UNIVERSITÄT DES SAARLANDES
Fachrichtung 6.1 - Mathematik
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Exercises Algebraic Geometry

Winterterm 2016/17

The solutions are collected on Tuesday, before the exercise session.
All further informations concerning the lecture can be found here: https://www.math.uni-sb.de/ag/schreyer/index.php/teaching

Sheet 8 02.01.2017

Exercise 1 (4.3.10)**.** Let R be a local Noetherian integral domain with maximal ideal $\mathfrak{m}$. Suppose that R contains a field L such that the composite map $L \to R \to R/\mathfrak{m}$ is an isomorphism. Then all quotients $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ are L-vector spaces. In this situation, show that R is a DVR iff one of the following two equivalent conditions holds:

(1) $\dim_L \mathfrak{m}^k/\mathfrak{m}^{k+1} = 1$ for all $k \geq 0$;
(2) $\dim_L R/\mathfrak{m}^k = k$ for all $k \geq 1$.

Exercise 2 (4.3.19)**.** Let $f \in \Bbbk[x, y]$ be a square-free polynomial, let $C = \mathrm{V}(f) \subset \mathbb{A}^2$ be the corresponding plane curve, and let $p \in C$ be a point.

(1) Suppose that p is a double point at which C has precisely one tangent line L. Show that, then, $i(C, L; p) \geq 3$. We say that p is a **cusp** of C if $i(C, L; p) = 3$.
(2) If p is the origin, and L is the x-axis, show that p is a cusp of C with tangent line L iff f is of type $f = ay^2 + bx^3 +$ other terms of degree ≥ 3, where $ab \neq 0$. □

Exercise 3 (4.4.6)**.** Let R be a ring and $\mathfrak{m} \subset R$ an ideal. Show that the $\mathfrak{m}$-adic topology is Hausdorff if $\bigcap_{k=0}^{\infty} \mathfrak{m}^k = 0$

Exercise 4 (4.4.9)**.** Let S be a ring which is complete with respect to some ideal $\mathfrak{m}$. Given $s_1, \ldots, s_n \in \mathfrak{m}$, show that there exists a unique homomorphism $\Phi : \Bbbk[[x_1, \ldots, x_n]] \to S$ such that $\Phi(x_i) = s_i$ for all i. In fact, Φ is the map which sends a a power series f to the series $f(s_1, \ldots, s_n) \in S$. As in the polynomial case, we refer to Φ as a **substitution homomorphism**.