FREE CALCULUS

ROLAND SPEICHER

Summer School on 'Quantum Probability',
Grenoble, 1998

1. Basic definitions and facts

Let me first recall the basic definitions and some fundamental realizations of freeness. For a more extensive review, I refer to the course of Biane or to the books [18, 17] (see also the survey [16]).

1.1. **Definitions.** 1) A **(non-commutative) probability space** consists of a pair $(\mathcal{A}, \varphi)$, where

- $\mathcal{A}$ is a unital algebra
- $\varphi : \mathcal{A} \to \mathbb{C}$ is a unital linear functional, i.e. in particular $\varphi(1) = 1$

2) Unital subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_n \subset \mathcal{A}$ are called **free**, if we have

$$\varphi(a_1 \dots a_k) = 0$$

whenever

$$\begin{gathered} a_i \in \mathcal{A}_{j(i)} \qquad (i = 1, \dots, k) \\ j(1) \neq j(2) \neq \dots \neq j(k) \\ \varphi(a_i) = 0 \qquad (i = 1, \dots, k) \end{gathered}$$

3) Random variables $x_1, \dots, x_n \in \mathcal{A}$ are called **free**, if $\mathcal{A}_1, \dots, \mathcal{A}_n$ are free, where $\mathcal{A}_i$ is the unital algebra generated by x_i.

A canonical realization of free random variables is given on the full Fock space

1.2. **Definitions.** Let $\mathcal{H}$ be a Hilbert space.
1) The **full Fock space over** $\mathcal{H}$ is the Hilbert space

$$\mathcal{F}(\mathcal{H}) := \mathbb{C}\Omega \oplus \bigoplus_{n \geq 1} \mathcal{H}^{\otimes n},$$

where Ω is a distinguished unit vector, called **vacuum**.
2) The **vacuum expectation** is the state

$$A \mapsto \langle \Omega, A\Omega \rangle.$$

3) For each $f \in \mathcal{H}$ we define the **(left) annihilation operator** $l(f)$ and the **(left) creation operator** $l^*(f)$ by

$$l(f)\Omega = 0$$
$$l(f) f_1 \otimes \cdots \otimes f_n = \langle f, f_1 \rangle f_2 \otimes \cdots \otimes f_n$$

and

$$l^*(f) f_1 \otimes \cdots \otimes f_n = f \otimes f_1 \otimes \cdots \otimes f_n.$$

One should note that in regard of whether we denote by $l(f)$ the annihilation or the creation operator we follow the opposite tradition as Voiculescu.

1.3. **Proposition.** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and put $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$. Consider the full Fock space over $\mathcal{H}$ and the corresponding creation and annihilation operators $l(f)$ and $l^*(f)$ for $f \in \mathcal{H}$. Put

$$\mathcal{A}_1 := C^*(l(f) \mid f \in \mathcal{H}_1), \qquad \mathcal{A}_2 := C^*(l(f) \mid f \in \mathcal{H}_2).$$

Then $\mathcal{A}_1$ and $\mathcal{A}_2$ are free with respect to the vacuum expectation.

If we replace the C^*-algebras by von Neumann algebras or by $*$-algebras, then the analogue statements are also true. The proof consists of direct checking for the case of $*$-algebras and then extending the assertion to uniform or weak closure by approximation arguments.

Whereas freeness is just modelled according to the situation on the full Fock space – hence its appearing in this context is not very surprising – there is another realization of freeness in a totally different context: freeness can also be thought of as the mathematical structure of $N \times N$ random matrices in the limit $N \to \infty$. We will not need this connection in our considerations, but it is always good to keep in mind that all our constructions also have some meaning in terms of random matrices.

Such random matrices X_N are $N \times N$ matrices whose entries are classical random variables and usually one is interested in the averaged eigenvalue distribution $\operatorname{distr}(X_N)$ of these matrices corresponding to a state given by the averaged normalized trace.

The following theorem is due to Voiculescu [15] (see also [10]).

1.4. **Theorem. 1) Gaussian random matrices**
Let

$$X_N = (a_{ij}^{(N)})_{i,j=1}^N, \qquad Y_N = (b_{ij}^{(N)})_{i,j=1}^N$$

be symmetric random $N \times N$ matrices with

- $a_{ij}^{(N)}$ $(1 \le i \le j \le N)$ are mutually independent and normally distributed with mean zero and variance $1/N$.

