Algebraic Geometry, Lecture 20

Frank-Olaf Schreyer

Saarland University, Perugia 2021

Overview

1. The blow-up
2. Resolution of singularities
3. A birational map between $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$
4. Cremona transformations

The blow-up

Definition. Let $X \subset \mathbb{P}^1 \times \mathbb{A}^2$ be defined by

$$\det \begin{pmatrix} z_0 & z_1 \\ x & y \end{pmatrix} \in K[z_0, z_1, x, y]$$

and let $\sigma : X \to \mathbb{A}^2$ denote the projection onto the second component. σ is called the **blow-up** of $\mathbb{A}^2$ at the origin o.
X is covered by two affine charts $U_j = X \cap (U_{z_j} \times \mathbb{A}^2)$ which are both isomorphic to $\mathbb{A}^2$.

$$K[U_0] \cong K[z, x, y]/(y - xz) \cong K[x, z]$$

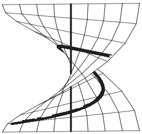
and the map $\sigma|_{U_0} : U_0 \to \mathbb{A}^2$ is given by $(x, z) \mapsto (x, xz)$. Similary

$$K[U_1] \cong K[w, y] \text{ and } \sigma|_{U_1} : U_1 \to \mathbb{A}^2, (w, y) \mapsto (wy, y).$$

The fiber of σ over $o = (0, 0) \in \mathbb{A}^2$ is $E = \mathbb{P}^1 \times \{o\} \cong \mathbb{P}^1$. E is called the **exceptional curve of** σ. Outside E the map σ restricts to an isomorphism $X \setminus E \cong \mathbb{A}^2 \setminus \{o\}$.

Strict and total transform

$X \setminus E \subset \mathbb{P}^1 \times (\mathbb{A}^2 \setminus \{o\})$ is isomorphic to the graph of the morphism



$$\mathbb{A}^2 \setminus \{o\}, (x, y) \mapsto [x : y].$$

In other words we may think of

$$X = V(\det \begin{pmatrix} z_0 & z_1 \\ x & y \end{pmatrix}) \subset \mathbb{P}^1 \times \mathbb{A}^2$$

as obtained from $\mathbb{A}^2$ by replacing the origin o by the projective space $E \cong \mathbb{P}^1$ of lines through o.

Definition. Let $C \subset \mathbb{A}^2$ be a plane curve. Then the closure $C' = \overline{\sigma^{-1}(C \setminus \{o\})} \subset X$ is called the **strict transform** of C. The **total transform** is $\sigma^{-1}(C)$.

Proposition. *Let $C = V(f)$ be a curve of multiplicity m at the origin. Then the strict transform $C' \subset X$ intersects E in precisely m points counted with multiplicities.*

Proof of the proposition

Suppose $f = f_m + \ldots + f_d \in K[x, y]$ with f_j homogeneous of degree j. The total transform of C in the chart U_0 is defined

$$f(x, xz) = x^m(f_m(1, z) + xf_{m+1}(1, z) + \ldots + x^{d-m} f_d(1, z)) = 0.$$

The exceptional curve E is defined by $x = 0$ on U_0. So the strict transform C' is defined by

$$f_m(1, z) + xf_{m+1}(1, z) + \ldots + x^{d-m} f_d(1, z) = 0.$$

Thus the intersection point of $E \cap C'$ contained in U_0 are defined by $V(f_m(1, z), x)$. Let $f_m = \prod_{i=1}^r \ell_i^{e_i}$ be the factorization of f_m into distinct linear factors. The intersection multiplicity

$$i(C', E; p_i) = e_i$$

at the point $p_i = [a_i : b_i] \in \mathbb{P}^1 = E$ corresponding to the tangent line $V(\ell_i)$ with $\ell_i = b_i x - a_i y$ since the factors ℓ_j for $j \neq i$ are units in $\mathcal{O}_{X,p_i}$. The result follows because $\sum_{i=1}^r e_i = m$. $\square$

The effect of the blow-up on curves

Corollary. *If o is an ordinary m-fold point, then E and C' have transversal intersections and C' is non-singular at the intersection points.* $\square$

Since X is covered by charts isomorphic to $\mathbb{A}^2$ we can iterate this process.

