COMBINATORIAL ASPECTS OF FREE PROBABILITY THEORY

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4. Applications of Free Probability to Random Matrices

4.1. **Making matrices free by random rotations.** Let us first recall one of the most important realizations of freeness by random matrices: if A_N and B_N are two sequences of constant matrices, each of which has a limit distribution with respect to the normalized trace, and if U_N is a Haar unitary $N \times N$-random matrix, then A_N and $U_N B_N U_N^*$ are, for $N \to \infty$, asymptotically free.

Note that conjugating by U_N does not change the distribution of B_N. The effect of this "random rotation" is that any possible relation between A_N and B_N is destroyed, and these two matrices are brought in "generic" position. This gives maybe the most concrete picture for freeness. $N \times N$-matrices are asymptotically free, if their eigenspaces are in generic position, have no definite relation between them.

The idea of bringing matrices in a generic, free position by random unitary conjugation also works more general for *random* matrices if we have in addition independence between the involved ensembles.

Theorem 4.1. *Let U be a sequence of Haar distributed unitary $N \times N$-random matrices. Suppose that $A_1, \dots, A_s$ and $B_1, \dots, B_t$ are sequences of $N \times N$-random matrices each of which has a limit distribution. Furthermore, assume that $\{A_1, \dots, A_s\}$, $\{B_1, \dots, B_t\}$, and $\{U\}$ are independent. Then the sequences $\{A_1, \dots, A_s\}$ and $\{UB_1U^*, \dots, UB_tU^*\}$ are asymptotically free.*

4.2. **Using free probability for random matrix calculations.** Instead of using random matrices for modelling operator algebras (which resulted in many new results about the free group factors) one can also revert the direction and use our free probability machinery in order to calculate eigenvalue distributions of random matrices in the limit

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$N \to \infty$. Such calculations are one important part of random matrix theory (in particular, in applications to engineering) and free probability theory has added quite a bit to this. Here are some examples what free probability theory can do.

- If we have two random matrices which are free (e.g., of the form A, UBU^* for a Haar unitary random matrix U), then we can use our R-transform or S-transform machinery for calculating the asymptotic eigenvalue distribution of the sum or of the product of these matrices.
- There exist also formulas (due to Nica and Speicher) describing the distribution of the commutator $ab - ba$ of two free random variables. This can be used for calculating the distribution of the commutator of two random matrices in generic position.
- We have also seen in the last lecture that we have a nice description of the free compression of random variables. However, this has also a very concrete meaning for random matrices. If we have a randomly rotated matrix UAU^*, then this is asymptotically free from diagonal projections, and taking the compression with these diagonal projections just means to take corners of the original matrix. Thus free probability tells us in this case how to calculate the distribution of corners of matrices out of the distribution of the whole matrix.

In the following we want to present some pictures which give an impression of how accurate free probability matches the random matrix results. One should in particular note that although free probability is only exact for random matrices in the limit $N \to \infty$, empirical evidence suggests that a 10×10-matrix is often "good enough" to corroborate the limiting predictions of free probability.

The following calculations and pictures are provided by Raj Rao from MIT. Together with Alan Edelman he is developing a "free" or "random matrix calculator" which allows to do many of the calculations involving Cauchy, R, or S-transform in a symbolic way on a computer. The main idea there is that one should work with the algebraic equations which these transforms satisfy and not with the explicit solutions. In general, there are only rare instances of explicit solutions for the free additive or multiplicative convolution, even if one starts with an explicit distribution. However, if one starts from Cauchy transforms which satisfy an algebraic equation then this property is preserved under the usual operations. Thus it is a good idea to work with the polynomial equations instead of their solutions, and solve the equations only at the very end, which can then be done numerically. This implementation allows also to extract the moments of the considered distributions very easily symbolically (i.e., also in dependence on some parameters). In the following examples this will be illustrated by giving the low moments of the resulting probability distributions.