- $b_{ij}^{(N)}$ $(1 \leq i \leq j \leq N)$ are mutually independent and normally distributed with mean zero and variance $1/N$.
- all $a_{ij}^{(N)}$ are independent from all $b_{kl}^{(N)}$.

Then X_N and Y_N converge in distribution to a semicircular family in the limit $N \to \infty$ with respect to

$$\varphi(\cdot) := \langle \frac{1}{N}\mathrm{tr}(\cdot)\rangle_{ensemble}.$$

2) randomly rotated matrices

Let A_N and B_N be symmetric deterministic (e.g. diagonal) $N \times N$ matrices with

$$\lim_{N\to\infty} \mathrm{distr}(A_N) = \mu, \qquad \lim_{N\to\infty} \mathrm{distr}(B_N) = \nu$$

for some compactly supported probability measures μ and ν. Let U_N be a random unitary matrix from the ensemble

$$\Omega_N = (\mathcal{U}(N), Haar\ measure).$$

Consider now

$$X_N := A_N, \qquad Y_N := U_N B_N U_N^*.$$

Then X_N and Y_N become free in the limit $N \to \infty$ with respect to

$$\varphi(\cdot) := \langle \frac{1}{N}\mathrm{tr}(\cdot)\rangle_{\Omega_N}.$$

2. Combinatorial aspects of freeness: the concept of cumulants

'Freeness' of random variables is defined in terms of mixed moments; namely the defining property is that very special moments (alternating and centered ones) have to vanish. This requirement is not easy to handle in concrete calculations. Thus we will present here another approach to freeness, more combinatorial in nature, which puts the main emphasis on so called 'free cumulants'. These are some polynomials in the moments which behave much better with respect to freeness than the moments. The nomenclature comes from classical probability theory where corresponding objects are also well known and are usually called 'cumulants' or 'semi-invariants'. There exists a combinatorial description of these classical cumulants, which depends on partitions of sets. In the same way, free cumulants can also be described combinatorially, the only difference to the classical case is that one has to replace all partitions by so called 'non-crossing partitions'.

In the case of one random variable, we will also indicate the relation of this combinatorial description with the analytical one presented in the course of Biane; namely our cumulants are in this case just the coefficients of the R-transform of Voiculescu (in the classical case the cumulants are the coefficients of the logarithm of the Fourier transform). Thus we will obtain purely combinatorial proofs of the main results on the R-transform.

This combinatorial description of freeness is due to me [9, 11, 12] (see also [5]); in a series of joint papers with A. Nica [6, 7, 8] it was pursued very far and yielded a lot of new results in free probability theory. I will restrict here mainly to the basic facts; for applications one should consult the original papers or my survey [13]. A recent fundamental link between freeness and the representation theory of the permutation groups S_n in the limit $n \to \infty$, which rests also on the combinatorial description of freeness, is due to Biane [1].

2.1. **Definitions.** A **partition** of the set $S := \{1, \dots, n\}$ is a decomposition

$$\pi = \{V_1, \dots, V_r\}$$

of S into disjoint and non-empty sets V_i, i.e.

$$V_i \neq \emptyset \quad (i = 1, \dots, r) \qquad \text{and} \qquad S = \dot{\cup}_{i=1}^{r} V_i.$$

We denote the set of all partitions of S with $\mathcal{P}(S)$.
We call the V_i the **blocks** of π.
For $1 \leq p, q \leq n$ we write

$$p \sim_\pi q \qquad \text{if } p \text{ and } q \text{ belong to the same block of } \pi.$$

A partition π is called **non-crossing** if the following does not occur: There exist $1 \leq p_1 < q_1 < p_2 < q_2 \leq n$ with

$$p_1 \sim_\pi p_2 \not\sim_\pi q_1 \sim_\pi q_2.$$

The set of all non-crossing partitions of $\{1, \dots, n\}$ is denoted by $\mathbf{NC(n)}$. We denote the 'biggest' and the 'smallest' element in $NC(n)$ by $\mathbf{1}_n$ and $\mathbf{0}_n$, respectively:

$$\mathbf{1}_n := \{(1, \dots, n)\}$$
$$\mathbf{0}_n := \{(1), \dots, (n)\}.$$

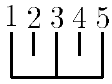
Non-crossing partitions were introduced by Kreweras [3] in a purely combinatorial context without any reference to probability theory.

