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UNIVERSITÄT DES SAARLANDES

Computer Algebra and Gröbner Bases

Winterterm 2020/21

All exercise sheets and course information can be found at: www.math.uni-sb.de/ag/schreyer/

Sheet 3 26. November 2020

Exercise 1. Prove that the algebraic sets $V(y-x^2)$ and $V(xy-1)$ in $\mathbb{A}^2$ are not isomorphic.

Exercise 2. An R-module M is called noetherian, if it satisfies analogous equivalent condition for submodules instead of ideals. Prove:

(1) Let
$$0 \longrightarrow M' \xrightarrow{\psi} M \xrightarrow{\varphi} M'' \to 0$$
be a short exact sequence of R-modules, i.e., the homomorphism ψ is injective, the homomorphism φ is surjective and $\ker\varphi = \operatorname{im}\psi$.
Then M is noetherian iff M' and M'' are noetherian.

(2) An R-module M over a noetherian ring R is noetherian iff M is finitely generated.

Exercise 3. Let R be a noetherian ring, let $\mathfrak{m}$ be a maximal ideal of R, and let I be any ideal of R. Show that the following are equivalent:

(1) I is $\mathfrak{m}$-primary.
(2) $\operatorname{rad}(I) = \mathfrak{m}$.
(3) $\mathfrak{m} \supset I \supset \mathfrak{m}^k$ for some $k \geq 1$.

Exercise 4.

(1) When is a monomial ideal a prime ideal?
(2) Characterize monomial primary ideals.
(3) Consider the monomial ideal $I = (xy, xz, yz) \subset \mathbb{Q}[x, y, z]$. Compute a primary decomposition of I and I^2.