Example. Consider $C = V(y^3 - x^5)$ the strict transform of C is contained in the chart U_0 where the total transform is defined by $x^3(z^3 - x^2)$.

The further blow-up $(x, z) = (uz, u)$ yields $u^5 z^3(u - z^2)$. Blowing-up the intersection point of the second exceptional curve $E_2 = \{u_2 = 0\}$ with C'' via $(u, z) = (wz, z)$ yields the local equation $w^5 z^9(w - z)$, and all curves intersect transversal.

Resolution of singularities

Theorem. *Let $C \subset \mathbb{P}^2$ be a plane algebraic curve. Then there exists a sequence*

$$X_r \xrightarrow{\sigma_r} X_{r-1} \longrightarrow \dots \longrightarrow X_1 \xrightarrow{\sigma_1} \mathbb{P}^2$$

of blow-ups, such that the strict transform $C^{(r)}$ of C in X_r is a non-singular curve.

The main difficulty in proving this theorem is to prove that some numerical invariant improves along the process of blow-ups. In the example above such invariant was the multiplicity of the singular points. However in general a more subtle invariant is needed.

Example

Example. Consider $y^2 - x^4 + x^6 = 0$. Substituting $(x, y) = (x, xz)$ leads to the strict transform $z^2 - x^2 + x^4 = 0$, which still has a double point at the origin. A second blow-up $(x, z) = (uz, z)$ gives the strict transform $u^2 - 1 + u^4 z^2 = 0$ which actually is now a smooth curve.

Blow-up of $\mathbb{P}^2$ and the cubic scroll

The blow-up of a point $p \in \mathbb{A}^2 \cong U_2 \subset \mathbb{P}^2$ glues with the U_0, U_1 to the blow-up $\mathbb{P}^2(p)$ of $\mathbb{P}^2$ at p. This is a projective surface which one can describe explicitely as follows: Consider the rational map

$$\mathbb{P}^2 \dashrightarrow \mathbb{P}^4, [x : y : z] \mapsto [x^2 : xy : y^2 : xz : yz].$$

The image is the cubic scroll defined by

$$\operatorname{rank} \begin{pmatrix} w_0 & w_1 & w_3 \\ w_1 & w_2 & w_4 \end{pmatrix} < 2.$$

It is the projection of the Veronese surface $V_{2,2} \subset \mathbb{P}^5$ from the point $\rho_{2,2}([0 : 0 : 1]) = [0 : \ldots : 0 : 1]$.
Note that we can identify the exceptional line

$$E = \mathbb{P}^1 = V(w_0, w_1, w_2)$$

with the projective tangent space $E \cong \mathbb{P}(T_p\mathbb{P}^2)$.

Blow-up of smooth projective surfaces

More generally one can define the blow-up $X(p)$ of a smooth projective surface $X \subset \mathbb{P}^n$ at a point p as the image of the composition of

$$X \hookrightarrow \mathbb{P}^n \hookrightarrow \mathbb{P}^{\binom{n+2}{2}-1} \dashrightarrow \mathbb{P}^{\binom{n}{2}-2}$$

of the 2-uple embedding $\rho_{n,2}$ with the projection π from the image point. The image is again a smooth projective surface $X(p)$ with an exceptional curve $E \cong \mathbb{P}(T_pX) \cong \mathbb{P}^1$ and

$$X(p) \setminus E \cong X \setminus p.$$

Remark. Frequently on can project $X \subset \mathbb{P}^n$ directly from p without first the Veronese re-embedding. This however does not work if X has a 3-secant line which passing through p, e.g,. if X contains a line which passes through p.

