

Algebraic Geometry, Lecture 22

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Overview

Today I will make a few remarks about the Hilbert scheme.

1. Subschemes of \mathbb{A}^n and \mathbb{P}^n
2. The Hilbert scheme
3. The Hilbert scheme $\text{Hilb}_{3t+1}(\mathbb{P}^3)$
4. Initial ideals and one-parameter subgroups of the torus

Subschemes of \mathbb{A}^n

We have frequently seen that non-radical ideals occur naturally.

Example. Consider the ideal $(x^2, xy) \subset K[x, y, z]$. It arises if we intersect the algebraic set $V(xz, yz)$ with the hyperplane $V(z - x)$. On the algebra side we have

$$K[x, y, z]/(xz, yz, z - x) \cong K[x, y]/(x^2, xy)$$

corresponding to this intersection.

It is thus natural to extend our algebra geometry dictionary beyond radical ideals as follows:

$$\{\text{radical ideals of } K[x_1, \dots, x_n]\} \xleftrightarrow{1:1} \{\text{algebraic subsets of } \mathbb{A}^n\}$$

$$\{\text{arbitrary ideals of } K[x_1, \dots, x_n]\} \xleftrightarrow{1:1} \{\text{subschemes of } \mathbb{A}^n\}.$$

The **coordinate ring** of the subscheme $X \subset \mathbb{A}^n$ corresponding to the ideal $J \subset K[x_1, \dots, x_n]$ is the ring $K[X] = K[x_1, \dots, x_n]/J$.

Morphism and local rings of a subscheme of \mathbb{A}^n

A morphism $\varphi : X \rightarrow Y$ between a subscheme $X \subset \mathbb{A}^n$ and a subscheme $Y \subset \mathbb{A}^m$ is by definition a ring homomorphism

$$\varphi^* : K[Y] \rightarrow K[X].$$

The local ring $\mathcal{O}_{X,p}$ of a subscheme at a point $p \in V(J)$ can be defined as the localization in the corresponding maximal ideal \mathfrak{m} of $K[X]$:

$$\mathcal{O}_{X,p} = K[X]_{\mathfrak{m}}.$$

The only disadvantage when considering subschemes is that $K[X]$ can no longer be regarded as a subring of the ring of functions

$$\{f : V(J) \rightarrow K\}.$$

In our example above with $K[X] = K[x, y]/(x^2, xy)$ the element $\bar{x} \in K[X]$ would correspond to the zero function because $(\bar{x})^2 = 0$. But $\bar{x} \neq 0 \in K[X]$. We allow non-trivial nilpotent elements in the coordinate ring of a subscheme.

Subschemes of \mathbb{P}^n

Definition. A **subscheme** X of \mathbb{P}^n is a collection of subschemes in the affine charts $U_i \cong \mathbb{A}^n$ that coincide in their intersections $U_i \cap U_j$.

Definition. A homogeneous ideal $I \subset K[x_0, \dots, x_n]$ is **saturated** if it coincides with its **saturation**

$$I_{sat} = I : (x_0, \dots, x_n)^\infty = \bigcup_{N=1}^{\infty} (I : (x_0, \dots, x_n)^N).$$

Proposition. *There is a bijection between saturated homogeneous ideals of $K[x_0, \dots, x_n]$ and subschemes of \mathbb{P}^n .*

Remark. A homogeneous ideal I is saturated iff the irrelevant ideal (x_0, \dots, x_n) is not an associated prime ideal of I .

The graded ring $K[X] = K[x_0, \dots, x_n]/I_{sat}$ is called the **homogeneous coordinate ring** of the subscheme $X \subset \mathbb{P}^n$ corresponding to the subscheme defined by I . The **Hilbert polynomial** $p_X \in \mathbb{Q}[t]$ is the Hilbert polynomial of $K[X]$.

The Hilbert scheme

A fundamental result in algebraic geometry due to Grothendieck is the following.

Theorem. *Let $p \in \mathbb{Q}[t]$ be a polynomial of degree $< n$. The set of subschemes*

$$\mathrm{Hilb}_p(\mathbb{P}^n) = \{X \subset \mathbb{P}^n \mid p_X = p\}$$

carries the structure of a projective scheme.

