Computer Algebra and Gröbner Bases

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Overview

Today I will make few remarks about the Hilbert

1. The Hilbert scheme
2. Smooth and singular points
3. Bertini's theorem
4. Grassmanians

Subschemes of $\mathbb{A}^n$

We have frequently seen that non-radical ideals occur naturally.

Example. Consider the ideal $(x^2, xy) \subset K[x, y, z]$. It arises if we intersect the algebraic set $V(xz, yz)$ with the hyperplane $V(z - x)$. On the algebra side we have

$$K[x, y, z]/(xz, yz, z - x) \cong K[x, y]/(x^2, xy)$$

corresponding to this intersection.

It is thus naturally to extend our algebra geometry dictionary beyond radical ideals as follows:

$$\{\text{radical ideals of} K[x_1, \dots, x_n]\} \overset{1:1}{\longleftrightarrow} \{\text{algebraic subsets of} \mathbb{A}^n\}$$

$$\{\text{arbitrary ideals of} K[x_1, \dots, x_n]\} \overset{1:1}{\longleftrightarrow} \{\text{subsschemes of } \mathbb{A}^n\}.$$

The **coordinate ring** of the subscheme $X \subset \mathbb{A}^n$ corresponding to the ideal $J \subset K[x_1, \dots, x_n]$ is the ring $K[X] = K[x_1, \dots, x_n]/J$.

Morphism and local rings of a subschemes of $\mathbb{A}^n$

A morphism $\varphi : X \to Y$ between a subscheme $X \subset \mathbb{A}^n$ and a subscheme $Y \subset \mathbb{A}^m$ are by definition a ring homomorphism

$$\varphi^* : K[Y] \to K[X].$$

The local ring $\mathcal{O}_{X,p}$ of a subscheme at a point $p \in V(J)$ can be defined as the localization in the corresponding maximal ideal $\mathfrak{m}$ of $K[X]$:

$$\mathcal{O}_{X,p} = K[X]_{\mathfrak{m}}.$$

The only disadvantage when considering subschemes, is that $K[X]$ can no longer be regarded as a subring of the functions

$$\{f : V(J) \to K\}.$$

In our example above with $K[X] = K[x, y]/(x^2, xy)$ the element $\overline{x} \in K[X]$ would correspond to the zero function, because $(\overline{x})^2 = 0$. But $\overline{x} \neq 0 \in K[X]$. We allow non-trivial nilpotent elements in the coordinate ring of a subscheme.

Subschemes of $\mathbb{P}^n$

Definition. A **subscheme** X of $\mathbb{P}^n$ is a collection of subschemes in the affine charts $U_i \cong \mathbb{A}^n$ that coincide in their intersections $U_i \cap U_j$.

Definition. A homogeneous ideal $I \subset K[x_0, \ldots, x_n]$ is **saturated** if it coincides with its **saturation**

$$I_{sat} = I : (x_0, \ldots, x_n)^\infty = \bigcup_{N=1}^{\infty} (I : (x_0, \ldots, x_n)^N).$$

Proposition. *There is a bijection between saturated homogeneous ideals of $K[x_0, \ldots, x_n]$ and subschemes of $\mathbb{P}^n$.*

Remark. A homogeneous ideal I is saturated iff the irrelevant ideal $(x_0, \ldots, x_n)$ is not an associated prime ideal of I.

The graded ring $K[X] = K[x_0, \ldots, x_n]/I_{sat}$ is the called the **homogeneous coordinate ring** of the subscheme $X \subset \mathbb{P}^n$ corresponding to the subscheme defined by I. The **Hilbert polynomial** $p_X \in \mathbb{Q}[t]$ is the Hilbert polynomial of $K[X]$.

The Hilbert scheme

A fundamental result in algebraic geometry due Grothendiek is the following.

Theorem. *Let* $p \in \mathbb{Q}[t]$ *be a polynomial of degree* $< n$. *The set of subschemes*

$$\mathrm{Hilb}_p(\mathbb{P}^n) = \{X \subset \mathbb{P}^n \mid p_X = p\}$$

carries the structure of a projective scheme.

Idea of the proof.

