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Exercises for the Algebraic Geometry course

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First week

Exercise 1. Let K be an infinite field and $f \in K[x_1, \dots, x_n]$ a non-zero polynomial. Show that there exists a point $a \in \mathbb{A}^n(K)$ such that $f(a) \neq 0$.

Exercise 2. Let $K[x]$ be a polynomial ring in one variable over a field. Prove that $K[x]$ is a principal ideal domain, that is, every ideal $I \subset K[x]$ is generated by a single polynomial.

Exercise 3. Implement the computer algebra system Macaulay2
https://faculty.math.illinois.edu/Macaulay2/
on your machine.

Exercise 4. Let $I \subset K[x_1, \dots, x_n]$ be an ideal. Prove

$$V(I) \subset \mathbb{A}^n \text{ is finite} \iff K[x_1, \dots, x_n]/I \text{ is a finite-dimensional } K\text{-vector space.}$$

Exercise 5. A binomial $f \in K[x_1, \dots, x_n]$ is a polynomial which has exactly two terms

$$f = ax^\alpha - bx^\beta.$$

A binomial ideal is an ideal generated by binomials and monomials. Prove:
Binomial ideals have a Gröbner basis consisting of binomials and monomials.

Exercise 6. (Key property of $>_{\text{lex}}$.)

(1) Suppose $f \in K[x_1, \dots, x_n]$ and $1 \leq j \leq n-1$. Then

$$\mathrm{Lt}_{\text{lex}}(f) \in K[x_{j+1}, \dots, x_n] \iff f \in K[x_{j+1}, \dots, x_n]$$

(2) Let $f_1, \dots, f_r$ be a Gröbner basis of $I \subset K[x_1, \dots, x_n]$ with respect to $>_{\text{lex}}$. Then

$$\{f_s \mid \mathrm{Lt}_{\text{lex}}(f_s) \in K[x_{j+1}, \dots, x_n]\}$$

is a Gröbner basis of $I_j = I \cap K[x_{j+1}, \dots, x_n]$.

Exercise 7. Consider the ideal $I = (xy(x+y)+1) \subset \mathbb{F}_2[x, y]$. Determine coordinates in which I satisfies the extra hypothesis of the projection theorem. Show that the extra hypothesis cannot be achieved by means of a linear change of coordinates.

Exercise 8. Prove that the algebraic sets $V(y-x^2)$ and $V(xy-1)$ in $\mathbb{A}^2$ are not isomorphic.

Exercise 9. Let $>$ be a monomial order and let $M \subset K[x_1, \dots, x_n]$ be a finite set of monomials. Prove that there exists a weight order $>_w$ (with $\mathbb{Q}$-linearly independent weights) which induces the same order on the monomials of M as $>$.

Hint: Consider the convex hull C of the set

$$\{\alpha - \beta \mid x^\alpha, x^\beta \in M \text{ with } x^\alpha \geq x^\beta\} \subset \mathbb{R}^n$$

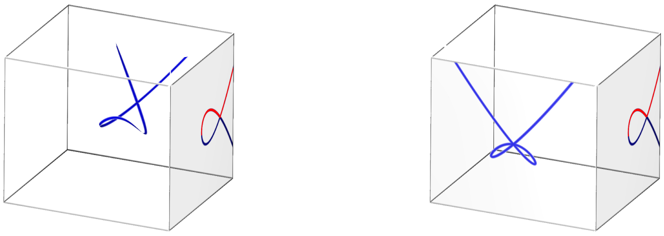
and prove that $0 \in C$ is a vertex of C, i.e., 0 is not a linear combination of other points in C with strictly positive coefficients.

