]$ be the product of these polynomials and $U_f = Y \setminus V(f)$ the corresponding non-empty open subset. We claim that for a point $q \in U_f$ the ideal

$$I_q = (\{f(x,q) | f \in I\}) \subset K[x_0, \ldots, x_n]$$

defines an algebraic set of dimension $\dim X - \dim Y$.
Indeed the computation of the Gröbner basis of I_q follows the same steps as the computation for $J = (I)$. We simply have to substitute q into the rational functions in $K(Y)$ which are the coefficients. Since each coefficient has a representation as a fraction with power of f in the denominator, the coefficients can be evaluated in q. Thus J and I_q have the same lead ideal.

The proof continued

Hence $K(Y)[x_0, \ldots, x_n]/J$ and $K[x_0, \ldots, x_n]/I_q$ have the the same Hilbert polynomial. In particular

$$\dim X_q = \dim V(J) = \operatorname{trdeg}_{K(Y)} K(X) = \dim X - \dim Y$$

holds for all $q \in Y$. Since Y is irreducible hence U_f is dense Y, we obtain

$$\dim X_q \geq \dim X - \dim Y$$

from the semi-continuity of the fiber dimension. □

As a corollary of the proof we note

Corollary. *Let Y be an affine variety and $I \subset K[Y][x_0, \ldots, x_n]$ be an ideal which is homogeneous in $x_0, \ldots, x_n$. Then there exist a non-empty open subset $U \subset Y$ such that the ideals*

$$I_q = (\{f(x, q) \mid f \in I\}) \subset K[x_0, \ldots, x_n]$$

have the same Hilbert function for all $q \in U$. □

Gröbner basis over prime fields

Theorem. *Let* $f_1, \ldots, f_r \in \mathbb{Z}[x_0, \ldots, x_n]$ *be homogeneous polynomials. Let* $I_{\mathbb{Q}} \subset \mathbb{Q}[x_0, \ldots, x_n]$ *and* $I_p \subset \mathbb{F}_p[x_0, \ldots, x_n]$ *for* p *a prime number denote the ideals generated by* $f_1, \ldots, f_r$ *in these rings. Then for all but finitely many primes the lead ideals*

$$\mathrm{Lt}(I_p) \text{ and } \mathrm{Lt}(I_{\mathbb{Q}})$$

are generated by the same monomials. In particular their Hilbert polynomials coincide.

Proof. We compute a normalized Gröbner basis of $I_{\mathbb{Q}}$. In this process we have to divide by finitely many leading coefficients, and the Gröbener basis of the ideal I_p where p does not divide any of the leading coefficients, is obtained by mapping the coefficients $\frac{a}{b} \in \mathbb{Q}$ to $ab^{-1} \in \mathbb{F}_p$. □

Gröbner basis over prime fields

Remark. Notice that a Gröbner basis over $\mathbb{Q}$ can have very large coefficients: In adding or multiplying two rational numbers

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \text{ or } \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

one often obtains numbers with twice the number of digits in the numerator and denominator.
By passing to a finite prime field this effect is avoided. If we are only interested say in the degree and the dimension of the $V(I_{\mathbb{Q}})$, then the result does not change for almost all primes. This is frequently used in experiments in algebraic geometry.