Idea of the proof.

Step 1. Let $p \in \mathbb{Q}[t]$. There exists an integer r such that the saturated ideal $I = I_X$ of any subscheme $X \subset \mathbb{P}^n$ with $p_X = p$ satisfies the following:

$$I_d \subset K[x_0, \dots, x_n]_d \text{ has codimension } p(d) \text{ for all } d \geq r. \quad (\dagger)$$

Key point of this step is that the bound r depends only on p . This is proved with the concept of the Castelnuovo-Mumford regularity, which builds upon coherent sheaves and their cohomology.

The Hilbert scheme

As additional property one obtains

$$I_{\geq d} = \bigoplus_{d' \geq d} I_{d'} = (I_d) \text{ for } d \geq r. \quad (\ddagger)$$

Remark. Since the homogeneous ideals J and $J \cap (x_0, \dots, x_n)^N$ have the same Hilbert polynomial the statements (\dagger) and (\ddagger) fail if we consider non-saturated ideals.

Step 2. We fix now a $d \geq r$. For each polynomial $p \in \mathbb{Q}[t]$ we thus have an injection

$$\{X \subset \mathbb{P}^n \mid p_X = p\} \hookrightarrow \mathbb{G}\left(\binom{n+d}{n} - p(d), \binom{d+n}{n}\right)$$

into a Grassmannian by mapping the subscheme X to the degree d part of its homogeneous ideal I_X

$$X \mapsto (I_X)_d \subset K[x_0, \dots, x_n]_d.$$

This is a codimension $p(d)$ subvector space, hence a point in the Grassmannian above.

The scheme structure of the Hilbert scheme

To get equations of the Hilbert scheme, we note that a subspace W lies in the image if and only if the ideal (W) generated by W has Hilbert function values $h_{(W)}(d') = \binom{n+d'}{n} - p(d')$ for $d' \geq d$.

Representing W by a $(\binom{n+d}{d} - p(d)) \times \binom{n+d}{n}$ -coefficient matrix, we see that for $d' = d + 1$ this gives the condition that a

$$(n+1) \left(\binom{n+d}{n} - p(d) \right) \times \binom{n+d+1}{n} \text{-matrix}$$

in these coefficients has rank precisely $\binom{n+d+1}{n} - p(d+1)$. Thus the corresponding minors give equations for the Hilbert scheme.

We leave it open to explain why the rank for any W satisfying these equations cannot be smaller and why these equations define the Hilbert scheme correctly from a conceptual point of view.

The key concept for the later is the notion of a representable functor.

$\text{Hilb}_{3t+1}(\mathbb{P}^3)$

$p(t) = 3t + 1$ is the Hilbert polynomial of the rational normal curve in \mathbb{P}^3 . We give a description of the open part of this Hilbert scheme of all saturated ideals with lead ideal $\text{Lt}(I) = (x_0^2, x_0x_1, x_1^2)$. If I is such an ideal, then its equation take the form

$$\begin{aligned} & x_0^2 + a_1x_0x_2 + a_2x_1x_2 + a_3x_0x_3 + a_4x_1x_3 + a_5x_2^2 + a_6x_2x_3 + a_7x_3^2 \\ & x_0x_1 + b_1x_0x_2 + b_2x_1x_2 + b_3x_0x_3 + b_4x_1x_3 + b_5x_2^2 + b_6x_2x_3 + b_7x_3^2 \\ & x_1^2 + c_1x_0x_2 + c_2x_1x_2 + c_3x_0x_3 + c_4x_1x_3 + c_5x_2^2 + c_6x_2x_3 + c_7x_3^2 \end{aligned}$$

with coefficients a_1, \dots, c_7 . The Buchberger test gives $2 \cdot 10$ equations for the coefficients obtained from the possible 10 different remainders of the division by x_0^2, x_0x_1, x_1^2 in degree 3. As it turns out the ideal is generated by only nine equations

$$a_5 + a_2b_1 - a_1b_2 + b_2^2 - a_2c_2, \dots, c_7 + b_3^2 - a_3c_3 + b_4c_3 - b_3c_4$$

which say that a_5, \dots, c_7 are quadratic polynomials in a_1, \dots, c_4 .