Exercise 10. Consider the curve $C = V(f_1, f_2) \subset \mathbb{A}^3(\mathbb{C})$, where

$$\begin{aligned} f_1 &= y^3z - 2y^2z - z^3 + x^2 + z, \\ f_2 &= xy^3z - 2xy^2z - xz^3 + x^3 + y^3 - 2y^2 + xz - z^2 + y. \end{aligned}$$

Prove that the reduced lexicographic Gröbner basis for the ideal (f_1, f_2) with variables ordered as $x > y > z$, is

$$x^2 - yz + z, \qquad y^3 - 2y^2 + y - z^2.$$



If we reorder the variables as $y > z > x$, the reduced lexicographic Gröbner basis for $\langle f_1, f_2\rangle$ consists of five polynomials:

$$\begin{array}{lll} y^3 - 2y^2 + y - z^2, & y^2x^2 - yx^2 - z^3, & yz - z - x^2, \\ yx^4 - z^4, & z^5 - zx^4 - x^6. & \end{array}$$

Consider $C_1 = V(y^3 - 2y^2 + y - z^2) \subset \mathbb{A}^2$ and $C_2 = V(z^5 - zx^4 - x^6) \subset \mathbb{A}^2$ and the projections $C \to C_1$ and $C \to C_2$ onto the curve in the yz- and xz-plane. How many preimage points have points in C_1 and C_2?

Second week

Exercise 11. An R-module M is called noetherian if it satisfies the analogous equivalent conditions for submodules instead of ideals. Prove:

(1) Let

$$0 \longrightarrow M' \xrightarrow{\psi} M \xrightarrow{\varphi} M'' \to 0$$

be a short exact sequence of R-modules, i.e., the homomorphism ψ is injective, the homomorphism φ is surjective and $\ker\varphi = \operatorname{im}\psi$.
Then M is noetherian iff M' and M'' are noetherian.

(2) An R-module M over a noetherian ring R is noetherian iff M is finitely generated.

Exercise 12. Let R be a noetherian ring, let $\mathfrak{m}$ be a maximal ideal of R, and let I be any ideal of R. Show that the following are equivalent:

(1) I is $\mathfrak{m}$-primary.
(2) $\operatorname{rad}(I) = \mathfrak{m}$.
(3) $\mathfrak{m} \supset I \supset \mathfrak{m}^k$ for some $k \geq 1$.

Exercise 13.

(1) When is a monomial ideal a prime ideal?
(2) Characterize monomial primary ideals.
(3) Consider the monomial ideal $I = (xy, xz, yz) \subset \mathbb{Q}[x, y, z]$. Compute a primary decomposition of I and I^2.

Exercise 14. Consider $R = C^0[0,1] = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\}$, the ring of continuous function on the interval $[0,1]$. Prove that R is neither an integral domain nor noetherian.

Exercise 15.

(1) Let $R = K[A]$ be the coordinate ring of a variety and $f \in R$ be an element which is not a unit. Prove that R_f is isomorphic to the coordinate ring of a variety as well.
(2) Prove that $K[x]_{(x)}$, the localization of the polynomial ring in one variable at the maximal ideal (x), is not isomorphic to the coordinate ring of a variety.

Exercise 16. Let $A = B \cup C$ be a decomposition of an algebraic set into proper algebraic subsets. Let $p \in A \setminus C$ be a point and $\mathfrak{m} = \mathrm{I}(p)$ be the corresponding maximal ideal. Prove

$$K[A]_{\mathfrak{m}} \cong K[B]_{\mathfrak{m}}.$$

Exercise 17. Consider the ring extension

$$R = \mathbb{R}[e_2, e_3] \hookrightarrow T = \mathbb{R}[t_1, t_2]$$

defined by $e_2 \mapsto t_1t_2 - (t_1 + t_2)^2, e_3 \mapsto t_1t_2(-t_1 - t_2)$.

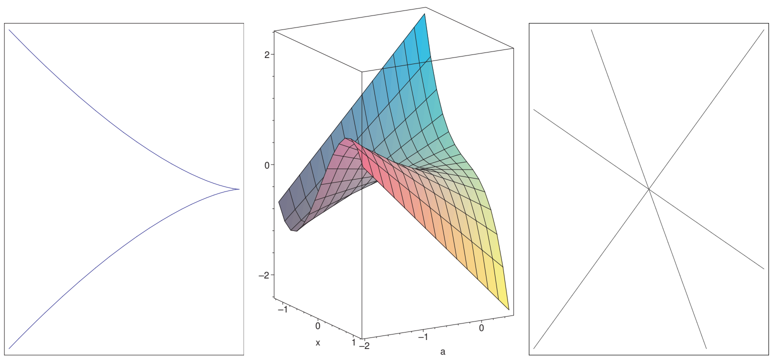
(1) Prove that $S = R[t_1] \cong R[x]/(x^3 + e_2x + e_3)$ and conclude that

