Fakultät MI, Fachrichtung Mathematik
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UNIVERSITÄT DES SAARLANDES

Computer Algebra and Gröbner Bases

Winterterm 2020/21

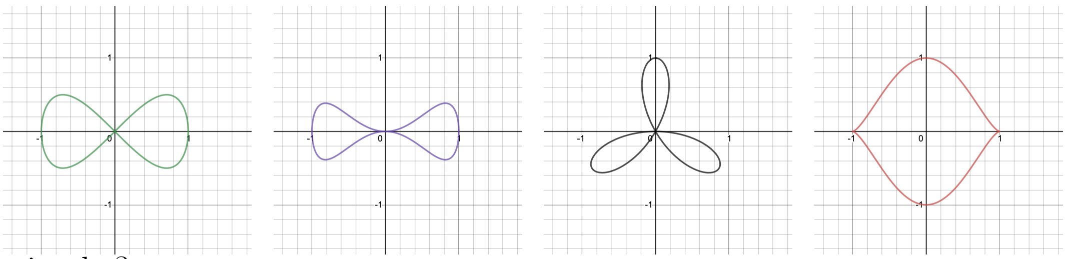
All exercise sheets and course information can be found at: www.math.uni-sb.de/ag/schreyer/

Sheet 6 7. January 2021

Exercise 1. Consider the plane curves defined by

$$y^2 = (1-x^2)^3, \quad y^2 = x^4 - x^6, \quad y^3 - 3x^2y = (x^2+y^2)^2, \quad y^2 = x^2 - x^4$$

Their real points are one of the following:



Who is who?

Exercise 2. Implement the computer algebra system Macaulay2
https://faculty.math.illinois.edu/Macaulay2/
on your machine.

Exercise 3. Describe the free resolution of $k \cong k[x_1, \ldots, x_n]/(x_1, \ldots, x_n)$ as an $K[x_1, \ldots, x_n]$-module.

Exercise 4.

(1) Let

$$\begin{aligned} f &= a_0x^d + a_1x^{d-1} + \ldots + a_d \\ g &= b_0x^d + b_1x^{d-1} + \ldots + b_e \end{aligned}$$

be two polynomials in $K[x]$ of degree d and e. Consider the $(d+e)\times(d+e)$ **Sylvester matrix**

$$\mathrm{Syl}(f,g) = \begin{pmatrix} a_0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\ a_1 & a_0 & & \vdots & b_1 & b_0 & & \vdots \\ \vdots & a_1 & \ddots & \vdots & \vdots & b_1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_0 & \vdots & \vdots & \ddots & b_0 \\ a_d & & & a_1 & b_e & & & b_1 \\ 0 & a_d & & \vdots & 0 & b_e & & \vdots \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_d & 0 & 0 & \cdots & b_e \end{pmatrix}$$

There e columns with entries the a_i's and d columns with entries b_j's. Prove f and g have a common root if and only if the **resultant**

$$\mathrm{Res}(f,g) = \det \mathrm{Syl}(f,g) = 0$$

of f and g vanishes.

(2) With notation similar as in Exercise 3, suppose that a_i and b_j are independent variables of degree i and j respectively. Prove that the resultant

$$\mathrm{Res}(f,g) \in \mathbb{Z}[a_i,b_j]$$

is a homogeneous polynomial of degree $d\cdot e$.