$\text{Hilb}_{3t+1}(\mathbb{P}^3)$

To phrase this differently: The open subset of $\text{Hilb}_{3t+1}(\mathbb{P}^3)$ corresponding to subschemes with lead ideal $\text{Lt}(I) = (x_0^2, x_0x_1, x_1^2)$ is isomorphic to a \mathbb{A}^{12} and the best way to phrase the equations in the coefficients is to say that the three generators of I can be written as the 2×2 -minors of the matrix

$$\begin{pmatrix} -x_1 - x_2b_1 - x_3b_3 & -x_2c_1 - x_3c_3 \\ x_0 + x_2a_1 + x_3a_3 - x_2b_2 - x_3b_4 & -x_1 + x_2b_1 + x_3b_3 - x_2c_2 - x_3c_4 \\ x_2a_2 + x_3a_4 & x_0 + x_2b_2 + x_3b_4 \end{pmatrix}.$$

Indeed, for example, the coefficients of a_5 of x_2^2 in the minor with lead term x_0^2 is $a_5 = b_2(a_1 - b_2) - a_2(b_1 - c_2)$.

Thus $\text{Hilb}_{3t+1}(\mathbb{P}^3)$ has a 12-dimensional component in which the $\text{PGL}(4, K)$ -orbit of the rational normal curve is dense. Note that

$$\dim \text{PGL}(4, K) - \dim \text{PGL}(2, K) = 15 - 3 = 12$$

Components of $\text{Hilb}_{3t+1}(\mathbb{P}^3)$

Actually, there are two components:

$$\text{Hilb}_{3t+1}(\mathbb{P}^3) = H_{12} \cup H_{15},$$

the component of dimension 12 from above and a further component of dimension 15. A general subscheme $X \in H_{15}$ is the union $X = E \cup \{p\}$ of a plane cubic curve E and a disjoint point. The choice of E depends on $3 + 9$ parameters, since we have to specify a plane in \mathbb{P}^3 and a cubic equation in that plane. The choice of p gives further 3 parameters. This fits with $15 = 3 + 9 + 3$.

Intersection of the components of $\text{Hilb}_{3t+1}(\mathbb{P}^3)$

To see the intersection of both components we look at the open chart of the Hilbert scheme of X with lead ideal

$$\text{Lt}(I_X) = (x_0^2, x_0x_1, x_0x_2, x_1^3).$$

This time we introduce $3 \cdot 7 + 10$ variables for the possible coefficients. The Buchberger test computation yields a system of equations for the coefficients. Its primary decomposition give two smooth components isomorphic to \mathbb{A}^{12} and \mathbb{A}^{15} intersecting in smooth subvariety isomorphic to \mathbb{A}^{14} . A general scheme corresponding to a point in the intersection consist of a nodal plane curve together with an embedded point at the node sticking out of the plane.

Initial ideals

Let $I = I_{\text{sat}}$ be a saturated homogeneous ideal in $K[x_0, \dots, x_n]$.
The Hilbert scheme allows us to interpret the association

$$I \rightsquigarrow \text{Lt}(I)$$

in a more conceptual way. Let $>_w$ be a weight order which has the same lead terms as the given monomial order for the Gröbner basis of I . We may assume that the weight (w_0, \dots, w_n) are integers and all positive, since adding the vector $(1, \dots, 1)$ does not change the ordering for monomials of a given degree. Consider the one-parameter subgroup

$$T = \left\{ \begin{pmatrix} t^{w_0} & & 0 \\ & \ddots & \\ 0 & & t^{w_n} \end{pmatrix} \mid t \in K^* \right\}$$

of the group of diagonal matrices in $\text{GL}(n+1, K)$.

We ask how the ideal moves under the substitution $x_i \mapsto t^{-w_i} x_i$.