$$R \subset S \subset T$$

is a tower of finite extensions.
(2) Compute the degrees of the field extensions

$$Q(R) \subset Q(S) \subset Q(T).$$

(3) Let $(b_2, b_3) \in \mathbb{A}^2(\mathbb{R})$ be a point. How many maximal ideals $\mathfrak{P}$ in S can lie over the maximal ideal of $\mathfrak{p} = (e_2 - b_2, e_3 - b_3) \subset R$? How many maximal ideals $\mathfrak{P}'$ in T can lie over $\mathfrak{p}$?
(4) What residue fields $S/\mathfrak{P}$ and $T/\mathfrak{P}'$ do occur?



R, S and T, and branch loci.

Exercise 18. If $\psi : L \to M$ and $\varphi : M \to N$ are two R-module homomorphism with $\varphi \circ \psi = 0$, then

$$H = \frac{\ker(\varphi)}{\operatorname{im} \psi}$$

is called the **homology** of the complex

$$L \xrightarrow{\psi} M \xrightarrow{\varphi} N$$

at M. Let

$$\begin{array}{ccccccc} E_1 & \xrightarrow{a} & E_0 & \longrightarrow & L & \longrightarrow & 0 \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_0} & & \downarrow{\scriptstyle \psi} & & \\ F_1 & \xrightarrow{b} & F_0 & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi} & & \\ G_1 & \xrightarrow{c} & G_0 & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

be free presentations of ψ and φ. Prove the correctness of the following algorithm.

1. Compute the syzyzgy matrix of $(c|\varphi_0)$

$$H_0 \xrightarrow{\begin{pmatrix} g_0 \\ h_0 \end{pmatrix}} G_1 \oplus F_0 \xrightarrow{(c|\varphi_0)} G_0 \ .$$

2. Compute the syzyzgy matrix of $(h_0|b|\psi_0)$

$$H_1 \xrightarrow{\begin{pmatrix} h_1 \\ g_1 \\ f_1 \end{pmatrix}} H_0 \oplus F_1 \oplus E_0 \xrightarrow{(h_0|b|\psi_0)} F_0 \ .$$

3. Then

$$H_1 \xrightarrow{h_1} H_0 \longrightarrow H \longrightarrow 0$$

is a presentation of H.

Exercise 19. A 3SAT formula with m clauses and n logical variables is an expression of the form

$$(a_1 \vee b_1 \vee c_1) \wedge (a_2 \vee b_2 \vee c_2) \wedge \ldots \wedge (a_m \vee b_m \vee c_m)$$

with

$$a_i, b_i, c_i \in \{z_1, \neg z_1, \ldots, z_n, \neg z_n\}.$$

It is satisfiable if there exist values $z_i \in \{true, false\}$ which makes the formula true.
We translate this formula as follows into the square-free monomial ideal $I \subset K[x_1, \ldots, y_n]$ in $2n$ variables. There are n degree two monomials

$$x_1 y_1, \ldots, x_n y_n$$

and m monomials of degree three. Each clause gives a degree three monomial, where x_i and y_i correspond to z_i and $\neg z_i$ respectively. For example

$$(z_1 \vee \neg z_3 \vee z_4) \longleftrightarrow x_1 y_3 x_4.$$

Thus altogether I has $n + m$ generators. Prove:

1) $\dim V(I) = n$ iff the formula is satisfyable. Here a solution with $x_i = 0$ corresponds to a SAT-solution with $z_i = true$, and $y_i = 0$ corresponds to a solution with $z_i = false$.
2) The number of points in $V(I + J)$ where $J = (x_1 + y_1 - 1, \ldots, x_n + y_n - 1)$ coincides with the number of solutions of the 3SAT formula.
3) Define $I_0 = I$ and recursively

$$I_k = I_{k-1} : (x_i + y_i).$$

Then the formula is not satisfyable iff $I_n = (1)$.

Thus computing the dimension of monomial ideals is NP-hard. The algorithm in 3) is a variant of the well-known resolution algorithm of J.A. Robinson (1963) for logical formulas.