2.2. **Examples.** We will also use a graphical notation for our partitions; the term 'non-crossing' will become evident in such a notation. Let

$$S = \{1, 2, 3, 4, 5\}.$$

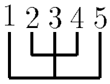
Then the partition

$$\pi = \{(1, 3, 5), (2), (4)\} \qquad \hat{=}$$



is non-crossing, whereas

$$\pi = \{(1, 3, 5), (2, 4)\} \qquad \hat{=}$$



is crossing.

2.3. **Remarks.** 1) In an analogous way, non-crossing partitions $NC(S)$ can be defined for any linearly ordered set S; of course, we have

$$NC(S_1) \cong NC(S_2) \qquad \text{if} \qquad \#S_1 = \#S_2.$$

2) In most cases the following recursive description of non-crossing partitions is of great use: a partition π ist non-crossing if and only if at least one block $V \in \pi$ is an interval and $\pi \backslash V$ is non-crossing; i.e. $V \in \pi$ has the form

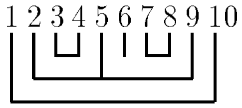
$$V = (k, k+1, \dots, k+p) \qquad \text{for some } 1 \leq k \leq n \text{ and } p \geq 0,\ k+p \leq n$$

and we have

$$\pi \backslash V \in NC(1, \dots, k-1, k+p+1, \dots, n) \cong NC(n-(p+1)).$$

Example: The partition

$$\{(1, 10), (2, 5, 9), (3, 4), (6), (7, 8)\} \qquad \hat{=}$$



can, by successive removal of intervals, be reduced to

$$\{(1, 10), (2, 5, 9)\} \hat{=} \{(1, 5), (2, 3, 4)\}$$

and finally to

$$\{(1, 5)\} \hat{=} \{(1, 2)\}.$$

3) By writing a partition π in the form

$$\pi = \{V_1, \dots, V_r\}$$

we will always assume that the elements within each block V_i are ordered in increasing order.

2.4. **Definition.** Let $(\mathcal{A}, \varphi)$ be a probability space, i.e. $\mathcal{A}$ is a unital algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ is a unital linear functional. We define the **(free or non-crossing) cumulants**

$$k_n : \mathcal{A}^n \to \mathbb{C} \qquad (n \in \mathbb{N})$$

(indirectly) by the following system of equations:

$$\varphi(a_1 \dots a_n) = \sum_{\pi \in NC(n)} k_\pi[a_1, \dots, a_n] \qquad (a_1, \dots, a_n \in \mathcal{A}),$$

where k_π denotes a product of cumulants according to the block structure of π:

$$k_\pi[a_1, \dots, a_n] := k_{V_1}[a_1, \dots, a_n] \dots k_{V_r}[a_1, \dots, a_n] \qquad \text{for } \pi = \{V_1, \dots, V_r\} \in NC(n)$$

and

$$k_V[a_1, \dots, a_n] := k_{\#V}[a_{v_1}, \dots, a_{v_l}] \qquad \text{for} \qquad V = (v_1, \dots, v_l).$$

2.5. **Remarks and Examples.** 1) Note: the above equations have the form

$$\varphi(a_1 \dots a_n) = k_n(a_1, \dots, a_n) + \sum_{\substack{\pi \in NC(n) \\ \pi \neq \mathbf{1}_n}} k_\pi[a_1, \dots, a_n],$$

and thus they can be resolved for the $k_n(a_1, \dots, a_n)$ in a unique way.
2) Examples:

- $n = 1$

$$\varphi(a_1) = k_{|}[a_1] = k_1(a_1),$$

thus

$$k_1(a_1) = \varphi(a_1).$$

- $n = 2$

$$\begin{aligned} \varphi(a_1 a_2) &= k_{\sqcup}[a_1, a_2] + k_{|\,|}[a_1, a_2] \\ &= k_2(a_1, a_2) + k_1(a_1) k_1(a_2), \end{aligned}$$

