Algebraic Geometry, Lecture 23

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Saarland University, Perugia 2021

Overview

Today we prove the theorem of Bertini.

1. Bertini's theorem and the geometric interpretation of the degree
2. The dual variety
3. Dynamical interpretation of intersection numbers
4. The topological genus of a complex projective curve

The dual projective space

Definition. Let $\mathbb{P}^n$ be a projective space. Then the projective space of hyperplanes $H \subset \mathbb{P}^n$ is a called the **dual projective space**

$$\check{\mathbb{P}}^n = \{H \subset \mathbb{P}^n \mid H \text{ is a hyperplane}\}.$$

Remark. For a point $p \in \mathbb{P}^n$ the space of hyperplanes passing through p

$$H_p = \{H \in \check{\mathbb{P}} \mid p \in H\} \subset \check{\mathbb{P}}^n$$

is a hyperplane in $\check{\mathbb{P}}^n$, and any hyperplane in $\check{\mathbb{P}}^n$ arises this way: The subvariety

$$\mathbb{F} = V(a_0 x_0 + \ldots + a_n x_n) \subset \mathbb{P}^n \times \check{\mathbb{P}}^n$$

can be interpreted in two ways

$$\mathbb{F} = \{(p, H) \in \mathbb{P}^n \times \check{\mathbb{P}}^n \mid p \in H\} = \{(p, H) \in \mathbb{P}^n \times \check{\mathbb{P}}^n \mid H \in H_p\}.$$

The fibers of the projection $\mathbb{F} \to \check{\mathbb{P}}^n$ onto the second factor are hyperplanes in $\mathbb{P}^n$, and the fibers of the projection to the first factor $\mathbb{F} \to \mathbb{P}^n$ are hyperplanes in $\check{\mathbb{P}}^n$.

Bertini's theorem

Theorem. *Let $X \subset \mathbb{P}^n$ be a projective variety of dimension d. Let X_{sing} denote its set of singular points. There exists an non-empty open subset $U \subset \check{\mathbb{P}}^n$ of hyperplanes such that $X \cap H$ is smooth outside $X_{sing} \cap H$ for every $H \in U$.* In particular if X is smooth, then $X \cap H$ is smooth as well for all $H \in U$.

Proof. Consider the open set $X^* = X \setminus X_{sing}$ of smooth points of X and the variety

$$\begin{tikzcd} D^* = \{(p, H) \in X^* \times \check{\mathbb{P}}^n \mid T_pX \subset H\} \arrow[r] \arrow[d, "\pi_1"'] & \check{\mathbb{P}}^n \\ X^* & \end{tikzcd}$$

with its two projections. A point $(p, H) \in D^*$ is a pair such that $X \cap H$ is singular in p.

Proof of Bertini's theorem

The fiber of $\pi_1 : D^* \to X^*$ over a point $p \in X^*$ is a projective space of dimension $n - d - 1$

$$\{H \subset \mathbb{P}^n \mid H \supset T_p(X)\} \cong \mathbb{P}^{n-d-1}$$

because H is contained in the fiber iff H is defined by a linear combination of the $n - d$ equations of $T_pX \cong \mathbb{P}^d \subset \mathbb{P}^n$.
Thus $\dim D^* = d + n - d - 1 = n - 1$. We take

$$D = \overline{D^*} \subset X \times \check{\mathbb{P}}^n \subset \mathbb{P}^n \times \check{\mathbb{P}}^n.$$

Then $\dim D = \dim D^*$ and the projection $\pi_2(D) \subset \check{\mathbb{P}}^n$ is a Zariski closed subset of dimension

$$\dim \pi_2(D) \leq \dim D = n - 1$$

and $U = \check{\mathbb{P}}^n \setminus \pi_2(D)$ is the desired open subset. $\square$

Geometric interpretation of the degree

Theorem. *Let $X \subset \mathbb{P}^n$ be a projective variety of dimension d. Then a general linear subspace $\mathbb{P}^{n-d} \subset \mathbb{P}^n$ intersects X in $\deg X$ many distinct points transversally.*

Proof. Let $H \subset \mathbb{P}^n$ be a general hyperplane. In particular H does not contain any component of X_{sing}. Let $C_1 \cup \ldots \cup C_r = X \cap H$ be the irreducible components. Then

$$\deg X = \sum_{j=1}^{r} i(X, H; C_j) \deg C_j$$

holds by Bézout's theorem. By Bertini's theorem the intersection is smooth. In particular the intersection is transversal at smooth points of each C_j, and the intersection multiplicity is 1. The result follows now by induction. A general complementary $\mathbb{P}^{n-d}$ is the intersection of d general hyperplanes $H_1 \cap \ldots \cap H_d$ such that H_i intersects each component of $X \cap H_1 \cap \ldots \cap H_{i-1}$ transversally. $\square$

The dual variety

Remark. Actually the intersection $X \cap H$ is irreducible for general H, and $X \cap \mathbb{P}^{n-d+1}$ is an irreducible smooth curve for a general linear subspace $\mathbb{P}^{n-d+1}$ of nearly complementary dimension.