Third week

Exercise 20. The group $\mathrm{GL}(n+1, K)$ acts on $\mathbb{P}^n(K)$ via the action induced from

$$\mathrm{GL}(n+1, K) \times K^{n+1} \to K^{n+1}, (A, x) \mapsto Ax.$$

Prove: The scalar multiples of the identity matrix act trivially on $\mathbb{P}^n$, and the quotient group

$$\mathrm{PGL}(n+1, \overline{K}) = \mathrm{GL}(n+1, \overline{K})/\overline{K}^*$$

is a quasi-projective subvariety of $\mathbb{P}^{n^2+2n}$.

Let $p_0, \ldots, p_{n+1}, p_{n+2} \in \mathbb{P}^n$ be $n+3$ points of which no subset of $n+2$ points lie in a hyperplane. Then there exists a unique automorphism $\varphi : \mathbb{P}^n \to \mathbb{P}^n$ such that

$$\varphi(p_i) = [0 : \ldots : 1 : \ldots 0]$$

is the i**th coordinate point** for $i \leq n+1$ and $\varphi(p_{n+2}) = [1 : \ldots : 1]$. For this reason $[1 : \ldots : 1]$ is sometimes called the **scaling point**. Conclude that $\mathrm{PGL}(n+1, K)$ acts faithfully on $\mathbb{P}^n$.

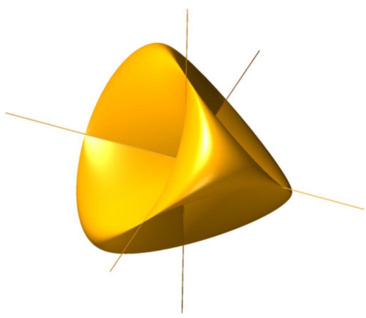
Exercise 21. Compute the homogeneous ideal of the projective closure of the affine curve parametrized by

$$\mathbb{A}^1 \to \mathbb{A}^3, t \mapsto (t, t^3, t^4).$$

Exercise 22. Consider the map

$$S^2 \to \mathbb{R}^3 = \mathbb{A}^3(\mathbb{R}), (x, y, z) \mapsto (yz, xz, xy).$$

Prove that the map factors over $\mathbb{P}^2(\mathbb{R})$ and compute the equation of the algebraic closure in $\mathbb{A}^3$.

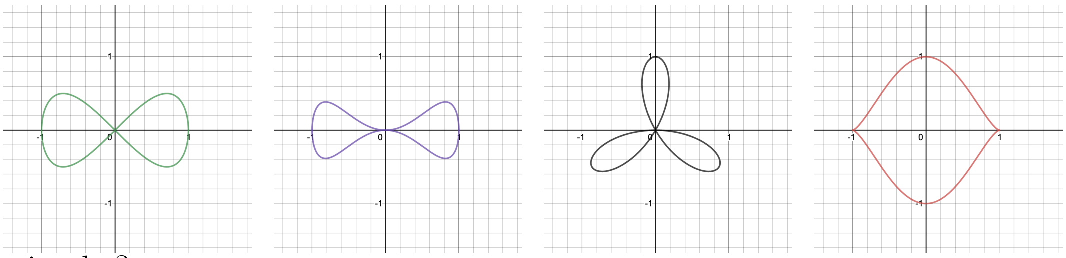


The image above was created with the program surfer https://imaginary.org/de/program/surfer. This surface is known under the name Steiner surface or Roman surface.

Exercise 23. Consider the plane curves defined by

$$y^2 = (1-x^2)^3, \quad y^2 = x^4 - x^6, \quad y^3 - 3x^2y = (x^2+y^2)^2, \quad y^2 = x^2 - x^4$$

Their real points are one of the following:



Who is who?

Exercise 24. Compute the ranks of the free modules F_i and the maps between them in the free resolution F of the following $R = K[x_1, \dots, x_n]$-modules:

1) $K \cong K[x_1, \dots, x_n]/(x_1, \dots, x_n)$ and
2) $K[x_1, \dots, x_n]/(x_1, \dots, x_n)^2$

Hint: The permutation group S_n of the n letters $x_1, \dots, x_n$ acts on R and on each F_i.