thus

$$k_2(a_1, a_2) = \varphi(a_1 a_2) - \varphi(a_1)\varphi(a_2).$$

- $n = 3$

$$\begin{aligned} \varphi(a_1 a_2 a_3) &= k_{\sqcup\!\sqcup}[a_1, a_2, a_3] + k_{|\,\sqcup}[a_1, a_2, a_3] + k_{\sqcup\,|}[a_1, a_2, a_3] \\ &\qquad + k_{\sqcup\!\!\!\!\sqcup}[a_1, a_2, a_3] + k_{|\,|\,|}[a_1, a_2, a_3] \\ &= k_3(a_1, a_2, a_3) + k_1(a_1) k_2(a_2, a_3) + k_2(a_1, a_2) k_1(a_3) \\ &\qquad + k_2(a_1, a_3) k_1(a_1) + k_1(a_1) k_1(a_2) k_1(a_3), \end{aligned}$$

and thus

$$k_3(a_1,a_2,a_3) = \varphi(a_1a_2a_3) - \varphi(a_1)\varphi(a_2a_3) - \varphi(a_1a_3)\varphi(a_2) \\ - \varphi(a_1a_2)\varphi(a_3) + 2\varphi(a_1)\varphi(a_2)\varphi(a_3).$$

3) For $n = 4$ we consider the special case where all $\varphi(a_i) = 0$. Then we have

$$k_4(a_1,a_2,a_3,a_4) = \varphi(a_1a_2a_3a_4) - \varphi(a_1a_2)\varphi(a_3a_4) - \varphi(a_1a_4)\varphi(a_2a_3).$$

4) The k_n are multi-linear functionals in their n arguments.

The meaning of the concept 'cumulants' for freeness is shown by the following theorem.

2.6. **Theorem.** Let $(\mathcal{A},\varphi)$ be a probability space and consider unital subalgebras $\mathcal{A}_1,\dots,\mathcal{A}_m \subset \mathcal{A}$. Then the following two statements are equivalent:

i) $\mathcal{A}_1,\dots,\mathcal{A}_m$ are free.
ii) We have for all $n \geq 2$ and for all $a_i \in \mathcal{A}_{j(i)}$ with $1 \leq j(1),\dots,j(n) \leq m$:

$$k_n(a_1,\dots,a_n) = 0 \qquad \text{if there exist } 1 \leq l,k \leq n \text{ with } j(l) \neq j(k).$$

2.7. **Remarks.** 1) This characterization of freeness in terms of cumulants is the translation of the definition of freeness in terms of moments – by using the relation between moments and cumulants from Definition 2.4. One should note that in contrast to the characterization in terms of moments we do not require that $j(1) \neq j(2) \neq \dots \neq j(n)$ or $\varphi(a_i) = 0$. Hence the characterization of freeness in terms of cumulants is much easier to use in concrete calculations.
2) Since the unit 1 is free from everything, the above theorem contains as a special case the statement:

$$k_n(a_1,\dots,a_n) = 0 \qquad \text{if } n \geq 2 \text{ and } a_i = 1 \text{ for at least one } i.$$

This special case will also present an important step in the proof of Theorem 2.6 and it will be proved separately as a lemma.
3) Note also: for $n = 1$ we have

$$k_1(1) = \varphi(1) = 1.$$

Proof. (i) $\Longrightarrow$ (ii): If all a_i are centered, i.e. $\varphi(a_i) = 0$, and alternating, i.e. $j(1) \neq j(2) \neq \dots \neq j(n)$, then the assertion follows directly by the definition of freeness and by the relation

$$\varphi(a_1 \dots a_n) = k_n(a_1,\dots,a_n) + \sum_{\substack{\pi \in NC(n) \\ \pi \neq \mathbf{1}_n}} k_\pi[a_1,\dots,a_n],$$

because at least one factor of k_π for $\pi \neq \mathbf{1}_n$ is of the form

$$k_{p+1}(a_l, a_{l+1}, \dots, a_{l+p}) \qquad \text{with } p+1 < n$$

and thus the assertion follows by induction.
The essential part of the proof consists in showing that on the level of cumulants the assumption 'centered' is not needed and 'alternating' can be weakened to 'mixed'.
Let us start with getting rid of the assumption 'centered'. For this we will need the following lemma – which is of course a special case of our theorem.