4.3. **Free Addition Examples.** Start with a symmetric Bernoulli:

$$\mu = 0.5\,\delta_{-1} + 0.5\,\delta_1.$$

The free additive convolution

$$\nu = \mu \boxplus \mu$$

results in the spectral measure ν whose Cauchy transform:

$$g(z) = \int \frac{1}{z-x}\nu(dx)$$

is the solution of the algebraic equation

$$g^2(z^2-4) - 1 = 0.$$

With appropriate normalization, this is the famous arc-sine law. Figure 1 compares theory with simulation.

Now consider a generalized Bernoulli distribution parameterized by p given as:

$$\mu_p = p\delta_1 + (1-p)\delta_{-1}.$$

The spectral measure

$$\nu = \mu_p \boxplus \mu_q$$

has a Cauchy transform $g(z)$ which is the solution to the algebraic equation

$$\left(z^4 - 4\,z^2\right) g^2 + \left(4\,z^2 - 4\,qz^2 - 4\,pz^2\right) g - 8\,qp - z^2 + 4\,q^2 + 4\,p^2 = 0.$$

The first 5 moments of this spectral measure are, respectively:

$$\begin{bmatrix} 2\,p - 2 + 2\,q \\ 4 - 4\,q - 4\,p + 8\,qp \\ 8\,p - 8 + 8\,q \\ 16 - 16\,q - 16\,p + 32\,q^2p + 32\,qp^2 - 32\,q^2p^2 \\ -32 + 32\,q + 32\,p \end{bmatrix}$$

Note that this reduces to the moments of the arc-sine law when $p = q = 0.5$, in particular, the odd moments vanish then.

4.4. **Free Multiplication Examples.**

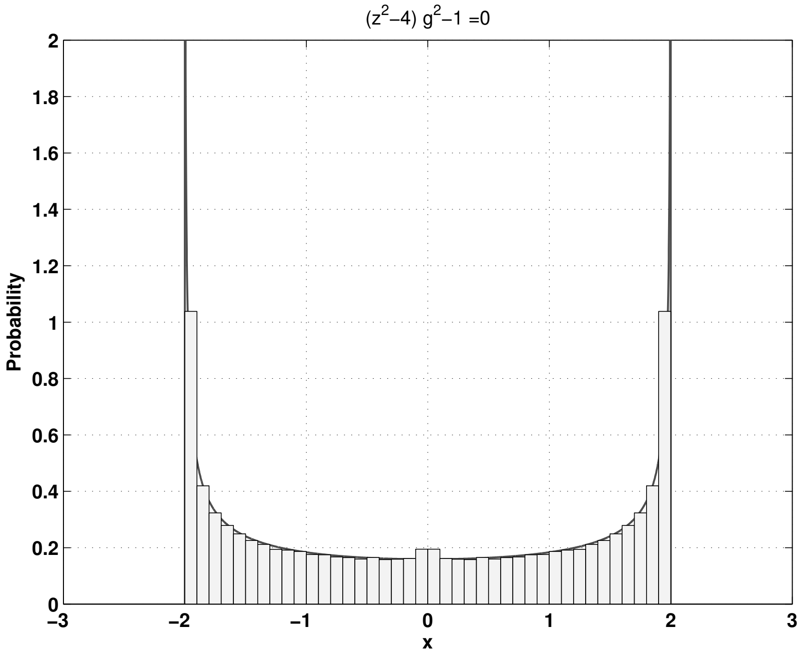

FIGURE 1. Histogram of the eigenvalues of 1000 realizations of 50×50 sized random matrices of the form $A + UAU^*$ where A is a diagonal matrix half of whose entries are 1 with the other half -1. U is a random unitary matrix with Haar measure.

4.4.1. *Product of Projections.* Start with a projection:

$$\mu = 0.5\,\delta_0 + 0.5\,\delta_1.$$

The free multiplicative convolution

$$\nu = \mu \boxtimes \mu$$

results in the spectral measure ν whose Cauchy transform $g(z)$ is the solution of the algebraic equation:

$$-1/4 + \left(z^3 - z^2\right) g^2 - \left(-z + z^2\right) g = 0.$$

Note that ν will contain an atom of mass 0.5 at zero. With appropriate normalization, this is the famous arc-sine law. Figure 2 compares theory with simulation.