Birational maps between smooth surfaces

A second place where the blow-up plays a crucial role is in the description of birational maps between smooth surfaces.

Theorem(Castelnuovo)

1. *Let $\varphi : Z \to X$ be a birational morphism between smooth projective surfaces. Then there exists a sequence of blow-ups*

$$X_r \xrightarrow{\sigma_r} X_{r-1} \longrightarrow \dots \longrightarrow X_1 \xrightarrow{\sigma_1} X$$

such that $Z \cong X^{(r)}$

2. *Every birational map $Y \dashrightarrow X$ between smooth projective surfaces can be factored into birational morphisms from a smooth projective surface Z as follows:*

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ Y & & X \end{tikzcd} .$$

where both morphisms are sequences of blow-ups.

The birational projection of $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$

Consider a point $p = (a, b) \in \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ and the rational map

$$\pi_p : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$$

which maps a point $q \in \mathbb{P}^1 \times \mathbb{P}^1$ to the line $\overline{pq} \in \mathbb{P}^2$ where we identify $\mathbb{P}^2$ with the set of lines in $\mathbb{P}^3$ through p. Its factorization is

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ \mathbb{P}^1 \times \mathbb{P}^1 & & \mathbb{P}^2. \end{tikzcd}$$

where $Z \to \mathbb{P}^1 \times \mathbb{P}^1$ is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ in p and $Z \to \mathbb{P}^2$ collapses the strict transforms of the lines $\mathbb{P}^1 \times \{b\}$ and $\{a\} \times \mathbb{P}^1$ to two points $p_1, p_2 \in \mathbb{P}^2$. The exceptional curve E over p is mapped to the line $\overline{p_1 p_2}$.

Projection of $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$

$$\begin{aligned}
\mathbb{P}^2 \setminus \overline{p_1 p_2} &= \mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1 \\
&\cong (\mathbb{P}^1 \setminus \{a\}) \times (\mathbb{P}^1 \setminus \{b\}) \\
&= \mathbb{P}^1 \times \mathbb{P}^1 \setminus (\{a\} \times \mathbb{P}^1 \cup \mathbb{P}^1 \times \{b\})
\end{aligned}$$

The quadratic transformation

Definition. The birational map

$$q : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad [x : y : z] \mapsto [\frac{1}{x} : \frac{1}{y} : \frac{1}{z}] = [yz : xz : yz]$$

is called the quadratic transformation. The map is not defined in the **fundamental points** $p_0 = [1 : 0 : 0], p_1 = [0 : 1 : 0]$ and $p_2 = [0 : 0 : 1]$. To visualize the map it is convenient to choose coordinates such that all three fundamental point lie in an affine chart:

The graph of the quadratic transformation

Proposition. *The graph of the quadratic transformation is isomorphic to the blow-up of $\mathbb{P}^2$ in the 3 fundamental points. The projection onto the second factor is again a blow-up of three points, the strict transforms of the 3 fundamental lines $L_{ij} = \overline{p_i p_j}$ are the exceptional curves of the second projection.*

Proof. The graph G is defined as the closure of the graph of the morphism $U \to \mathbb{P}^2$, which represents q. We use coordinates $[x : y : z]$ and $[u : v : w]$ on $\mathbb{P}^2 \times \mathbb{P}^2$. The graph is defined by

$$\operatorname{rank} \begin{pmatrix} yz & xz & xy \\ u & v & w \end{pmatrix} < 2$$

outside the fundamental points. However, over the fundamental points we need additional equations. If J denotes the ideal generated by the 2×2-minors of the matrix above. Then

$$I = J : (xy, xz, yz) = (vy - ux, ux - wz)$$

gives the defining ideal of G.