Exercise 25. Let

$$f = a_0x^d + a_1x^{d-1} + \dots + a_d$$
$$g = b_0x^d + b_1x^{d-1} + \dots + b_e$$

be two polynomials in $K[x]$ of degree d and e. Consider the $(d+e) \times (d+e)$ **Sylvester matrix**

$$\mathrm{Syl}(f,g) = \begin{pmatrix} a_0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\ a_1 & a_0 & & \vdots & b_1 & b_0 & & \vdots \\ \vdots & a_1 & \ddots & \vdots & \vdots & b_1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_0 & \vdots & \vdots & \ddots & b_0 \\ a_d & & & a_1 & b_e & & & b_1 \\ 0 & a_d & & \vdots & 0 & b_e & & \vdots \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_d & 0 & 0 & \cdots & b_e \end{pmatrix}$$

There are e columns with entries a_i's and d columns with entries b_j's. Prove

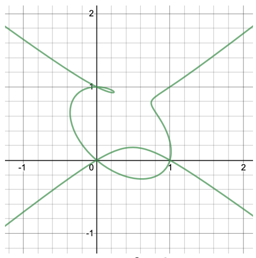
1) f and g have a common root if and only if the **resultant**
$$\mathrm{Res}(f,g) = \det \mathrm{Syl}(f,g) = 0$$
of f and g vanishes.
2) Suppose that a_i and b_j are independent variables of degree i and j respectively. Prove that the resultant
$$\mathrm{Res}(f,g) \in \mathbb{Z}[a_i, b_j]$$
is a homogeneous polynomial of degree $d \cdot e$.

Exercise 26. Give examples of two smooth plane conics which intersect in points with multiplicities

(a) $1, 1, 1, 1$
(b) $2, 1, 1$
(c) $2, 2$
(d) $3, 1$
(e) 4

Exercise 27. Use Macaulay2 to compute the following:

1) The rational parametrization of the curve defined by $f = -3x^5 - 2x^4y - 3x^3y^2 + xy^4 + 3y^5 + 6x^4 + 7x^3y + 3x^2y^2 - 2xy^3 - 6y^4 - 3x^3 - 5x^2y + xy^2 + 3y^3$ from Lecture 15.
2) A rational parametrization of the plane quartic curve $V(f) \subset \mathbb{A}^2$ defined by $f = -2x^4 - 2x^3y + x^2y^2 + 3xy^3 + 4y^4 + 4x^3 + x^2y - 4xy^2 - 8y^3 - 2x^2 + xy + 4y^2$.
Hint: $V(f)$ contains the points with coordinates $(0,0), (1,0), (0,1)$ and $(1,1)$,



3) The equation of the image C of

$$\varphi : \mathbb{P}^1 \to \mathbb{P}^2, [t_0 : t_1] \mapsto [t_0^4 : t_0^3t_1 - t_0t_1^3 : t_1^4].$$

Where are the singular points of C?

Exercise 28. Let R be a noetherian ring, and let M and N be finitely generated R-module and $F_\bullet$ and $G_\bullet$ free resolutions of M and N respectively. Prove:

1) Every R-module homomorphism $\varphi : M \to N$ extends to a **map of complexes**

$$\begin{array}{ccccccccc}
\dots \xrightarrow{\partial_3} & F_2 & \xrightarrow{\partial_2} & F_1 & \xrightarrow{\partial_1} & F_0 & \longrightarrow & M & \longrightarrow 0 \\
& \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi} & \\
\dots \xrightarrow[\partial_3']{} & G_2 & \xrightarrow[\partial_2']{} & G_1 & \xrightarrow[\partial_1']{} & G_0 & \longrightarrow & N & \longrightarrow 0,
\end{array}$$

i.e., all squares in this diagram commute in particular $\varphi_{i-1}\partial_i = \partial_i'\varphi_i$ for all $i \geq 1$.
2) Two extensions $(\varphi_i)_{i\in\mathbb{N}}$ and $(\varphi_i')_{i\in\mathbb{N}}$ of φ differ by a **homotopy**, i.e., $\exists\ (h_i)_{i\in\mathbb{N}}$

$$\begin{array}{ccccccccc}
\dots \xrightarrow{\partial_3} & F_2 & \xrightarrow{\partial_2} & F_1 & \xrightarrow{\partial_1} & F_0 & \longrightarrow & M & \longrightarrow 0 \\
{\scriptstyle h_2}\swarrow & \downarrow & {\scriptstyle h_1}\swarrow & \downarrow & {\scriptstyle h_0}\swarrow & \downarrow & & \downarrow{\scriptstyle \varphi} & \\
\dots \xrightarrow[\partial_3']{} & G_2 & \xrightarrow[\partial_2']{} & G_1 & \xrightarrow[\partial_1']{} & G_0 & \longrightarrow & N & \longrightarrow 0
\end{array}$$

such that

$$\varphi_0 - \varphi_0' = \partial_1'h_0 \quad \text{and} \quad \varphi_i - \varphi_i' = h_{i-1}\partial_i + \partial_{i+1}'h_i \quad \text{for } i \geq 1.$$