Now consider a generalized Bernoulli distribution parameterized by p given as:

$$\mu_p = p\delta_1 + (1-p)\delta_0.$$

The spectral measure given by:

$$\nu = \mu_p \boxtimes \mu_q$$

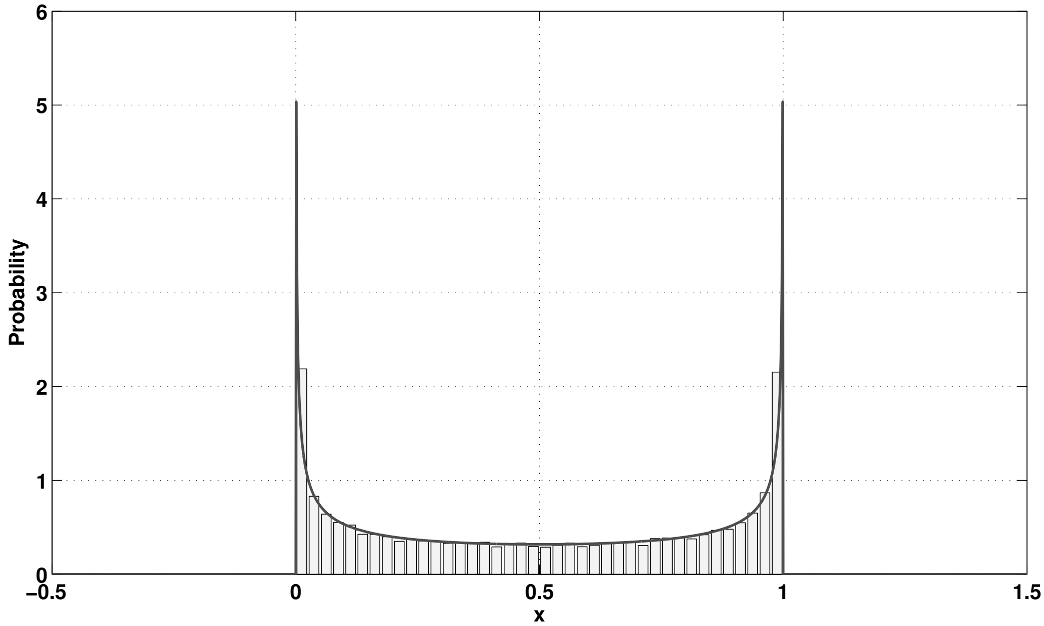

FIGURE 2. Histogram of the eigenvalues of 1000 realizations of 50×50 sized random matrices of the form $AUAU^*$ where A is a diagonal matrix half of whose entries are 1 with the other half 0. U is a random unitary matrix with Haar measure. The δ-peak at 0 is surpressed.

has a Cauchy transform $g(z)$ which is the solution to the algebraic equation

$$\left(z^3 - z^2\right) g^2 - \left(-2\,z + zp + zq + z^2\right) g + q + p - 1 - pq = 0$$

The first 4 moments of this spectral measure are, respectively:

$$\begin{bmatrix} pq \\ -p^2q^2 + p^2q + q^2p \\ p^3q - 3\,p^3q^2 - 3\,p^2q^3 + q^3p + 3\,p^2q^2 + 2\,p^3q^3 \\ 6\,p^3q^2 + 6\,p^2q^3 + 10\,p^4q^3 + p^4q + q^4p - 5\,p^4q^4 - 6\,q^4p^2 + 10\,p^3q^4 - 6\,p^4q^2 - 16\,p^3q^3 \end{bmatrix}$$

4.4.2. *Product of Semi-Circle with Free Poisson.* The free multiplicative convolution of a semi-circular distribution with a free Poisson with parameter c gives us a spectral measure whose Cauchy transform is the solution of the algebraic equation:

$$1 + g^4z^2 - (2\,z - 2\,cz)\,g^3 + \left(-2\,c + c^2 + 1\right) g^2 - gz = 0.$$

