# COMBINATORIAL ASPECTS OF FREE PROBABILITY THEORY

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## 2. Combinatorial and Analytical Theory of Free Convolution and R-Transform

2.1. **From moments to probability measures.** Before we give a systematic combinatorial description of free independence let us make a general remark about the analytical side of our "distributions". For a random variable $a$ we have up to now considered its distribution in a very algebraic, combinatorial way, namely just as the collection of all its moments $\varphi(a^n)$. Of course, in classical probability theory distributions are much more analytical objects, namely probability measures. However, if $a$ is a selfadjoint bounded operator then we can identify its distribution in our algebraic sense with a distribution in classical sense; namely, there exists a compactly supported probability measure on $\mathbb{R}$, which we denote by $\mu_a$ and which is uniquely determined by

$$\varphi(a^n) = \int_{\mathbb{R}} t^n d\mu_a(t).$$

(This is more or less a combination of the Riesz representation theorem and Stone-Weierstrass.) Of course, the same is also true more general for normal operators $b$, then the $*$-distribution of $b$ can be identified with a probability measure $\mu_b$ on the spectrum of $b$, via

$$\varphi(b^m b^{*n}) = \int_{\sigma(b)} z^m \bar{z}^n d\mu_b(z) \qquad \text{for all } m, n \in \mathbb{N}.$$

This raises of course the question how we can determine effectively a probability measure out of its moments. The Stieltjes inversion formula for the Cauchy transform is a classical recipe for doing this in the case of a real probability measure.

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**Definition 2.1.** For a probability measure $\mu$ on $\mathbb{R}$ the Cauchy transform $G$ is defined by

$$G(z) := \int_{\mathbb{R}} \frac{1}{z-t} d\mu(t).$$

This is an analytic function in the upper complex half-plane.

If we observe that

$$G(z) = \sum_{n=0}^{\infty} \frac{\int_{\mathbb{R}} t^n d\mu(t)}{z^{n+1}}$$

(and, for compactly supported $\mu$, this sum converges for sufficiently large $|z|$), then we see that the Cauchy transform of a measure is more or less the generating power series in its moments. So if we are given a sequence of moments $m_n$ ($n \in \mathbb{N}$), we build out of them their generating power series

$$M(z) = \sum_{n=0}^{\infty} m_n z^n,$$

and in many interesting cases one is able to calculate this $M(z)$ out of combinatorial information about the $m_n$. Thus we get, via

$$G(z) = \frac{1}{z} M(\frac{1}{z})$$

the Cauchy transform of the measure which is behind the moments. So the main question is whether we can recover a measure from its Cauchy transform. The affirmative answer is given by Stieltjes inversion theorem which says that

$$d\mu(t) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \Im G(t + i\varepsilon),$$

where $\Im$ stands for the operation of taking the imaginary part of a complex number. The latter limit is to be understood in the weak topology on the space of probability measures on $\mathbb{R}$.

*Example* 2.2. Let us use Stieltjes inversion formula to calculate the probability distribution of a semi-circular element $s$ of variance 1. Recall that

$$\varphi(s^k) = \begin{cases} c_n, & \text{if } k = 2n \\ 0, & \text{if } k \text{ odd.} \end{cases}$$

where $c_n$ are the Catalan numbers. Let us now see whether we can get the corresponding probability measure out of those moments. The recurrence relation

$$c_n = \sum_{k=1}^{n} c_{k-1} c_{n-k}$$

for the Catalan numbers results in the quadratic equation

$$G(z)^2 - zG(z) + 1 = 0$$

for the corresponding Cauchy transform $G$, which has the solution

$$G(z) = \frac{z \pm \sqrt{z^2 - 4}}{2}.$$

Since we know that Cauchy transforms must behave like $1/z$ for $z$ going to infinity, we have to choose the "$-$"-sign; applying the Stieltjes inversion formula to this gives us for $\mu_s$ a probability measure on $[-2, 2]$ with density