Step 1. Let $p \in \mathbb{Q}[t]$. There exists an integer r such that the saturated ideal $I = I_X$ of any subscheme $X \subset \mathbb{P}^n$ with $p_X = p$ satisfies the following:

$$I_d \subset K[x_0, \ldots, x_n]_d \text{ has codimension } p(d) \text{ for all } d \geq r. \qquad (\dagger)$$

Key point of this step is that the bound r depends only on p. This is proved with the concept of the Castelnuovo-Mumford regularity, which builds upon coherent sheaves and their cohomology.

The Hilbert scheme

As additional property one obtains

$$I_{\geq d} = \bigoplus_{d' \geq d} I_{d'} = (I_d) \text{ for } d \geq r. \tag{‡}$$

Remark. Since a homogeneous ideal J and $J \cap (x_0, \ldots, x_n)^N$ have the same Hilbert polynomial the statements (†) and (‡) fail if we consider non-saturated ideals.

Step 2. We fix now an $d \geq r$. For each polynomial $p \in \mathbb{Q}[t]$ we thus have an injection

$$\{X \subset \mathbb{P}^n \mid p_X = p\} \hookrightarrow \mathbb{G}(\binom{n+d}{n} - p(d), \binom{d+n}{n})$$

into a Grassmannian by mapping the subscheme X to the degree d part of ist homogenous ideal I_X

$$X \mapsto (I_X)_d \subset K[x_0, \ldots, x_n]_d.$$

This is a codimension $p(d)$ subvectorspace, hence a point in the Grassmannian above.

The scheme structure of the Hilbert scheme

To get equations of the Hilbert scheme, we note that a subspace W lies in the image if and only if the ideal (W) generated by W has Hilbert function values $h_{(W)}(d') = \binom{n+d'}{n} - p(d')$ for $d' \geq d$.

Representing W by a $(\binom{n+d}{d} - p(d)) \times \binom{n+d}{n}$ coefficient matrix, we see that for $d' = d+1$ this gives the condition that a

$$(n+1)(\binom{n+d}{n} - p(d)) \times \binom{n+d+1}{n}\text{-matrix}$$

in these coefficients has rank precisely $\binom{n+d+1}{n} - p(d+1)$. Thus the corresponding minors give equations for the Hilbert scheme.

We leave it open to explain why the rank for any W satisfying these equations cannot be smaller, and why these equations define the Hilbert scheme correctly from a conceptual point of view. The key concept for the later is the notion of a representable functor.

$\mathrm{Hilb}_{3t+1}(\mathbb{P}^3)$

$p(t) = 3t + 1$ is the Hilbert polynomial of the rational normal curve in $\mathbb{P}^3$. We give a description of the open part of this Hilbert scheme of all saturated ideals with lead ideal $\mathrm{Lt}(I) = (x_0^2, x_0x_1, x_1^2)$. If I is such an ideal then its equation take the form

$$x_0^2 + a_1x_0x_2 + a_2x_1x_2 + a_3x_0x_3 + a_4x_1x_3 + a_5x_2^2 + a_6x_2x_3 + a_7x_3^2$$
$$x_0x_1 + b_1x_0x_2 + b_2x_1x_2 + b_3x_0x_3 + b_4x_1x_3 + b_5x_2^2 + b_6x_2x_3 + b_7x_3^2$$
$$x_1^2 + c_1x_0x_2 + c_2x_1x_2 + c_3x_0x_3 + c_4x_1x_3 + c_5x_2^2 + c_6x_2x_3 + c_7x_3^2$$

with coefficients $a_1, \ldots, c_7$. The Buchberger test gives $2 \cdot 10$ equations for the coefficients obtained from the possible 10 different remainders of the division by x_0^2, x_0x_1, x_1^2 in degree 3. As it turns out the ideal is generated by only nine equations

$$a_5 + a_2b_1 - a_1b_2 + b_2^2 - a_2c_2, \ldots, c_7 + b_3^2 - a_3c_3 + b_4c_3 - b_3c_4$$

which say that $a_5, \ldots, c_7$ are quadratic polynomials in $a_1, \ldots, c_4$.