Exercise 29. Let $(R, \mathfrak{m})$ be a local noetherian ring and M a finitely generated R-module. A free resolution

$$\dots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial} M \xrightarrow{\partial_{-1}} 0$$

is **minimal** if at each step we choose a minimal set of generators of $\ker \partial_{i-1}$ and a free module F_i whose basis maps to these generators. Prove:

1) A resolution $F_\bullet$ is minimal if and only if the matrices describing ∂_i for $i \geq 1$ have entries in $\mathfrak{m}$.
2) The minimal free resolution of M is uniquely determined up to an isomorphism of complexes.

Exercise 30. Let $R = \oplus_{d\geq 0} R_d$ be a finitely generated graded ring k-algebra with $\mathfrak{m} = R_+ = \oplus_{d>0} R_d$ a maximal ideal with residue field $k = R_0$. Let M be a finitely generated graded R-module.

1) Prove Nakayama's Lemma in the graded case: If $N \subset M$ is a graded submodule, then
$$M = N + \mathfrak{m}M \implies N = M.$$
2) Conclude that the minimal graded free resolution of M is unique up to isomorphism.
3) Deduce that graded Betti numbers β_{ij} of the minimal free resolution $F_\bullet$ of a finitely generated graded module M over the standard graded polynomial ring $S = K[x_0, \ldots, x_n]$ are invariants of M.

Exercise 31. A monomial ideal $I \subset S = k[x_0, \ldots, x_n]$ is called **Borel fixed** if
$$x_i x^\alpha \in I \implies x_j x^\alpha \in I \quad \text{holds for all monomials } x^\alpha \text{ and all } j < i.$$
Prove that the algorithm from Lecture 14 computes the minimal free resolution. Eliahou and Kevaire (1990) even gave a description of the matrices in the minimal free resolution.

Exercise 32. 1) Consider the ideal
$$I = (x_0x_1x_2,\ x_1x_2x_3,\ x_0x_1x_4,\ x_0x_3x_4,\ x_2x_3x_4,\ x_0x_2x_5,\ x_0x_3x_5,\ x_1x_3x_5,\ x_1x_4x_5,\ x_2x_4x_5)$$
Prove that the minimal free resolution of I as an $K[x_0, \ldots, x_5]$-module depends on the characteristic of the ground field. The ranks of the free modules in $\mathrm{char}(K) = 2$ and $\mathrm{char}(K) \neq 2$ are different.

An explanation of this phenomenon is given by Hochster's theory (1977) of Stanley-Reisner rings. Let Δ be a simplicial complex with $n+1$ vertices. We may regard Δ as a subcomplex of the standard n simplex Δ_n, which by definition is the convex hall of the coordinate vectors $e_i \in \mathbb{R}^{n+1} = \mathbb{A}^{n+1}(\mathbb{R})$. The square-free monomial ideal I_Δ of Δ is the vanishing ideal of the cone over Δ with vertex $0 \in \mathbb{R}^{n+1}$. Since I_Δ is monomial ideal its generators generate an ideal $I_\Delta^K \subset K[x_0, \ldots, x_n]$ for any field K. The coordinate ring $K[\Delta] = K[x_0, \ldots, x_n]/I_\Delta^K$ is called the **Stanley-Reisner** ring of Δ over K. Conversely any square-free monomial ideal $J \subset \mathbb{Q}[x_0, \ldots, x_n]$ defines a simplicial complex by intersecting the cone $C(J) \subset \mathbb{A}^{n+1}(\mathbb{C})$ over $V(J) \subset \mathbb{P}^n(\mathbb{C})$ with the standard n-simplex $\Delta_n \subset \mathbb{R}^{n+1} \subset \mathbb{C}^{n+1} = \mathbb{A}^{n+1}(\mathbb{C})$.