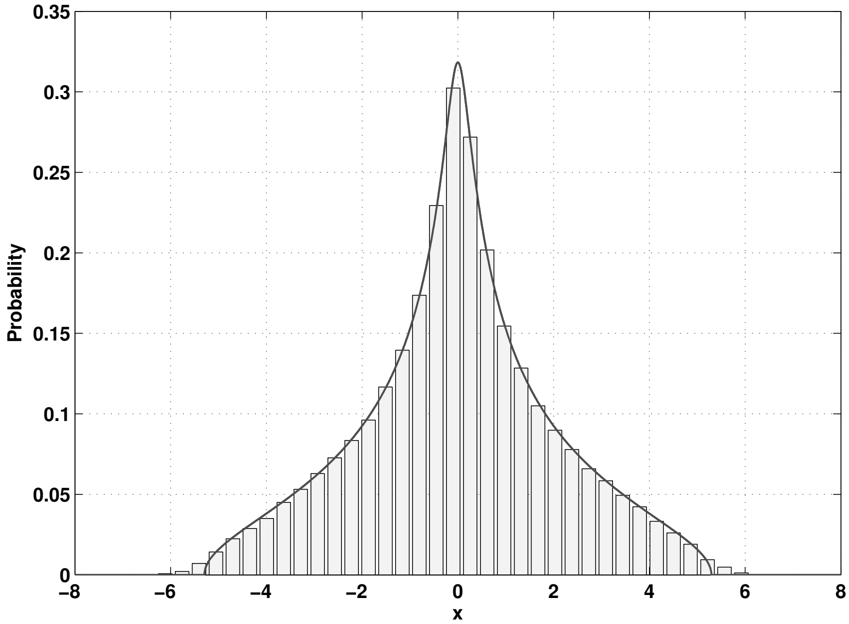

FIGURE 3. Histogram of the eigenvalues of 1000 realizations of 50×50 sized random matrices of the form AB where A is a Wigner matrix and B is an independent Wishart Matrix with parameter $c = 2$

Note the agreement of theory with simulation (for $c = 2$) as in Figure 3. The first 10 moments of this spectral measure are:

$$\begin{bmatrix} 0 \\ c^2 \\ 0 \\ 2\,c^4 + 2\,c^3 \\ 0 \\ 12\,c^5 + 5\,c^6 + 5\,c^4 \\ 0 \\ 56\,c^7 + 14\,c^5 + 56\,c^6 + 14\,c^8 \\ 0 \\ 240\,c^7 + 240\,c^9 + 42\,c^{10} + 42\,c^6 + 405\,c^8 \end{bmatrix}$$

4.4.3. *Product of Free Poisson with Shifted Free Poisson.* The free multiplicative convolution of a free Poisson of parameter c shifted by $-c$ (which has vanishing first moment) with a free Poisson with parameter

c gives us a spectral measure whose Cauchy transform is the solution of the algebraic equation:

$$1+g^4cz^2-\left(2\,cz-z^2-2\,c^2z\right)g^3+\left(c^3+cz+c-2\,z-2\,c^2\right)g^2-(c+z-1)\,g=0$$

Note the agreement of theory with simulation (for $c=2$) as in Figure 4. The first 10 moments of this spectral measure are given by:

$$\begin{bmatrix} 0 \\ c^3 \\ c^4 \\ 2\,c^6+3\,c^5 \\ 6\,c^6+5\,c^7 \\ 5\,c^9+15\,c^7+21\,c^8 \\ 63\,c^9+21\,c^{10}+36\,c^8 \\ 112\,c^{11}+14\,c^{12}+91\,c^9+204\,c^{10} \\ 621\,c^{11}+84\,c^{13}+444\,c^{12}+232\,c^{10} \\ 603\,c^{11}+1755\,c^{13}+540\,c^{14}+1900\,c^{12}+42\,c^{15} \end{bmatrix}$$