2.6.1. *Lemma.* Let $n \geq 2$ und $a_1, \dots, a_n \in \mathcal{A}$. Then we have:

$$\text{there exists a } 1 \leq i \leq n \text{ with } a_i = 1 \qquad \Longrightarrow \qquad k_n(a_1, \dots, a_n) = 0.$$

Proof. To simplify notation we consider the case $a_n = 1$, i.e. we want to show

$$k_n(a_1, \dots, a_{n-1}, 1) \stackrel{!}{=} 0.$$

We will prove this by induction on n.
$n = 2$: the assertion is true, since

$$k_2(a, 1) = \varphi(a1) - \varphi(a)\varphi(1) = 0.$$

$n-1 \to n$: Assume we have proved the assertion for all $k < n$. Then we have

$$\begin{aligned} \varphi(a_1 \dots a_{n-1} 1) &= \sum_{\pi \in NC(n)} k_\pi[a_1, \dots, a_{n-1}, 1] \\ &= k_n(a_1, \dots, a_{n-1}, 1) + \sum_{\substack{\pi \in NC(n) \\ \pi \neq \mathbf{1}_n}} k_\pi[a_1, \dots, a_{n-1}, 1]. \end{aligned}$$

According to our induction hypothesis only such $\pi \neq \mathbf{1}_n$ contribute to the above sum which have the property that $\{n\}$ is a one-element block of π, i.e. which have the form

$$\pi = \sigma \cup (n) \qquad \text{with} \qquad \sigma \in NC(n-1).$$

Then we have

$$\begin{aligned} k_\pi[a_1, \dots, a_{n-1}, 1] &= k_\sigma[a_1, \dots, a_{n-1}] k_1(1) \\ &= k_\sigma[a_1, \dots, a_{n-1}], \end{aligned}$$

hence

$$\begin{aligned} \varphi(a_1 \dots a_{n-1} 1) &= k_n(a_1, \dots, a_{n-1}, 1) + \sum_{\sigma \in NC(n-1)} k_\sigma[a_1, \dots, a_{n-1}] \\ &= k_n(a_1, \dots, a_{n-1}, 1) + \varphi(a_1 \dots a_{n-1}). \end{aligned}$$

Since

$$\varphi(a_1 \dots a_{n-1} 1) = \varphi(a_1 \dots a_{n-1}),$$

we obtain

$$k_n(a_1, \dots, a_{n-1}, 1) = 0.$$

□

Let $n \geq 2$. Then this lemma implies that we have for arbitrary $a_1, \dots, a_n \in \mathcal{A}$ the relation

$$k_n(a_1, \dots, a_n) = k_n\big(a_1 - \varphi(a_1)1, \dots, a_n - \varphi(a_n)1\big),$$

i.e. we can center the arguments of our cumulants k_n ($n \geq 2$) without changing the value of the cumulants.
Thus we have proved the following statement: Consider $n \geq 2$ and $a_i \in \mathcal{A}_{j(i)}$ ($i = 1, \dots, n$) with $j(1) \neq j(2) \neq \dots \neq j(n)$. Then we have

$$k_n(a_1, \dots, a_n) = 0.$$

It remains to weaken the assumption 'alternating' to 'mixed'. For this we will need the following lemma.

2.6.2. *Lemma.* Consider $n \geq 2$, $a_1, \dots, a_n \in \mathcal{A}$ and $1 \leq p \leq n-1$. Then we have

$$\begin{aligned} k_{n-1}(a_1, \dots, a_{p-1}, a_p a_{p+1}, a_{p+2}, \dots, a_n) &= k_n(a_1, \dots, a_p, a_{p+1}, \dots, a_n) \\ &+ \sum_{\substack{\pi \in NC(n) \\ \#\pi = 2 \\ p \not\sim_\pi p+1}} k_\pi[a_1, \dots, a_p, a_{p+1}, \dots, a_n]. \end{aligned}$$

Examples:

$$k_2(a_1 a_2, a_3) = k_3(a_1, a_2, a_3) + k_1(a_1) k_2(a_2, a_3) + k_2(a_1, a_3) k_1(a_2)$$

$$\begin{aligned} k_3(a_1, a_2 a_3, a_4) = k_4(a_1, a_2, a_3, a_4) &+ k_1(a_2) k_3(a_1, a_3, a_4) \\ &+ k_2(a_1, a_2) k_2(a_3, a_4) + k_3(a_1, a_2, a_4) k_1(a_3) \end{aligned}$$