$\mathrm{Hilb}_{3t+1}(\mathbb{P}^3)$

To phrase this differently: The open subset of $\mathrm{Hilb}_{3t+1}(\mathbb{P}^3)$ corresponding to subschemes with lead ideal $\mathrm{Lt}(I) = (x_0^2, x_0x_1, x_1^2)$ is isomorphic to a $\mathbb{A}^{12}$ and the best way to phrase the equations in the coefficients is to say that the three generators of I can be written as the 2×2-minors of the matrix

$$\begin{pmatrix} -x_1 - x_2b_1 - x_3b_3 & -x_2c_1 - x_3c_3 \\ x_0 + x_2a_1 + x_3a_3 - x_2b_2 - x_3b_4 & -x_1 + x_2b_1 + x_3b_3 - x_2c_2 - x_3c_4 \\ x_2a_2 + x_3a_4 & x_0 + x_2b_2 + x_3b_4 \end{pmatrix}.$$

Indeed, for example the coefficients of a_5 of x_2^2 in the minor with lead term x_0^2 is $a_5 = b_2(a_1 - b_2) - a_2(b_1 - c_2)$.

Thus $\mathrm{Hilb}_{3t+1}(\mathbb{P}^3)$ has a 12-dimensional component in which the $\mathrm{PGL}(4, K)$-orbit of the rational normal curve is dense. Note that

$$\dim \mathrm{PGL}(4, K) - \dim \mathrm{PGL}(2, K) = 15 - 3 = 12$$

Components of $\mathrm{Hilb}_{3t+1}(\mathbb{P}^3)$

Actually, there are two components:

$$\mathrm{Hilb}_{3t+1}(\mathbb{P}^3) = H_{12} \cup H_{15},$$

the component of dimension 12 from above and a further component of dimension 15. A general subscheme $X \in H_{15}$ is the union $X = E \cup \{p\}$ of a plane cubic curve E and a disjoint point. The choice of E depends on $3 + 9$ parameters, since we have to specify a plane in $\mathbb{P}^3$ and a cubic equation in that plane. The choice of p gives further 3 parameters. This fits with $15 = 3 + 9 + 3$.

Intersection of the components of $\mathrm{Hilb}_{3t+1}(\mathbb{P}^3)$

To see the intersection of both components we look at the open chart of the Hilbert scheme of X with lead ideal

$$\mathrm{Lt}(I_X) = (x_0^2, x_0x_1, x_0x_2, x_1^3).$$

This time we introduce $3 \cdot 7 + 10$ variables for the possible coefficients. The Buchberger test computation, yields a system of equations for the coefficients. Its primary decomposition give two smooth components isomorphic to $\mathbb{A}^{12}$ and $\mathbb{A}^{15}$ intersecting in smooth subvariety isomorphic to $\mathbb{A}^{14}$. A general scheme corresponding to a point in the intersection consist of a nodal plane curve together with an embedded point at the node sticking out of the plane.

Initial ideals

Let $I = I_{sat}$ be a saturated homogeneous ideal in $K[x_0, \ldots, x_n]$. The Hilbert scheme allows us to interpret the association

$$I \rightsquigarrow \mathrm{Lt}(I)$$

in a more conceptual way. Let $>_w$ be a weight order which has the same lead terms as the given monomial order for the Gröbner basis of I. We may assume that the weight $(w_0, \ldots, w_n)$ are integers and all positive, since adding the vector $(1, \ldots, 1)$ does not change ordering for mononmials of a given degree. Consider the one parameter subgroup

$$T = \{ \begin{pmatrix} t^{w_0} & & 0 \\ & \ddots & \\ 0 & & t^{w_n} \end{pmatrix} \mid t \in K^* \}$$

of the group of diagonal matrices in $\mathrm{GL}(n+1, K)$.
We ask how the ideal moves under the substitution $x_i \mapsto t^{-w_i} x_i$.