$$d\mu_s(t) = \frac{1}{2\pi}\sqrt{4 - t^2}dt.$$

*Remark* 2.3. Quite often one is only interested in the norm of a (let's say, selfadjoint) operator $a$ and not so much in its distribution $\mu_a$. Clearly the knowlege of the latter contains in particular (provided our state $\varphi$ is faithful, which will usually be the case) the information about the spectrum of $a$ and thus also about its norm. Thus, the explicit formula of the semi-circle distribution shows that the spectrum of a semi-circular element with variance 1 is $[-2, 2]$ and thus its norm is 2. However, one can also get this more directly from the growth of the moments, namely if $\varphi$ is faithful and $a$ selfadjoint then one has

$$\|a\| = \lim_{n\to\infty} \|a\|_{2n} = \lim_{n\to\infty} \big(\varphi(a^{2n})\big)^{1/2n}.$$

Thus, for a semi-circular element $s$ with variance 1, we could also get the norm more directly from the fact that the Catalan numbers $c_n$ grow asymptotically like $4^n$.

2.2. **Free convolution.** In classical probability theory the distribution of the sum of independent random variables is given by the convolution of the two distributions. Much of the basic body of classical probability theory centers around the understanding of this operation. So, if freeness wants to be a serious relative of independence then it better should also provide some interesting analogous theory of free convolution. The succesful treatment of this type of questions were the first steps of Voiculescu into the free probability world.

**Notation 2.4.** In analogy with the usual convolution we introduce the notion $\boxplus$ of *free convolution* as operation on probability measures on $\mathbb{R}$ by

$$\mu_{a+b} = \mu_a \boxplus \mu_b \qquad \text{if } a, b \text{ are free.}$$

Note that the moments of $a+b$ are just sums of mixed moments in $a$ and $b$, which, for $a$ and $b$ free, can be calculated out of the moments of $a$ and the moments of $b$. Thus it is clear that $\mu_{a+b}$ depends only on $\mu_a$ and $\mu_b$. In order to get a binary operation on all compactly supported probability measure we must be able to find for any pair of compactly supported probability measures $\mu$ and $\nu$ operators $a$ and $b$ which are free and such that $\mu = \mu_a$ and $\nu = \mu_b$. This can be achieved by some general free product construction (which is the free analogue of the construction of a product measure).

By approximation techniques one can extend $\boxplus$ also to all probability measures on $\mathbb{R}$.

Defining the free convolution is of course just the zeroth step, the crucial question is whether we can develop tools to deal with it successfully.

2.3. **From moments to cumulants.** "Freeness" of random variables is defined in terms of mixed moments; namely the defining property is that very special moments (alternating and centered ones) have to vanish. This requirement is not easy to handle in concrete calculations. Thus we will present here another approach to freeness, more combinatorial in nature, which puts the main emphasis on so called "free cumulants". These are some polynomials in the moments which behave much nicer with respect to freeness than the moments. The nomenclature comes from classical probability theory where corresponding objects are also well known and are usually called "cumulants" or "semi-invariants". There exists a combinatorial description of these classical cumulants, which depends on partitions of sets. In the same way, free cumulants can also be described combinatorially, the only difference to the classical case is that one has to replace all partitions by so called "non-crossing partitions".

**Definition 2.5.** A *partition* of the set $S := \{1, \dots, n\}$ is a decomposition

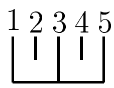
$$\pi = \{V_1, \dots, V_r\}$$

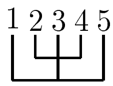
of $S$ into disjoint and non-empty sets $V_i$, i.e. for all $i, j = 1, \dots, r$ with $i \neq j$ we have

$$V_i \neq \emptyset, \qquad V_i \cap V_j = \emptyset$$

and
$$S = \dot{\cup}_{i=1}^{r} V_i.$$
We denote the set of all partitions of $S$ with $\mathcal{P}(S)$.
We call the $V_i$ the *blocks* of $\pi$.
For $1 \leq p, q \leq n$ we write
$$p \sim_\pi q \qquad \text{if } p \text{ and } q \text{ belong to the same block of } \pi.$$
A partition $\pi$ is called *non-crossing* if the following does not occur: There exist $1 \leq p_1 < q_1 < p_2 < q_2 \leq n$ with
$$p_1 \sim_\pi p_2 \not\sim_\pi q_1 \sim_\pi q_2.$$
The set of all non-crossing partitions of $\{1, \dots, n\}$ is denoted by $NC(n)$. We denote the "biggest" and the "smallest" element in $NC(n)$ by $\mathbf{1}_n$ and $\mathbf{0}_n$, respectively:
$$\mathbf{1}_n := \{(1, \dots, n)\}, \qquad \mathbf{0}_n := \{(1), \dots, (n)\}.$$

