Computer Algebra and Gröbner Bases

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Overview

A phenomenon unique to algebraic geometry is that algebraic sets occur naturally in families, which themselves carry structure of an algebraic set. The main point is that we can vary the coefficients of the defining equations.

1. The space of hypersurfaces of degree d
2. Linear systems of plane curves
3. Grassmanians

The family of hypersurfaces

The family of hypersurfaces $X \subset \mathbb{P}^n$ of degree d is a projective space:

Proposition. *Let $L(n,d) = K[x_0,\ldots,x_n]_d$ denote the K-vector space of polynomials of degree d. Then the $\binom{d+n}{n} - 1$-dimensional projective space $\mathbb{P}(L(n,d))$ is in bijection with the set of hypersurfaces of degree d.*

Proof. The equation f of an hypersurface $X = V(f)$ is uniquely determined up to a scalar at least in case that f has no multiple factors. In particular, the set

$$\{X \subset \mathbb{P}^n \mid X \text{ is an irreducible hypersurface of degree } d\}$$

is in bijection with a Zariski open subset of $\mathbb{P}(L(n,d))$. This gives this set the structure of a quasi-projective variety. We consider the projective space of all equations for simplicity, since the set of reducible or not square free polynomials has a complicated structure. □

The families of reducible hypersurfaces

Remark. Let $d = d_1 + d_2$. The set of reducible hypersurface

$$\{[f] \in \mathbb{P}(L(n,d)) \mid f = f_1 f_2 \text{ with } \deg f_i = d_i\}$$

is the image of the morphism

$$\mathbb{P}(L(n,d_1)) \times \mathbb{P}(L(n,d_2)) \to \mathbb{P}(L(n,d)), ([f_1],[f_2]) \mapsto [f_1 f_2].$$

Hence it is a projective variety. In case of $d_1 \neq d_2$ it is a birational linear projection from the Segre embedding of $\mathbb{P}(L(n,d_1)) \times \mathbb{P}(L(n,d_2))$ hence of large degree.

Example. The map

$$\mathbb{P}(L(n,1)) \to \mathbb{P}(L(n,d)), [\ell] \mapsto [\ell^d]$$

can be identified with the Veronese embedding

$$\rho_{n,d} : \mathbb{P}^n \to \mathbb{P}^{\binom{d+n}{n}-1}.$$

$V_{2,2}$ revisited

In the special case of plane conics we have the following:
We write the equation of a plane conic in the form

$$q(x,y,z) = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} w_0 & w_1 & w_3 \\ w_1 & w_2 & w_4 \\ w_3 & w_4 & w_5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and identify $\mathbb{P}^5 = \mathbb{P}(L(2,2))$. Then the Veronese surface $V_{2,2} \subset \mathbb{P}^5$ corresponds to the squares of linear forms, i.e., to **double lines**, and

$$V(\det \begin{pmatrix} w_0 & w_1 & w_3 \\ w_1 & w_2 & w_4 \\ w_3 & w_4 & w_5 \end{pmatrix}) \subset \mathbb{P}^5$$

corresponds to the set of reducible conics, i.e., to **pairs of lines**.

Linear systems of hypersurfaces

Definition. A **linear system** of hypersurfaces is a projective space $\mathbb{P}(L) \subset \mathbb{P}(L(n,d))$ for a linear subspace $L \subset L(n,d)$. We speak of a **pencil** if $\mathbb{P}(L) \cong \mathbb{P}^1$. A **net** or **web** is a linear system of dimension 2 and 3 respectively.

Example. A pencil of conics contains counted with multiplicity precisely three reducible conics unless all conics are reducible because

$$\deg \det \begin{pmatrix} w_0 & w_1 & w_3 \\ w_1 & w_2 & w_4 \\ w_3 & w_4 & w_5 \end{pmatrix} = 3.$$

A general net of conics contains no double lines because a general $\mathbb{P}^2 \subset \mathbb{P}^5$ does not intersect the Veronese surface $V_{2,2} \subset \mathbb{P}^5$.