The graph of the quadratic transformation 2

Now restricted to the open set $G_{21} = G \cap \{z = 1, v = 1\}$, we obtain

$$K[G_{21}] \cong K[x, y, u, w]/(y - ux, w - ux) \cong K[x, u]$$

and the projection onto the first factor is

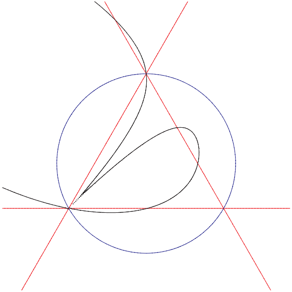
$$\mathbb{A}_2 \cong G_{21} \to U_0 \cong \mathbb{A}^2, (x, u) \mapsto (x, ux).$$

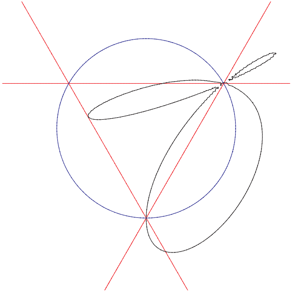
This is the chart of a blow-up. Since the sets G_{ij} for $i \neq j$ cover G the proposition follows. $\square$

Proposition. *Let C be a reduced plane curve of degree d which has multiplicity r_0, r_1, r_2 in the fundamental points p_0, p_1, p_2 of a quadratic transformation q. Then the strict transform $q(C)$ has degree $2d - r_0 - r_1 - r_2$ and three new singular points with multiplicity $d - r_1 - r_2, d - r_0 - r_2$ and $d - r_0 - r_1$.*

Example

Consider the curve $C = V()$. Then $q(C) = V()$.





Note that the circle on one side is the strict transform of the line at infinity of the visible chart on the other side. The original curve has a non-ordinary triple point, the strict transform has an ordinary triple point.

Cremona resolution

Theorem. $K = \overline{K}$. *Every irreducible plane curve can be transformed by a sequence of quadratic transformations into a plane curve with only ordinary singularities.*

The Cremona resolution allows to deduce the existence of a resolution of singularities.

Theorem. *Let* $C \subset \mathbb{P}^2$ *be a plane algebraic curve. Then there exists a sequence*

$$X_r \xrightarrow{\sigma_r} X_{r-1} \longrightarrow \dots \longrightarrow X_1 \xrightarrow{\sigma_1} \mathbb{P}^2$$

of blow-ups, such that the strict transform $C^{(r)}$ *of* C *in* X_r *is a non-singular curve.*

Proof. Let C be a plane curve. We first assume that C is irreducible and consider a Cremona resolution of C. Now instead of performing the quadratic transforms, we only blow-up the fundamental points.

Proof of resolution singularities

The fundamental points of the second step give points on the blown-up surface, and iterating this process we end up with a strict transform of C on an iterated blow-up of $\mathbb{P}^2$ which has only ordinary singular points. Blowing-up all those points we arrive at a smooth curve C'.

Now assume $C = C_1 \cup \ldots \cup C_s$ is reducible. We can perform this process in parallel for each component of C_i arriving at an iterated blow-up of $\mathbb{P}^2$ where all strict transforms C_i' are smooth. Thus the only singularities of $C' = C_1' \cup \ldots \cup C_s'$ are points where two components intersect.
We leave it as an exercise to prove that two smooth curves which intersect with multiplicity $i(C_1, C_2; p) = m$ at a point p get separated after precisely m blow-ups. $\square$

Ingredients of the proof for the Cremona resolution

The proof for the existence of a Cremona resolution needs two results.

1. We need Bertini's theorem, which will allow us to find good fundamental points for the quadratic transformations.
2. We have to introduce two invariants of plane curves which improve under a suitable chosen quadratic transformation. The first invariant is the number of non-ordinary singularities of C. The second invariant is the difference

$$\binom{d-1}{2} - \sum_{p \in C} \binom{r_p}{2}$$

   where r_p denotes the multiplicity of C at p.

We will return to this after the proof of Bertini's theorem.