Non-crossing partitions were introduced by Kreweras in 1972 in a purely combinatorial context without any reference to probability theory.

*Example* 2.6. We will also use a graphical notation for our partitions; the term "non-crossing" will become evident in such a notation. Let
$$S = \{1, 2, 3, 4, 5\}.$$
Then the partition
$$\pi = \{(1, 3, 5), (2), (4)\} \qquad \hat{=}$$


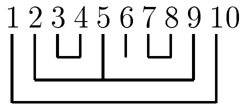

is non-crossing, whereas
$$\pi = \{(1, 3, 5), (2, 4)\} \qquad \hat{=}$$



is crossing.

*Remark* 2.7. 1) In an analogous way, non-crossing partitions $NC(S)$ can be defined for any linearly ordered set $S$; of course, we have
$$NC(S_1) \cong NC(S_2) \qquad \text{if} \qquad \#S_1 = \#S_2.$$
2) In most cases the following recursive description of non-crossing partitions is of great use: a partition $\pi$ ist non-crossing if and only if at least one block $V \in \pi$ is an interval and $\pi \backslash V$ is non-crossing; i.e. $V \in \pi$ has the form
$$V = (k, k+1, \dots, k+p) \qquad \text{for some } 1 \leq k \leq n \text{ and } p \geq 0,\ k+p \leq n$$
and we have
$$\pi \backslash V \in NC(1, \dots, k-1, k+p+1, \dots, n) \cong NC(n-(p+1)).$$

Example: The partition

$$\{(1,10),(2,5,9),(3,4),(6),(7,8)\} \qquad \hat{=}$$




can, by successive removal of intervals, be reduced to

$$\{(1,10),(2,5,9)\} \hat{=} \{(1,5),(2,3,4)\} \in NC(5)$$

and finally to

$$\{(1,5)\} \hat{=} \{(1,2)\} \in NC(2).$$

3) By writing a partition $\pi$ in the form $\pi = \{V_1, \dots, V_r\}$ we will always assume that the elements within each block $V_i$ are ordered in increasing order.

Now we can present the main object in our combinatorial approach to freeness.

**Definition 2.8.** Let $(\mathcal{A}, \varphi)$ be a probability space, i.e. $\mathcal{A}$ is a unital algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ is a unital linear functional. We define the *free cumulants* as a collection of multilinear functionals

$$k_n : \mathcal{A}^n \to \mathbb{C} \qquad (n \in \mathbb{N})$$

(indirectly) by the following system of equations (which we address as *moment-cumulant formula*):

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} k_\pi[a_1, \dots, a_n] \qquad (a_1, \dots, a_n \in \mathcal{A}),$$

where $k_\pi$ denotes a product of cumulants according to the block structure of $\pi$:

$$k_\pi[a_1, \dots, a_n] := k_{V_1}[a_1, \dots, a_n] \dots k_{V_r}[a_1, \dots, a_n]$$

for $\pi = \{V_1, \dots, V_r\} \in NC(n)$ and

$$k_V[a_1, \dots, a_n] := k_l(a_{v_1}, \dots, a_{v_l}) \qquad \text{for} \qquad V = (v_1, \dots, v_l).$$

Note: the above equations have the form

$$\varphi(a_1 \cdots a_n) = k_n(a_1, \dots, a_n) + \sum_{\substack{\pi \in NC(n) \\ \pi \neq \mathbf{1}_n}} k_\pi[a_1, \dots, a_n];$$

since the terms with $\pi \neq \mathbf{1}_n$ involve only lower order cumulants, this can be resolved for the $k_n(a_1, \dots, a_n)$ in a unique way.