Linear systems of plane curves

In the following we study linear systems of plane curves and abbreviate our notation:

$$L(d) = L(2, d) \quad (= K[x, y, z]_d).$$

Definition. Let $\mathbb{P}(L) \subset \mathbb{P}(L(d))$ be a linear system of plane curves. A point $p \in \mathbb{P}^2$ is called a **base point** of $\mathbb{P}(L)$ if $p \in V(f)$ for all $f \in L(d)$.
Let $p_1, \ldots, p_s \in \mathbb{P}^2$ be distinct points and let $r_1, \ldots, r_s$ be positive integers. Then we set

$$L(d; r_1 p_1, \ldots, r_s p_s) := \{f \in L(d) \mid f \text{ has multiplicity } r_i \text{ at } p_i \ \forall i\}.$$

$\mathbb{P}(L(d; r_1 p_1, \ldots, r_s p_s))$ is called the linear system of plane curves of degree d with **assigned base points** p_i of **multiplicity** r_i.

Dimensions of linear systems of plane curves

Proposition. *Let* $p_1, \dots, p_s \in \mathbb{P}^2$ *be distinct points and let* $r_1, \dots, r_s$ *be positive integers. Then*

$$\dim_K L(d; r_1 p_1, \dots, r_s p_s) \geq \binom{d+2}{2} - \sum_{i=1}^{s} \binom{r_i+1}{2}$$

and equality holds if $d > \sum_{i=1}^{s} r_i$*.*

Proof. Since $L(d; r_1 p_1, \dots, r_s p_s) = \bigcap_{i=1}^{s} L(d; r_i p_i)$ it suffices to prove that

$$L(d; rp) \subset L(d)$$

has codimension $\binom{r+1}{2}$ for the first statement. If $p = [0:0:1]$, then $f \in L(d; rp)$ iff in the affine equation

$$f(x_1, x_2, 1) = \sum_{|\alpha| \leq d} f_\alpha x^\alpha$$

the coefficients f_α vanish for $|\alpha| \leq r$. These are $\binom{r+1}{2}$ coefficients.

Dimensions of linear systems of plane curves

The second statement is proved by induction on $\sum r_i$. The key step is to prove that $L(d; r_1 p_1, \dots, r_s p_s) \subset L(d; (r_1 - 1)p_1, \dots, r_s p_s)$ has the maximal possible codimension $r_1 + 1$ in case $d > \sum_{i=1}^{s} r_i$. We leave this as an exercise. □

Example. The inequality might be strict if the points lie in special position. For example in case $p_1, \dots, p_4$ lie on a line we have

$$\dim \mathbb{P}(L(2; p_1, \dots, p_4)) = 2.$$

In all other cases $\mathbb{P}(L(2; p_1, \dots, p_4))$ is a pencil as expected $\binom{2+2}{2} - 4 \cdot 1 - 1 = 1$.

Dimensions of linear systems of plane curves in case of general points

$L(d; r_1 p_1, \ldots, r_s p_s) \subset L(d)$ is defined by a linear system of equations whose coefficients are polynomials in the coordinates $[a_i : b_i : c_i]$ of p_i. Thus there exists an open subset

$$U \subset \mathbb{P}^2 \times \ldots \times \mathbb{P}^2$$

of the product of s copies of $\mathbb{P}^2$ such that $\dim_K L(d; r_1 p_1, \ldots, r_s p_s)$ takes its minimal value for all tuples $(p_1, \ldots, p_s) \in U$.

The minimal value can be larger than $\binom{d+2}{2} - \sum \binom{r_i+1}{2}$.

Example. $\dim L(4; 2p_1, \ldots, 2p_5) \geq 1$ although $\binom{4+2}{2} - 5 \cdot 3 = 0$. The reason is that $L(2; p_1, \ldots, p_5) \geq 1$ and the equation q of a conic through the five points yields a non-zero quartic $q^2 \in L(4; 2p_1, \ldots, 2p_5)$.

Dimensions of linear systems in case of simple base points.

It is on-going reseach to characterize those multiplicities $r_1, \ldots, r_s$ for which $\dim L(d; r_1 p_1, \ldots, r_s p_s) = \binom{d+2}{2} - \sum_{i=1}^{s} \binom{r_i+1}{2}$ holds for a general collection of points $p_1, \ldots, p_s$. Ciro Ciliberto and Rick Miranda are leading experts in this line of research.

Proposition. *Let $p_1, \ldots, p_s$ be a general tuple of points in $\mathbb{P}^2$. Then*

$$\dim L(d; p_1, \ldots, p_s) = \binom{d+2}{2} - s$$

as long as the right hand side is non-negative.