The corresponding 1-parameter family of schemes

Let $f = \sum_{|\alpha|=d} f_\alpha x^\alpha \in I$ be a homogeneous element and let $f_\beta x^\beta = \text{Lt}(f)$. Then

$$\begin{aligned} f(t) &= t^{\langle \beta, w \rangle} f(t^{-w_0} x_0, \dots, t^{-w_n} x_n) \\ &= \sum_{\alpha} f_\alpha t^{\langle \beta - \alpha, w \rangle} x^\alpha \in K[t, x_0, \dots, x_n] \end{aligned}$$

because $x^\beta >_w x^\alpha$ for all $\alpha \neq \beta$ with $f_\alpha \neq 0$. Now consider a Gröbner basis f_1, \dots, f_r of I and the subscheme

$$Y \subset \mathbb{P}^n \times \mathbb{A}^1$$

defined by the ideal $J = (f_1^{(t)}, \dots, f_r^{(t)}) \subset K[t, x_0, \dots, x_n]$. The fibers Y_q for $q \in \mathbb{A}^1 \setminus \{0\}$ all isomorphic to $Y_1 = X$ which is our original scheme. The special fiber Y_0 is defined by $\text{Lt}(I)$.

Thus

$$\lim_{t \rightarrow 0} Y_t = X_0 \in \text{Hilb}_{p_X}(\mathbb{P}^n).$$

Degeneration to monomial ideals

Corollary. Let $I = I_{\text{sat}} \subset K[x_0, \dots, x_n]$ be a homogeneous saturated ideal, X the corresponding subscheme of \mathbb{P}^n and p_X its Hilbert polynomial. The set of possible lead ideals

$$\{\text{Lt}_{>}(I) \mid > \text{ is a monomial order}\}$$

coincides with the set of monomial limits of X in $\text{Hilb}_{p_X}(\mathbb{P}^n)$ along one-parameter subgroups of the torus

$$\left\{ \begin{pmatrix} a_0 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \mid a_i \in K^* \right\} \subset \text{GL}(n+1, K).$$



Since $\text{Hilb}_p(\mathbb{P}^n)$ is a projective scheme as a subscheme of a suitable Grassmannian, limits exist along arbitrary one-parameter families!

A non-monomial limit

Consider the rational normal curve in \mathbb{P}^3 defined by the 2×2 -minors of the matrix

$$\begin{pmatrix} x_3 & x_0 & x_1 \\ x_0 & x_1 + x_3 & x_2 \end{pmatrix}$$

Substituting $x_0 \mapsto x_0, x_1 \mapsto tx_1, x_2 \mapsto tx_2, x_3 \mapsto tx_3$ and computing the Gröbner basis of the ideal I_t with respect to the induced order we get

$x_0^2 - t^2 x_1 x_3 - t^2 x_3^2$	$-x_1$	$-x_2$	0	0
$x_0 x_1 - t x_2 x_3$	x_0	$tx_1 + tx_3$	$-x_2$	$-x_1^2 - x_1 x_3$
$x_0 x_2 - t x_1^2 - t x_1 x_3$	tx_3	x_0	x_1	$x_2 x_3$
$x_1^3 - x_2^2 x_3 + x_1^2 x_3$			t	x_0

A non-monomial limit

Thus

$$\begin{aligned}I_0 &= (x_0^2, x_0x_1, x_0x_2, x_1^3 - x_2^2x_3 + x_1^2x_3) \\ &= (x_0, x_1^3 - x_2^2x_3 + x_1^2x_3) \cap (x_0^2, x_1, x_2)\end{aligned}$$

is a nodal plane curve with an embedded point at the node

$$p = [0 : 0 : 0 : 1].$$

Stratification of the Hilbert scheme

Like the Grassmannian the Hilbert scheme allows a stratification. Fix a Hilbert polynomial p and a global monomial order, for example, $>_{\text{lex}}$ or $>_{\text{rlex}}$. There are only finitely many saturated monomial ideals J with Hilbert polynomial p . Choose one of these monomial ideals J . Then the set

$$S_J = \{I \in \text{Hilb}_p(\mathbb{P}^n) \mid \text{Lt}(I) = J\}$$

of saturated ideals I_{sat} with $\text{Lt}(I) = J$ are precisely those ideals which approach J under a suitable 1-parameter subgroup. The strata have a natural structure of affine schemes.

A famous theorem of Hartshorne says that the Hilbert schemes $\text{Hilb}_p(X)$ are connected. The proof builds upon the idea to connect these monomial ideals via a chain of 1-parameter families.