*Example* 2.9. The best way to understand this definition is by examples. Let me give the the cumulants for small $n$.

- $n = 1$
$$\varphi(a_1) = k_{|}[a_1] = k_1(a_1),$$
thus
$$k_1(a_1) = \varphi(a_1).$$
- $n = 2$
$$\begin{aligned}\varphi(a_1a_2) &= k_{\sqcup}[a_1, a_2] + k_{|\,|}[a_1, a_2] \\ &= k_2(a_1, a_2) + k_1(a_1)k_1(a_2),\end{aligned}$$
thus
$$k_2(a_1, a_2) = \varphi(a_1a_2) - \varphi(a_1)\varphi(a_2).$$
- $n = 3$
$$\begin{aligned}\varphi(a_1a_2a_3) = &\, k_{\sqcup\!\sqcup}[a_1, a_2, a_3] + k_{|\,\sqcup}[a_1, a_2, a_3] + k_{\sqcup\,|}[a_1, a_2, a_3] \\ &+ k_{\sqcup\!\!\!\!\!\;\cdot\;}[a_1, a_2, a_3] + k_{|\,|\,|}[a_1, a_2, a_3] \\ = &\, k_3(a_1, a_2, a_3) + k_1(a_1)k_2(a_2, a_3) + k_2(a_1, a_2)k_1(a_3) \\ &+ k_2(a_1, a_3)k_1(a_2) + k_1(a_1)k_1(a_2)k_1(a_3),\end{aligned}$$
and thus
$$\begin{aligned}k_3(a_1, a_2, a_3) = &\, \varphi(a_1a_2a_3) - \varphi(a_1)\varphi(a_2a_3) - \varphi(a_1a_3)\varphi(a_2) \\ &- \varphi(a_1a_2)\varphi(a_3) + 2\varphi(a_1)\varphi(a_2)\varphi(a_3).\end{aligned}$$

3) For $n = 4$ we consider only the special case where all $\varphi(a_i) = 0$ (this reduces the number of terms from 14 to 3). Then we have
$$k_4(a_1, a_2, a_3, a_4) = \varphi(a_1a_2a_3a_4) - \varphi(a_1a_2)\varphi(a_3a_4) - \varphi(a_1a_4)\varphi(a_2a_3).$$

2.4. **Freeness and vanishing of mixed free cumulants.** That this is actually a good definition in the context of free random variables is the content of the following basic theorem. Roughly it says that freeness is equivalent to the vanishing of mixed cumulants.

**Theorem 2.10.** *Let $(\mathcal{A}, \varphi)$ be a probability space and consider unital subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_m \subset \mathcal{A}$. Then $\mathcal{A}_1, \dots, \mathcal{A}_m$ are free if and only if we have the following: We have for all $n \geq 2$ and for all $a_i \in \mathcal{A}_{j(i)}$ with $1 \leq j(1), \dots, j(n) \leq m$:*
$$k_n(a_1, \dots, a_n) = 0 \qquad \text{if there exist } 1 \leq l, k \leq n \text{ with } j(l) \neq j(k).$$

An example of the vanishing of mixed cumulants is that for $a, b$ free we have $k_3(a, a, b) = 0$, which, by the definition of $k_3$ just means that
$$\varphi(aab) - \varphi(a)\varphi(ab) - \varphi(aa)\varphi(b) - \varphi(ab)\varphi(a) + 2\varphi(a)\varphi(a)\varphi(b) = 0.$$
This vanishing of mixed cumulants in free variables is of course just a reorganization of the information about joint moments of free variables – but in a form which is much more useful for many applications.

The above characterization of freeness in terms of cumulants is the translation of the definition of freeness in terms of moments – by using the moment-cumulant formula. One should note that in contrast to the characterization in terms of moments we do not require that $j(1) \neq j(2) \neq \cdots \neq j(m)$ or $\varphi(a_i) = 0$. (That's exactly the main part of the proof of that theorem: to show that on the level of cumulants the assumption "centered" is not needed and that "alternating" can be weakened to "mixed".) Hence the characterization of freeness in terms of cumulants is much easier to use in concrete calculations.