4.5. **Free Commutator Examples.**

4.5.1. *Commutator between Wishart and Shifted Wishart.* The spectral measure corresponding to the free commutator of the free Poisson with parameter c and another free Poisson of parameter c shifted by $-c$ (which has vanishing first moment) has a Cauchy transform that is a solution of the algebraic equation:

$$-4-g^5z^3+\left(-4\,z^2c+z^2\right)g^4-\left(-z+4\,zc^2\right)g^3+\left(4\,c-4\,c^2-1\right)g^2+4\,zg=0$$

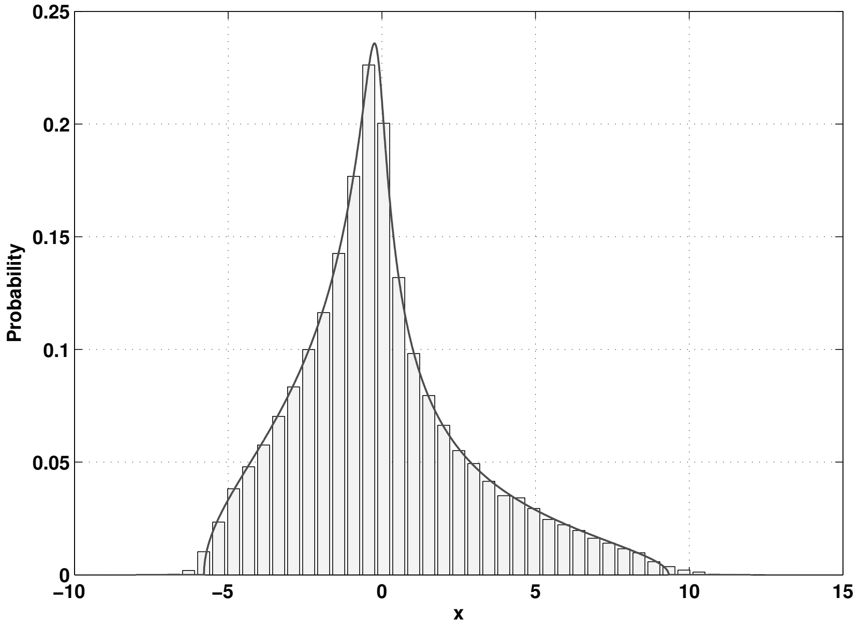

FIGURE 4. Histogram of the eigenvalues of 1000 realizations of 50×50 sized random matrices of the form AB where A is a Wishart matrix and B is another independent Wishart Matrix shifted by $-c$, both with $c = 2$

Figure 5 compares theory to simulation. Its first 10 moments are given by:

$$\begin{bmatrix} 0 \\ 2\,c^2 \\ 0 \\ 10\,c^4 + 4\,c^3 \\ 0 \\ 10\,c^4 + 66\,c^6 + 60\,c^5 \\ 0 \\ 280\,c^6 + 498\,c^8 + 28\,c^5 + 728\,c^7 \\ 0 \\ 84\,c^6 + 8280\,c^9 + 4066\,c^{10} + 5220\,c^8 + 1200\,c^7 \end{bmatrix}$$

4.5.2. *Commutator between two Wisharts.* The spectral measure corresponding to the free commutator of a free Poission with parameter c and the semi-circular distribution has a Cauchy transform that is a