Definition. $\check{X} = \pi_2(D)$ is called the **dual variety** of X.

For $C \subset \mathbb{P}^2$ be an irreducible curve which is not a line, the dual variety is again a curve $\check{C} \subset \check{\mathbb{P}}^2$.

Theorem. *Let $C \subset \mathbb{P}^2$ be irreducible curve over a field of characteristic 0. Then the double dual curve*

$$\check{\check{C}} = C$$

gives the original curve back.

Theorem of Brianchon

Theorem. *The three diagonals of a hexagon which is circumscribed around a conic intersect in a point*

This theorem follows via duality from Pascal's theorem.

Strange curves

If $\mathrm{char}(K) = p > 0$, it is possible that all tangent lines of an irreducible plane curve pass through a common point. Curves different from lines with this property are called **strange**.

Example. Consider $C = V(x^p - yz^{p-1}) \subset \mathbb{P}^2$. In the affine chart $U_{z=1}$ this curve has the parametrization

$$\mathbb{A}^1 \to C \cap U_{z=1} \subset \mathbb{A}^2, t \mapsto q = (t, t^p)$$

and equation $f = x^p - y$. Since $d_q f = pt^{p-1}(x - t) - 1(y - t^p)$ the projective tangent lines are $T_q C = V(-y + t^p z)$. These lines all pass through the point $V(y, z) = [1:0:0]$.

So the dual curves $\check{C} \subset \check{\mathbb{P}}^2$ is the set of lines $H_{[1:0:0]} \cong \mathbb{P}^1 \subset \check{\mathbb{P}}^2$ passing through $[1:0:0]$, and $\check{\check{C}} \neq C$.

Notice that a strange curve can have at most one 'strange point', because the dual curve is a line.

Degree of a morphism $f : C \to \mathbb{P}^1$

Let $C \subset \mathbb{P}^n$ be an irreducible smooth projective curve, and let $f \in K(C)$ a non-constant rational function. The rational map

$$C \dashrightarrow \mathbb{P}^1, p \mapsto [1 : f(p)]$$

extends to a morphism $f : C \to \mathbb{P}^1$, which we denote by the same letter.

Definition. The **degree** of f is

$$\deg f = \sum_{p \in C : v_p(f) > 0} v_p(f)$$

the number of preimage points of $[1 : 0]$ counted with multiplicities.

Proposition. *Counted with multiplicities each fiber $f^{-1}(\lambda)$ of $\lambda \in \mathbb{P}^1$ has precisely $\deg f$ many points.*

Proof

Since rational functions are given by quotients of homogeneous polynomials of the same degree on the ambient $\mathbb{P}^n$ the number of poles $\sum_{p\in C:v_p(f)<0} -v_p(f)$ coincides with the number of zeroes by Bézout's theorem. To see that the number of preimage points of $\lambda \in \mathbb{A}^1 = K$ coincides with $\deg f$, we note that f and $f - \lambda$ have the same poles. $\square$

Remark. One can show that $\deg f$ also coincides with the degree of the field extension $[K(C) : K(f)]$. Note that $K(f) \cong K(\mathbb{P}^1)$.

More generally for a morphism $\varphi : C \to E$ between smooth projective curves the **degree** can be defined as

$$\deg \varphi = [K(C) : K(E)],$$

and this number coincides with the number of preimage points of any point $p \in E$ counted with multiplicities.

Dynamical intersection numbers

We assume that $K = \mathbb{C}$. Let $f \in K[x, y, z]$ be a square-free polynomial of degree d, and let $g \in K[x, y, z]$ be a homogeneous polynomial of degree e which has no common factor with f. Then

$$d \cdot e = \sum_{p \in V(f,g)} i(f, g; p)$$

by Bézout's theorem. We will show that the intersection multiplicities can be interpreted dynamically.

As an application of Bertini's theorem we see that there exists a homogeneous polynomial g_1 of degree e such that $C = V(f)$ and $D = V(g_1)$ intersect transversally in $d \cdot e$ distinct points.

Indeed, consider the e-uple embedding

$$\rho_{2,e} : \mathbb{P}^2 \to \mathbb{P}^{\binom{e+2}{2}-1}.$$

Curves of degree e in $\mathbb{P}^2$ correspond to hyperplanes H in $\mathbb{P}^{\binom{e+2}{2}-1}$, and a general hyperplane H_1 intersects every component of $\rho_{2,e}(C)$ transversally in smooth points of $\rho_{2,e}(C)$.

Dynamical intersection numbers, 2

Let g_1 be the polynomial corresponding to the equation of H_1 and consider the pencil of curves of degree e

$$D = V(t_0 g + t_1 g_1) \in \mathbb{P}^1 \times \mathbb{P}^2.$$

All but finitely many fibers D_λ over $\lambda \in \mathbb{P}^1$ intersect C in $d \cdot e$ distinct points. Consider now the curve

$$X' = D \cap (\mathbb{P}^1 \times C)$$

and the union X of components which dominate $\mathbb{P}^1$. Let $\sigma : Y \to X$ be a birational morphism from a smooth projective curve and let Y_0 be the preimage of $[1:0] \in \mathbb{P}^1$ under $f = \pi_1 \circ \sigma$ where π_1 denotes the projection onto the first factor of $\mathbb{P}^1 \times \mathbb{P}^2$. Each point of Y_0 maps to a point of $V(f, g)$ under π_2.