Since the unit 1 is free from everything, the above theorem contains as a special case the following statement.

**Proposition 2.11.** *Let $n \geq 2$ und $a_1, \dots, a_n \in \mathcal{A}$. Then we have:*

$$\text{there exists a } 1 \leq i \leq n \text{ with } a_i = 1 \qquad \Longrightarrow \qquad k_n(a_1, \dots, a_n) = 0.$$

Note also: for $n = 1$ we have

$$k_1(1) = \varphi(1) = 1.$$

2.5. **Free cumulants of random variables.** Let us now specialize the information contained in the cumulants to one random variable.

**Notation 2.12.** For a random variable $a \in \mathcal{A}$ we put

$$k_n^a := k_n(a, \dots, a)$$

and call the sequence of numbers $(k_n^a)_{n \geq 1}$ the *free cumulants of $a$*.

Our main theorem on the vanishing of mixed cumulants in free variables specifies in this one-dimensional case to the linearity of the cumulants.

**Proposition 2.13.** *Let $a$ and $b$ be free. Then we have*

$$k_n^{a+b} = k_n^a + k_n^b \qquad \text{for all } n \geq 1.$$

*Proof.* We have

$$\begin{aligned} k_n^{a+b} &= k_n(a+b, \dots, a+b) \\ &= k_n(a, \dots, a) + k_n(b, \dots, b) \\ &= k_n^a + k_n^b, \end{aligned}$$

because cumulants which have both $a$ and $b$ as arguments vanish by our main theorem that freeness is the same as the vanishing of mixed cumulants. □

Thus, free convolution is easy to describe on the level of cumulants; the cumulants are additive under free convolution. It remains to make the connection between moments and cumulants as explicit as possible. On a combinatorial level, our definition specializes in the one-dimensional case to the following relation.

**Proposition 2.14.** *Let $(m_n)_{n\geq 1}$ and $(k_n)_{n\geq 1}$ be the moments and free cumulants, respectively, of some random variable. The connection between these two sequences of numbers is given by*

$$m_n = \sum_{\pi \in NC(n)} k_\pi,$$

*where*

$$k_\pi := k_{\#V_1} \cdots k_{\#V_r} \qquad \textit{for} \qquad \pi = \{V_1, \dots, V_r\}.$$

Example: For $n = 3$ we have

$$\begin{aligned} m_3 &= k_{\sqcup\!\sqcup} + k_{|\,\sqcup} + k_{\sqcup\,|} + k_{\sqcup\!\!\!\!\!|\;} + k_{|\,|\,|} \\ &= k_3 + 3k_1k_2 + k_1^3. \end{aligned}$$

*Example* 2.15. Our formula for the moments of a semi-circular element, together with the fact that the Catalan number $c_n$ counts also the number of non-crossing pairings of $2n$ elements (a pairing is a partition where each block has exactly 2 elements), gives for the cumulants of a semi-circular element $s$ of variance 1 the following:

$$k_n^s = \begin{cases} 1, & \text{if } n = 2 \\ 0, & \text{otherwise} \end{cases}$$

2.6. **Analytic description of free convolution: the $R$-transform machinery.** For concrete calculations, however, one would prefer to have a more analytical description of the relation between moments and cumulants. This can be achieved by translating the above relation to corresponding formal power series.

2.7. **Theorem.** Let $(m_n)_{n\geq 1}$ and $(k_n)_{n\geq 1}$ be two sequences of complex numbers and consider the corresponding formal power series

$$M(z) := 1 + \sum_{n=1}^{\infty} m_n z^n,$$

$$C(z) := 1 + \sum_{n=1}^{\infty} k_n z^n.$$

Then the following three statements are equivalent:

(i) We have for all $n \in \mathbb{N}$

$$m_n = \sum_{\pi \in NC(n)} k_\pi = \sum_{\pi = \{V_1,\dots,V_r\} \in NC(n)} k_{\#V_1} \dots k_{\#V_r}.$$