Proof. We have to prove that the Zariski-open subset $U \subset \mathbb{P}^2 \times \ldots \times \mathbb{P}^2$ where equality holds is non-empty. Suppose $\dim_K L(d; p_1, \ldots, p_{s-1}) \neq 0$. Choose a non-zero $f \in L(d; p_1, \ldots, p_{s-1})$ and a point $p_s \notin V(f)$. Then $L(d; p_1, \ldots, p_s) \subset L(d; p_1, \ldots, p_{s-1})$ has (the maximal possible) codimension 1, and $U \neq \emptyset$ follows by induction on s. $\square$

The Grassmannian

We now turn to the description of families of varieties of larger codimension. The first interesting case is perhaps the family of lines in $\mathbb{P}^3$ or, equivalently, two dimensional subvector spaces $W \subset K^4$.

Definition. Let $1 \leq d < n$ be two integers. As a set we define the Grassmannian

$$\mathbb{G}(d, n) = \{W \subset K^n \mid W \text{ is a subvector space of dimension } d\}.$$

If $M = \{A = (a_{ij}) \in K^{d \times n} \mid \operatorname{rank} A = d\} \subset \mathbb{A}^{dn}$ denotes the quasi-affine variety of $d \times n$ matrices of maximal rank d, then we can identify

$$\mathbb{G}(d, n) = M / \operatorname{GL}(d, K)$$

with the set of orbits under the action

$$\operatorname{GL}(d, n) \times M \to M, (B, A) \mapsto BA.$$

The Grassmannian

Indeed, map

$$M \to \mathbb{G}(d, n)$$

which maps the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{d1} & a_{d2} & \dots & a_{dn} \end{pmatrix}$$

to subspace $W \subset K^n$ spanned by the rows of A is surjective and the fibers correspond to the choices of a basis of W, i.e., to the points of in the orbit $\mathrm{GL}(d, K)A \subset M$.

To give $\mathbb{G}(d, n)$ the structure of a projective variety we consider the Plücker embedding. For a subset

$$I = \{i_1 < i_2 < \ldots < i_d\} \subset \{1, \ldots, n\}$$

of d elements we denote by A_I the $d \times d$-submatrix of A with columns i_k for $k = 1, \ldots, d$.

The Plücker embedding

Consider the map

$$\gamma : \mathbb{G}(d,n) \to \mathbb{P}^{\binom{n}{d}-1}, \quad [A] \mapsto [\det A_I]$$

induces by all $d \times d$-minors of A. This induces a well-defined map because the $d \times d$ minors of A and BA differ by the common factor $\det B \in K^*$ since $\det(BA)_I = \det B \det A_I$ and at least one minor is non-zero because $\operatorname{rank} A = d$

In algebraic terms we have a variable p_I in the homogeneous coordinate ring of $\mathbb{P}^{\binom{n}{d}-1}$ and we define $\mathbb{G}(d,n) \subset \mathbb{P}^{\binom{n}{d}-1}$ as the projective variety defined by the ideal $\ker(\gamma^*)$ of the ring homomorphism

$$\gamma^* : K[p_I] \to K[a_{ij}],\ p_I \mapsto \det A_I.$$

Proposition. *The affine charts $U_I = \{p_I \neq 0\}$ of $\mathbb{P}^{\binom{n}{d}-1}$ intersect the Gassmannian in affine varieties $\mathbb{G}(d,n) \cap U_I \cong \mathbb{A}^{d(n-d)}$. In particular it is a smooth projective variety of dimension $d(n-d)$.*

The charts of the Grassmannian

Proof. We consider $U_I \cap \mathbb{G}(d,n)$ for $I = \{1,\dots,d\}$. A points of $\gamma^{-1}(U_I)$ are represented by a matrices A' with $\det A'_I \neq 0$. Thus we have a distinguished representative $A = (A'_I)^{-1}A'$ of shape

$$A = \begin{pmatrix} 1 & & 0 & a_{1,d+1} & \dots & a_{1n} \\ & \ddots & & \vdots & & \vdots \\ 0 & & 1 & a_{d,d+1} & \dots & a_{dn} \end{pmatrix}.$$

On this chart we have

$$p_{(\{1,\dots,d\}\setminus\{i\})\cup\{j\}} = (-1)^{d-j} a_{ij} \text{ for } j > d.$$