2) Prove
$$\Delta = C(J) \cap \Delta_n$$
is a simplicial complex with $I_\Delta^{\mathbb{Q}} = J$.
3) Check that the monomial ideal
$$I \subset K[x_0, \ldots, x_5]$$
above corresponds to a triangulation of $\mathbb{P}^2(\mathbb{R})$. Hint: Compute a primary decomposition of I.

The fact that the homology of $\mathbb{P}^2(\mathbb{R})$ with coefficients in K is different for $\mathrm{char}(K) = 2$ explains Example 1) by Hochster's theory.

Exercise 33. Prove that the Segre product $\mathbb{P}^n \times \mathbb{P}^m \subset \mathbb{P}^N$ with $N = (n+1)(m+1) - 1$ has dimension $\dim \mathbb{P}^n \times \mathbb{P}^m = n + m$ and degree $\deg \mathbb{P}^n \times \mathbb{P}^m = \binom{n+m}{n}$.

Exercise 34. Consider the algebraic set $S(e,d) \subset \mathbb{P}^{d+e+1}$ for $d, e \geq 1$ defined by the 2×2-minors of the matrix
$$\begin{pmatrix} x_0 & x_1 & \ldots & x_{d-1} & y_0 & y_1 & \cdots & y_{e-1} \\ x_1 & x_2 & \ldots & x_d & y_1 & y_2 & \cdots & y_e \end{pmatrix}$$
where $x_0 \ldots y_e$ are the homogeneous coordinates on $\mathbb{P}^{d+e+1}$.

1) Prove that there exists a morphism $\pi : S(d,e) \to \mathbb{P}^1$ whose fibers are lines.
2) Let $\phi_1 : \mathbb{P}^1 \to \mathbb{P}^d = V(y_0, \dots, y_e)$ and $\phi_2 : \mathbb{P}^1 \to \mathbb{P}^e = V(x_0, \dots, x_d) \subset \mathbb{P}^{d+e+1}$ be the parametrisation of the rational normal curve of degree d and e in disjoint linear subspaces $\mathbb{P}^d \cup \mathbb{P}^e \subset \mathbb{P}^{d+e+1}$. Prove
$$S(e,d) \cong \bigcup_{p \in \mathbb{P}^1} \overline{\phi_1(p)\phi_2(p)}$$
where $\overline{\phi_1(p)\phi_2(p)}$ denotes the line joining $\phi_1(p)$ and $\phi_2(p)$.

Fourth week

Exercise 35. Let $\varphi : X \to Y$ be a projective morphism. Then
$$A_r = \{q \in Y \mid \dim X_q \geq r\} \subset Y$$
is a Zariski-closed subset of Y. Suppose that X and Y are varieties and that φ is surjective. Prove that
$$\dim A_r + r < \dim X$$
for r with $r > \dim X - \dim Y$.

Exercise 36. Let $X \subset \mathbb{P}^n$ be a projective variety. The secant variety of X is
$$\mathrm{Sec}(X) = \overline{\bigcup_{p,q \in X^*, p \neq q} \overline{pq}} \subset \mathbb{P}^n$$
where $\overline{pq}$ is the line spanned by the two points and $X^* = X \setminus X_{sing}$ is the open set of smooth points. Prove:

1) $$\dim \mathrm{Sec}(X) \leq 2 \dim X + 1.$$
2) If X is smooth and p a point not contained in $\mathrm{Sec}(X)$, then the projection
$$\pi_p : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$$
from p induces an isomorphism $X \cong \pi_p(X)$.

Conclude:

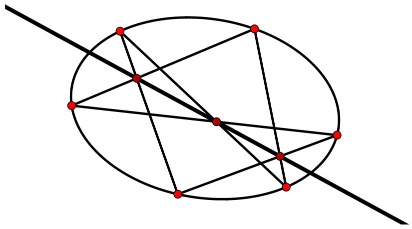
3) Every irreducible smooth projective curve can be embedded into $\mathbb{P}^3$.
4) Every irreducible smooth projective curve has a birational model in $\mathbb{P}^2$ with only nodes as singularities.