(ii) We have for all $n \in \mathbb{N}$ (where we put $m_0 := 1$)

$$m_n = \sum_{s=1}^{n} \sum_{\substack{i_1,\dots,i_s \in \{0,1,\dots,n-s\} \\ i_1+\dots+i_s=n-s}} k_s m_{i_1} \dots m_{i_s}.$$

(iii) We have

$$C[zM(z)] = M(z).$$

*Proof.* We rewrite the sum

$$m_n = \sum_{\pi \in NC(n)} k_\pi$$

in the way that we fix the first block $V_1$ of $\pi$ (i.e. that block which contains the element 1) and sum over all possibilities for the other blocks; in the end we sum over $V_1$:

$$m_n = \sum_{s=1}^{n} \sum_{V_1 \text{ with } \#V_1 = s} \sum_{\substack{\pi \in NC(n) \\ \text{where } \pi = \{V_1, \dots\}}} k_\pi.$$

If

$$V_1 = (v_1 = 1, v_2, \dots, v_s),$$

then $\pi = \{V_1, \dots\} \in NC(n)$ can only connect elements lying between some $v_k$ and $v_{k+1}$, i.e. $\pi = \{V_1, V_2, \dots, V_r\}$ such that we have for all $j = 2, \dots, r$: there exists a $k$ with $v_k < V_j < v_{k+1}$. There we put

$$v_{s+1} := n + 1.$$

Hence such a $\pi$ decomposes as

$$\pi = V_1 \cup \tilde{\pi}_1 \cup \dots \cup \tilde{\pi}_s,$$

where $\tilde{\pi}_j$ is a non-crossing partition of $\{v_j + 1, v_j + 2, \dots, v_{j+1} - 1\}$, i.e., with $i_j := v_{j+1} - v_j - 1$ we have $\tilde{\pi}_j \in NC(i_j)$. For such $\pi$ we have

$$k_\pi = k_{\#V_1} k_{\tilde{\pi}_1} \dots k_{\tilde{\pi}_s} = k_s k_{\tilde{\pi}_1} \dots k_{\tilde{\pi}_s},$$

and thus we obtain

$$\begin{aligned}
m_n &= \sum_{s=1}^{n} \sum_{\substack{i_1,\dots,i_s\in\{0,1,\dots,n-s\}\\ i_1+\dots+i_s+s=n}} \sum_{\substack{\pi=V_1\cup\tilde\pi_1\cup\dots\cup\tilde\pi_s\\ \tilde\pi_j\in NC(i_j)}} k_s k_{\tilde\pi_1}\dots k_{\tilde\pi_s}\\
&= \sum_{s=1}^{n} k_s \sum_{\substack{i_1,\dots,i_s\in\{0,1,\dots,n-s\}\\ i_1+\dots+i_s+s=n}} \big(\sum_{\tilde\pi_1\in NC(i_1)} k_{\tilde\pi_1}\big)\dots\big(\sum_{\tilde\pi_s\in NC(i_s)} k_{\tilde\pi_s}\big)\\
&= \sum_{s=1}^{n} \sum_{\substack{i_1,\dots,i_s\in\{0,1,\dots,n-s\}\\ i_1+\dots+i_s+s=n}} k_s m_{i_1}\dots m_{i_s}.
\end{aligned}$$

This yields the implication (i) $\Longrightarrow$ (ii).
We can now rewrite (ii) in terms of the corresponding formal power series in the following way (where we put $m_0 := k_0 := 1$):

$$\begin{aligned}
M(z) &= 1+\sum_{n=1}^{\infty} z^n m_n\\
&= 1+\sum_{n=1}^{\infty}\sum_{s=1}^{n} \sum_{\substack{i_1,\dots,i_s\in\{0,1,\dots,n-s\}\\ i_1+\dots+i_s=n-s}} k_s z^s m_{i_1} z^{i_1}\dots m_{i_s} z^{i_s}\\
&= 1+\sum_{s=1}^{\infty} k_s z^s \big(\sum_{i=0}^{\infty} m_i z^i\big)^s\\
&= C[zM(z)].
\end{aligned}$$

This yields (iii).
Since (iii) describes uniquely a fixed relation between the numbers $(k_n)_{n\geq 1}$ and the numbers $(m_n)_{n\geq 1}$, this has to be the relation (i). $\square$

If we rewrite the above relation between the formal power series in terms of the Cauchy-transform

$$G(z) := \sum_{n=0}^{\infty} \frac{m_n}{z^{n+1}}$$

and the so-called *R-transform*

$$R(z) := \sum_{n=0}^{\infty} k_{n+1} z^n,$$

then we obtain Voiculescu's basic results about free convolution.