Every Plücker coordinates $\det A_J$ is a polynomial in the a_{ij} with $j > d$. Thus interpreting these a_{ij}'s in terms of the Plücker coordinates $p_{(\{1,\dots,d\}\setminus\{i\})\cup\{j\}}$ above and homogenizing with respect to $p_{\{1,\dots,d\}}$ we obtain elements of $\ker(\gamma^*)$ which show that $\mathbb{G}(d,n) \cap U_{\{1,\dots,d\}}$ is isomorphic to $\mathbb{A}^{d(n-d)}$. The arguments in other charts is analogous. In particular we see that $\mathbb{G}(n,d)$ is covered by $\binom{n}{d}$ charts which are all needed to cover $\mathbb{G}(d,n)$. □

The Plücker quadric

$\mathbb{G}(2,4) \subset \mathbb{P}^5$ is a hypersurface. It is actually a quadric. In terms of coordinates $p_{12}, \ldots, p_{34}$ the ideal is generated by the Plücker quadric

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}.$$

We can see that this equations is satisfied for the minors of the 2×4 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$$

by expanding the determinant

$$0 = \det \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$$

with respect to the first two rows:

$$0 = \det A_{12} \det A_{34} - \det A_{13} \det A_{14} + \ldots + \det A_{34} \det A_{12}$$

Thus $2(\det A_{12} \det A_{34} - \det A_{13} A_{24} + \det A_{14} A_{23}) = 0 \in \mathbb{Z}[a_{ij}]$.

Stratification of the Grassmannians

$\mathbb{P}^n = \mathbb{G}(1, n+1)$ has a stratification by affine strata:

$$\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1} = \mathbb{A}^n \cup \mathbb{A}^{n-1} \cup \ldots \cup \mathbb{A}^1 \cup \mathbb{A}^0.$$

A similar stratification exists for the Grassmannians. We describe this for the case $\mathbb{G}(2, 4)$.

$$S_{12} = \{\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}\} \cong \mathbb{A}^4$$

$$S_{13} = \{\begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}\} \cong \mathbb{A}^3$$

$$S_{14} = \{\begin{pmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\} \cong \mathbb{A}^2 \qquad S_{23} = \{\begin{pmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}\} \cong \mathbb{A}^2$$

$$S_{24} = \{\begin{pmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\} \cong \mathbb{A}^1$$

$$S_{34} = \{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\} \cong \mathbb{A}^0$$

Stratification of the Grassmannians

In which strata a point $[A] \in \mathbb{G}(d, n)$ lies depends on the row echelon form of the matrix $A \in K^{d \times n}$. The closure of the strata

$$\overline{S_{i_1 \dots i_d}}$$

are called **Schubert varieties**. They are intensely studied objects of algebraic geometry. The closure of the strata can also characterized by how the corresponding linear subspace intersect the subspaces a complete flag of linear subspaces.
We illustrate this in case of $\mathbb{G}(2, 4)$. Consider the flag

$$p_0 \subset L_0 \subset P_0 \subset \mathbb{P}^3$$

of the point, line and plane defined by

$$V(x_0, x_1, x_2) \subset V(x_0, x_1) \subset V(x_0) \subset \mathbb{P}^3.$$

Stratification of the Grassmannian $\mathbb{G}(2,4)$

$$\overline{S_{12}} = \mathbb{G}(2,4)$$

$$\overline{S_{13}} = \{L \in \mathbb{G}(2,4) \mid L \cap L_0 \neq \emptyset\}$$

$$\overline{S_{14}} = \{L \mid p_0 \in L\} \qquad \overline{S_{23}} = \{L \mid L \subset P_0\}$$

$$\overline{S_{24}} = \{L \mid p_0 \in L \subset P_0\} \cong \mathbb{P}^1$$

$$\overline{S_{34}} = \{L_0\}$$

Corollary. *The set of lines L in affine three space is the quasi projective variety*

$$\{L \subset \mathbb{A}^3 \mid L \textit{ is a line}\} = \mathbb{G}(2,4) \setminus \overline{S_{23}}$$

where $\overline{S_{23}} = \mathbb{P}^2$ is the space of lines contained in the plane at infinity of $\mathbb{P}^3$. □

Schubert calculus

Hermann Schubert (1848-1911) developed a general machinery to solve enumerative problems. For example: How many lines intersect four given lines in $\mathbb{P}^3$. Here is how Schubert would argue. Take $L_1, \ldots, L_4$ into special position. Then there are visibly precisely 2 lines intersecting all four.

One of Schubert's most famous result is that 5 general conics are tangent to precisely 3264 smooth conics. To put Schubert calculus on a solid foundation was Hilbert's 15^{th} problem.