Exercise 37. Let $V_2 = \rho_{2,2}(\mathbb{P}^2) \subset \mathbb{P}^5$ be the Veronese surface, and let $p \in \mathbb{P}^5$ be a general point. Prove that the projection $\pi_p : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ induces an isomorphism
$$V_2 \cong \pi_p(V_2) \subset \mathbb{P}^4.$$
A famous result of Severi (1901) says: A smooth surface $X \subset \mathbb{P}^5$ has no isomorphic projection from a point into $\mathbb{P}^4$ unless X lies in a hyperplane or X is projectively equivalent to the Veronese surface.

Exercise 38. Let $L_1 \cup L_2 \cup L_3 \cup L_4 \subset \mathbb{P}^3$ be four general lines. Prove: Counted with multiplicities there are exactly two lines $L \subset \mathbb{P}^3$ which intersects all four lines.

Hint: Take the special case $L_1 = V(w,x), L_2 = V(y,z)$ and $L_3 = V(w+y, x+z)$ and prove that $L_1 \cup L_2 \cup L_3$ lies in a unique quadric hypersurface $Q \subset \mathbb{P}^3$ isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Exercise 39. Consider a conic $C \subset \mathbb{P}^2$ and six different points $p_1, \dots, p_6$ on C. Prove Pascal's theorem: *The opposite sides of the hexagon* $L_{12} = \overline{p_1p_2}, L_{23} = \overline{p_2p_3}, \dots, L_{56} = \overline{p_5p_6}, L_{61} = \overline{p_5p_6}$ *intersect in three points* $q_1 = L_{12} \cap L_{45}, q_2 = L_{23} \cap L_{56}, q_3 = L_{34} \cap L_{61}$ *which lie on a line.*

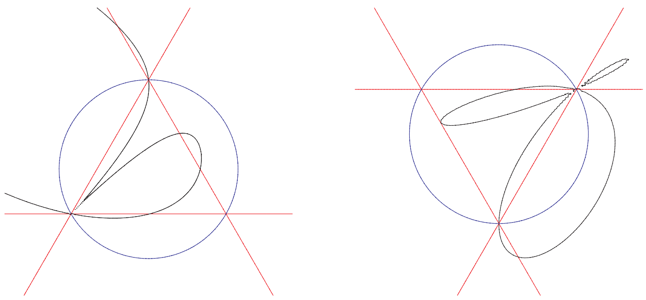


Hint: Consider the pencil of cubics

$$V(t_0 f + t_1 g) \subset \mathbb{P}^2 \text{ with } [t_0 : t_1] \in \mathbb{P}^1$$

where $f = \ell_{12}\ell_{34}\ell_{56}$ and $g = \ell_{23}\ell_{45}\ell_{61}$ are products of the equation ℓ_{ij} of L_{ij}.

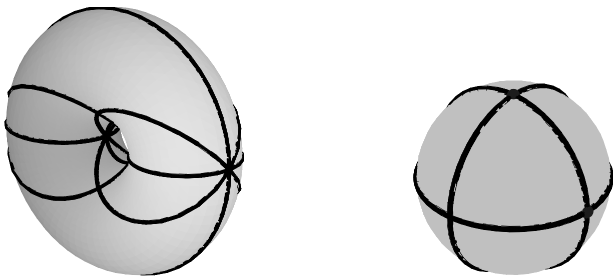
Exercise 40. Find the affine equations of the pair of quartics with non-ordinary respectively ordinary triple point related by a Cremona transformation as in the example from Lecture 20.



Exercise 41. Consider over $K = \mathbb{C}$ the projective closure E of the curve

$$V(y^2 - x^3 + x) \subset \mathbb{A}^2 \subset \mathbb{P}^2$$

and the projection $\pi : E \to \mathbb{P}^1$ from the point $o \in E$ at infinity onto the x-axis. Then π has degree 2 and branch points in $\{0, 1, -1, \infty\}$. Triangulate $\mathbb{P}^1 \approx S^2$ like an octahedron with $\{0, 1, -1, i, -i, \infty, \}$ as vertices. Describe the induced triangulation of E!



Exercise 42. Let $C \subset \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ be a curve defined by a form of bi-degree (d, e). Prove that C has degree $\deg C = d + e$ and arithmetic genus $p_a = (d-1)(e-1)$.