**Theorem 2.16.** *1) The relation between the Cauchy-transform $G(z)$ and the $R$-transform $R(z)$ of a random variable is given by*

$$G[R(z) + \frac{1}{z}] = z.$$

*2) The $R$-transform is additive for free random variables, i.e.,*

$$R^{a+b}(z) = R^a(z) + R^b(z) \qquad \textit{if } a \textit{ and } b \textit{ are free.}$$

*Proof.* 1) We just have to note that the formal power series $M(z)$ and $C(z)$ from the previous theorem and $G(z)$, $R(z)$, and $K(z) = R(z) + \frac{1}{z}$ are related by:

$$G(z) = \frac{1}{z} M(\frac{1}{z})$$

and

$$C(z) = 1 + zR(z) = zK(z), \qquad \text{thus} \qquad K(z) = \frac{C(z)}{z}.$$

This gives

$$K[G(z)] = \frac{1}{G(z)} C[G(z)] = \frac{1}{G(z)} C[\frac{1}{z} M(\frac{1}{z})] = \frac{1}{G(z)} M(\frac{1}{z}) = z,$$

thus $K[G(z)] = z$ and hence also

$$G[R(z) + \frac{1}{z}] = G[K(z)] = z.$$

2) This is just the fact that the cumulants as coefficients of the $R$-transform are additive for the sum of free variables. □

2.8. **Voiculescu's approach to the $R$-transform.** Our derivation of the basic properties of the $R$-transform was purely based on the combinatorics of non-crossing partitions. That was not the way Voiculescu found those results. His approach was more analytical, and used creation and annihilation operators on full Fock spaces. The relation between these two approaches comes from the fact that the calculation of moments for special polynomials in creation and annihilation operators leads very naturally to non-crossing partitions and the moment-cumulant formula.

Namely, with $l := l(f)$ (for a unit vector $f \in \mathcal{H}$) the creation operator on the full Fock space, consider operators of the form

$$b = l + \sum_{i=0}^{\infty} k_{i+1} l^{*i}$$

(take this as a formal sum, or consider only sums with finitely many non-vanishing coefficients) Then we have

$$\begin{aligned} m_n &= \langle \Omega, (l + \sum_{i=0}^{\infty} k_{i+1} l^{*i})^n \Omega \rangle \\ &= \sum_{i(1),\dots,i(n)\in\{-1,0,1,\dots,n-1\}} \langle \Omega, l^{*i(n)} \dots l^{*i(1)} \Omega \rangle k_{i(1)+1} \dots k_{i(n)+1}, \end{aligned}$$

where $l^{*-1}$ is identified with $l$, and $k_0 := 1$.

The sum is running over tuples $(i(1), \dots, i(n))$, which can be identified with paths in the lattice $\mathbb{Z}^2$:

$$\begin{aligned} i = -1 \qquad &\hat{=} \qquad \text{diagonal step upwards: } \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ i = 0 \qquad &\hat{=} \qquad \text{horizontal step to the right: } \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ i = k \quad (1 \le k \le n-1) \qquad &\hat{=} \qquad \text{diagonal step downwards: } \begin{pmatrix} 1 \\ -k \end{pmatrix} \end{aligned}$$