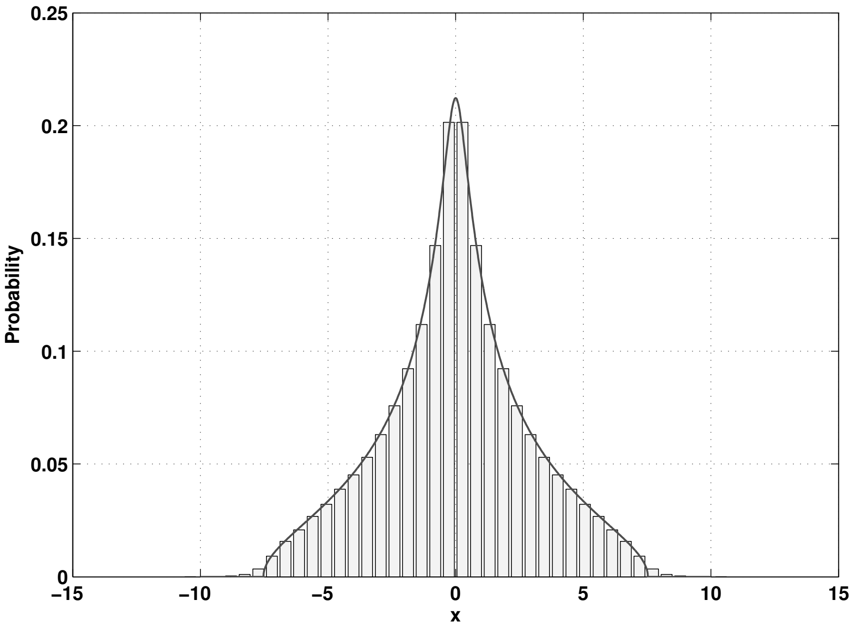

FIGURE 5. Histogram of the eigenvalues of 1000 realizations of 50×50 sized random matrix commutator formed from a Wishart matrix of parameter $c = 2$ and another independent Wishart matrix of same parameter shifted by $-c$.

solution of the algebraic equation:

$$2 + g^4 z^2 + 2\,cg^3 z + (-1 + 2\,c)\,g^2 - 2\,zg = 0.$$

Figure 6 compares theory with experiment. Its first 10 moments are:

$$\begin{bmatrix} 0 \\ 2\,c \\ 0 \\ 10\,c^2 + 2\,c \\ 0 \\ 2\,c + 66\,c^3 + 30\,c^2 \\ 0 \\ 2\,c + 364\,c^3 + 498\,c^4 + 60\,c^2 \\ 0 \\ 2\,c + 4066\,c^5 + 1160\,c^3 + 4140\,c^4 + 100\,c^2 \end{bmatrix}.$$

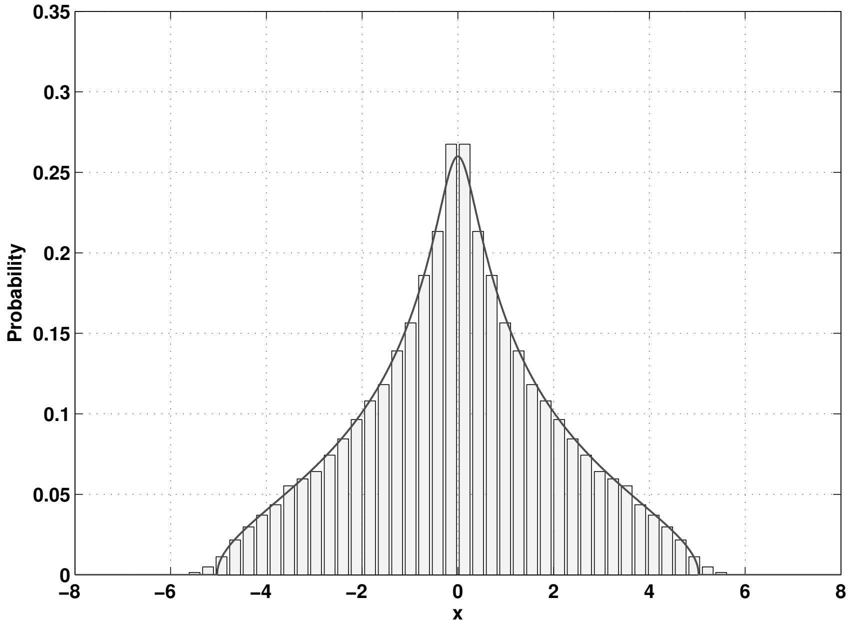

FIGURE 6. Histogram of the eigenvalues of 1000 realizations of 50×50 sized random matrix commutators formed from a Wishart matrix with parameter $c = 2$ and an independent Wigner matrix.

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