Dynamical intersection numbers, 3

Let $q \in Y_0$ be a point and $s \in \mathfrak{m}_{Y,q} \subset \mathcal{O}_{Y,q}$ a local generator. The rational function $t = t_1/t_0 \in \mathcal{O}_{\mathbb{P}^1,[1:0]}$ pulls back to $f = us^r$ with $r = v_q(f)$ and $u \in \mathcal{O}_{Y,q}$ a unit. For a point $\lambda \in \mathbb{A}^1 = \mathbb{C}$ with $|\lambda|$ small there are precisely r preimage points in the holomorphic chart defined by s with absolute value approximately $(\frac{|\lambda|}{|u(0)|})^{1/r}$. For $\lambda \to 0$ the images of these points in C approach the image of $p \in C \cap D_0$ of q.

Let $p_1, \ldots, p_k$ denote the distinct points of $C \cap V(g)$. Let q_{ij} for $j = 1, \ldots d_i$ denote the distinct preimages of p_i in Y and r_{ij} denote the ramification numbers as above. Then precisely $\sum_{j=1}^{d_i} r_{ij}$ images of the points in the fiber $f^{-1}(\lambda)$ approach p_i for $\lambda \to 0$.

Dynamical intersection numbers, 4

Thus we must have

$$i(f,g;p_i) = \sum_{j=1}^{d_i} r_{ij}.$$

This identity fits with the fact that $\sum_{i=1}^{k}\sum_{j=1}^{d_i} r_{ij} = d\cdot e$ counts the number of points in the fibers of $Y \to \mathbb{P}^1$.

To prove this identity one can use that $i(f,g;p_i)$ can also be computed as the multiplicity of the resultant $\mathrm{Res}_x(f,g) \in K[y,z]$ at the point $[b_i : c_i]$ for $p_i = [a_i : b_i : c_i]$ if our coordinate system is chosen general enough. For example, the $[b_i : c_i]$'s should be pairwise distinct. The resultant $\mathrm{Res}_x(f,g_\lambda)$ has precisely $\sum_{j=1}^{d_i} r_{ij}$ zeroes counted with multiplicities which approach $[b_i : c_i]$ for $\lambda \to 0$. $\square$

The topological genus

Let C be an irreducible smooth projective curve. Then C is also a compact Riemann surface, which turns out to be connected.

One proof of the connectedness builds upon analytic continuation and monodromy from one complex variable theory and Galois theory from algebra. This is beyond the scope of the course.

The underlying compact two-dimensional differential or topological real manifolds are orientable and classified by their **genus** $g \in \mathbb{N}$.

The genus can be computed from any triangulation: If C has a triangulation with c_0 vertices, c_1 edges and c_2 triangles, then the **topological Euler characteristic** satisfies

$$\chi_{top}(C) := c_0 - c_1 + c_2 \overset{!}{=} 2 - 2g.$$

Ramification and branch points

Let $\varphi : C \to E$ be a non-constant morphism of smooth projective curves defined over $\mathbb{C}$. Let $p \in C$ a point and $q = \varphi(p) \in E$ its image. Let $s \in \mathfrak{m}_{C,p} \subset \mathcal{O}_{C,p}$ and $t \in \mathfrak{m}_{E,q} \subset \mathcal{O}_{E,q}$ be generators of the maximal ideals. Then $\varphi^*(t) = us^r$ for some integer $r > 0$ and $u \in \mathcal{O}_{C,p}$ a unit.

Definition. With this notation the integer

$$e_p := r$$

is called the **ramification index** of φ at p. In case $e_p > 1$ we call p a **ramification point** and $q = \varphi(p) \in E$ a **branch point** of φ.

$$R = \sum_{p \in C} (e_p - 1)$$

is called the **total ramification number** of φ. Note that the left hand side is a finite sum, since the ramification points are isolated in C.

The Riemann-Hurwitz formula

Theorem. *Let $\varphi : C \to E$ be a non-constant morphism of smooth projective curves defined over $K = \mathbb{C}$ of $d = \deg \varphi$. Let g_C and g_E denote the topological genus of C and E respectively, and let R be the total ramification number of φ. Then*

$$2 - 2g_C = d(2 - 2g_E) - R.$$

Proof. Consider a triangulation of the underlying real manifold of E with c_0 vertices, c_1 edges and c_2 triangles. We take the triangulation fine enough, such that each triangle contains at most one branch point, which if present is a vertex of the triangle. Moreover each triangle should have precisely d preimage triangles in C which are disjoint except for possible ramification points. So the preimages give a triangulation of C which has dc_2 triangles, dc_1 edges but only $dc_0 - R$ vertices because of the ramification. Hence

$$\begin{aligned} 2 - 2g_C &= \chi_{top}(C) = dc_0 - R - dc_1 + dc_2 \\ &= d\chi_{top}(E) - R = d(2 - 2g_E) - R. \end{aligned}$$

□