We have now

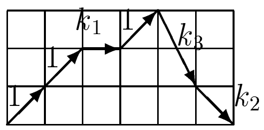
$$\langle \Omega, l^{*i(n)} \dots l^{*i(1)} \Omega \rangle = \begin{cases} 1, & \text{if } i(1) + \dots + i(m) \le 0 \ \forall\, m = 1, \dots, n \text{ and} \\ & \qquad i(1) + \dots + i(n) = 0 \\ 0, & \text{otherwise} \end{cases}$$

and thus

$$m_n = \sum_{\substack{i(1),\dots,i(n)\in\{-1,0,1,\dots,n-1\} \\ i(1)+\dots+i(m)\le 0 \quad \forall\, m=1,\dots,n \\ i(1)+\dots+i(n)=0}} k_{i(1)+1} \dots k_{i(n)+1}.$$

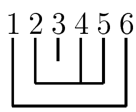
Hence only such paths from $(0,0)$ to $(n,0)$ contribute which stay always above the $x$-axis. Each such path is weighted in a multiplicative way (using the $k$'s) with the length of its steps.

Example:




$$\hat{=} \qquad \langle \Omega, l^{*1}\, l^{*2}\, l\, l^{*0}\, l\, l\, \Omega \rangle k_1 k_3 k_2$$

The paths appearing in the above sum can be identified with non-crossing partitions, e.g., the example above would correspond to



In this way the above summation can be rewritten in terms of a summation over non-crossing partitions, leading exactly to the moment-cumulant formula.

2.9. **A concrete calculation of a free convolution.** The $R$-transform is now the solution to the problem of calculating the free convolution $\mu \boxplus \nu$ of two probability measures on $\mathbb{R}$. Namely, we calculate for each of them the Cauchy transform, from this the $R$-transform, add up the $R$-transforms, then go back to the Cauchy transform and get finally, by Stieltjes inversion, the measure $\mu \boxplus \nu$. Of course, not all steps can be done always explicitely (the biggest problem is usually solving for the inverse under composition of the Cauchy or $R$-transform). But let me show a non-trivial example, where everything can be calculated explicitly. Let

$$\mu = \nu = \frac{1}{2}(\delta_{-1} + \delta_{+1}).$$

Then we have

$$G_\mu(z) = \int \frac{1}{z-t} d\mu(t) = \frac{1}{2}\big(\frac{1}{z+1} + \frac{1}{z-1}\big) = \frac{z}{z^2-1}.$$

Put

$$K_\mu(z) = \frac{1}{z} + R_\mu(z).$$

Then $z = G_\mu[K_\mu(z)]$ gives

$$K_\mu(z)^2 - \frac{K_\mu(z)}{z} = 1,$$

which has as solutions

$$K_\mu(z) = \frac{1 \pm \sqrt{1+4z^2}}{2z}.$$

Thus the $R$-transform of $\mu$ is given by

$$R_\mu(z) = K_\mu(z) - \frac{1}{z} = \frac{\sqrt{1+4z^2} - 1}{2z}$$

(Note: $R_\mu(0) = k_1(\mu) = m_1(\mu) = 0$, thus we have to take the $+$-sign in the above solution.) Hence we get

$$R_{\mu\boxplus\mu}(z) = 2R_\mu(z) = \frac{\sqrt{1+4z^2}-1}{z},$$

and

$$K(z) := K_{\mu\boxplus\mu}(z) = R_{\mu\boxplus\mu}(z) + \frac{1}{z} = \frac{\sqrt{1+4z^2}}{z},$$

which allows to determine $G := G_{\mu\boxplus\mu}$ via

$$z = K[G(z)] = \frac{\sqrt{1+4G(z)^2}}{G(z)}$$

as

$$G(z) = \frac{1}{\sqrt{z^2-4}}$$

From this we can calculate the density

$$\frac{d(\mu\boxplus\mu)(t)}{dt} = -\frac{1}{\pi}\lim_{\varepsilon\to 0}\Im\frac{1}{\sqrt{(t+i\varepsilon)^2-4}} = -\frac{1}{\pi}\Im\frac{1}{\sqrt{t^2-4}},$$

so that we finally get the arcsine distribution in this case:

$$\frac{d(\mu\boxplus\mu)(t)}{dt} = \begin{cases} \frac{1}{\pi\sqrt{4-t^2}}, & |t|\le 2 \\ 0, & \text{otherwise} \end{cases}$$

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