The corresponding 1-parameter family of schemes

Let $f = \sum_{|\alpha|=d} f_\alpha x^\alpha \in I$ be a homogeneous element and let $f_\beta x^\beta = \mathrm{Lt}(f)$. Then

$$f^{(t)} = t^{<\beta,w>} f(t^{-w_0}x_0, \ldots, t^{-w_n}x_n) = \sum_\alpha f_\alpha t^{<\beta-\alpha,w>} x^\alpha \in K[t, x_0, \ldots, x$$

because $x^\beta >_w x^\alpha$ for all $\alpha \neq \beta$ with $f_\alpha \neq 0$. Now consider a Gröbner basis $f_1, \ldots, f_r$ of I and the subscheme

$$Y \subset \mathbb{P}^n \times \mathbb{A}^1$$

the ideal $J = (f_1^{(t)}, \ldots, f_r^{(t)}) \subset K[t, x_0, \ldots, x_n]$. The fibers Y_q for $q \in \mathbb{A}^1 \setminus \{o\}$ all isomorphic to $Y_1 = X$ our original scheme, and special fiber Y_0 is defined corresponds to $\mathrm{Lt}(I)$.
Thus

$$\lim_{t\to 0} Y_t = X_0 \in \mathrm{Hilb}_{p_X}(\mathbb{P}^n).$$

Degeneration to monomial ideals

Corollary. *Let* $I = I_{sat} \subset K[x_0, \ldots, x_n]$ *be homogeneous saturated ideal,* X *the corresponding subsscheme of* $\mathbb{P}^n$ *and* p_X *its Hilbert polynomial. The set of possible lead ideals*

$$\{\mathrm{Lt}_>(I) \mid \ > \ \textit{is a monomial order}\}$$

coincides with the set of monomial limits of X *in* $\mathrm{Hilb}_{p_X}(\mathbb{P}^n)$ *along one parameter subgroups of the torus*

$$\{\begin{pmatrix} a_0 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \mid a_i \in K^*\} \subset \mathrm{GL}(n+1, K).$$

□

A non-monomial limit

Consider the rational normal curve in $\mathbb{P}^3$ defined by the 2×2-minors of the matrix

$$\begin{pmatrix} x_3 & x_0 & x_1 \\ x_0 & x_1 + x_3 & x_2 \end{pmatrix}$$

Substituting $x_0 \mapsto x_0, x_1 \mapsto tx_1, x_2 \mapsto tx_2, x_3 \mapsto tx_3$ and computing the Gröbner basis of the ideal I_t with respect to the induced order we get

$x_0^2 - t^2 x_1 x_3 - t^2 x_3^2$	$-x_1$	$-x_2$	0	0
$x_0 x_1 - t\, x_2 x_3$	x_0	$tx_1 + tx_3$	$-x_2$	$-x_1^2 - x_1 x_3$
$x_0 x_2 - t\, x_1^2 - t\, x_1 x_3$	tx_3	x_0	x_1	$x_2 x_3$
$x_1^3 - x_2^2 x_3 + x_1^2 x_3$			t	x_0

A non-monomial limit

Thus

$$I_0 = (x_0^2, x_0x_1, x_0x_2, x_1^3 - x_2^2x_3 + x_1^2x_3) = (x_0, x_1^3 - x_2^2x_3 + x_1^2x_3) \cap (x_0^2, x_1, x_2)$$

a nodal plane curve with an embedded point at node

$$p = [0 : 0 : 0 : 1].$$

Stratification of the Hilbert scheme

Like the Grassmannian the Hilbert scheme allows a stratification. Fix a Hilbert polynomial p and a global monomial order, for example $>_{\mathrm{lex}}$ or $>_{\mathrm{rlex}}$. There are only finitely many saturated monomial ideals J with Hilbert polynomial p. Choose one of these monomial ideal J. Then the set

$$S_J = \{I \in \mathrm{Hilb}_p(\mathbb{P}^n) \mid \mathrm{Lt}(I) = J\}$$

of saturated ideals I_{sat} with $\mathrm{Lt}(I) = J$ are precisely those ideals which approach J under a suitable 1-parameter subgroup. The strata have a natural structure of an affine scheme.

A famous theorem of Hartshorne says that the Hilbert schemes $\mathrm{Hilb}_p(X)$ are connected. The proof builds upon the idea to connect these monomial ideals via a chain of 1-parameter families.