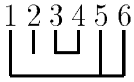
Proof. For $\pi \in NC(n)$ we denote by $\pi|_{p=p+1} \in NC(n-1)$ that partition which is obtained by identifying p and $p+1$, i.e. for $\pi = \{V_1, \dots, V_r\}$ we have

$$\pi|_{p=p+1} = \{V_1, \dots, (V_k \cup V_l) \backslash \{p+1\}, \dots, V_r\}, \qquad \text{if } p \in V_k \text{ and } p+1 \in V_l.$$

(If p and $p+1$ belong to different blocks, then $\pi|_{p=p+1}$ has one block less than π; if p and $p+1$ belong to the same block, then the number of blocks does not change; of course, we identify partitions of the set $\{1, \dots, p, p+2, \dots, n\}$ with partitions from $NC(n-1)$; the property

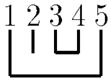
'non-crossing' is preserved under the transition from π to $\pi|_{p=p+1}$.)
Example: Consider

$$\pi = \{(1,5,6),(2),(3,4)\} \qquad \hat{=}$$



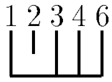
Then we have

$$\pi|_{5=6} = \{(1,5),(2),(3,4)\} \qquad \hat{=}$$



and

$$\pi|_{4=5} = \{(1,3,4,6),(2)\} \hat{=} \{(1,3,4,5),(2)\} \qquad \hat{=}$$



With the help of this definition we can state our assertion more generally for k_σ for arbitrary $\sigma \in NC(n-1)$: Assume that our assertion is true for all $l < n$, i.e.

$$\begin{aligned} k_{l-1}(a_1,\dots,a_pa_{p+1},\dots,a_l) &= k_{\mathbf{1}_{l-1}}[a_1,\dots,a_pa_{p+1},\dots,a_l] \\ &= \sum_{\substack{\pi\in NC(l)\\ \pi|_{p=p+1}=\mathbf{1}_{l-1}}} k_\pi[a_1,\dots,a_p,a_{p+1},\dots,a_l]. \end{aligned}$$

Then it is quite easy to see that we have for arbitrary $\sigma \in NC(n-1)$ with $\sigma \neq \mathbf{1}_{n-1}$:

$$k_\sigma[a_1,\dots,a_pa_{p+1},\dots,a_n] = \sum_{\substack{\pi\in NC(n)\\ \pi|_{p=p+1}=\sigma}} k_\pi[a_1,\dots,a_p,a_{p+1},\dots,a_n].$$

We will now prove the assertion of our lemma by induction n.
$n = 2$: The assertion is true because

$$\begin{aligned} k_1(a_1a_2) &= \varphi(a_1a_2) \\ &= (\varphi(a_1a_2) - \varphi(a_1)\varphi(a_2)) + \varphi(a_1)\varphi(a_2) \\ &= k_2(a_1,a_2) + k_1(a_1)k_1(a_2). \end{aligned}$$

$n-1 \to n$: Let the assertion be proven for all $l < n$, which implies, as indicated above, that we have also for all $\sigma \in NC(n-1)$ with $\sigma \neq \mathbf{1}_{n-1}$

$$k_\sigma[a_1,\dots,a_pa_{p+1},\dots,a_n] = \sum_{\substack{\pi\in NC(n)\\ \pi|_{p=p+1}=\sigma}} k_\pi[a_1,\dots,a_p,a_{p+1},\dots,a_n].$$

Then we have

$$
\begin{aligned}
&k_{n-1}(a_1,\dots,a_pa_{p+1},\dots,a_n)\\
&=\varphi(a_1\dots(a_pa_{p+1})\dots a_n)-\sum_{\substack{\sigma\in NC(n-1)\\ \sigma\neq\mathbf{1}_{n-1}}} k_\sigma[a_1,\dots,a_pa_{p+1},\dots,a_n]\\
&=\varphi(a_1\dots a_pa_{p+1}\dots a_n)\\
&\qquad\qquad-\sum_{\substack{\sigma\in NC(n-1)\\ \sigma\neq\mathbf{1}_{n-1}}}\ \sum_{\substack{\pi\in NC(n)\\ \pi|_{p=p+1}=\sigma}} k_\pi[a_1,\dots,a_p,a_{p+1},\dots,a_n]\\
&=\varphi(a_1\dots a_pa_{p+1}\dots a_n)-\sum_{\substack{\pi\in NC(n)\\ \pi|_{p=p+1}\neq\mathbf{1}_{n-1}}} k_\pi[a_1,\dots,a_p,a_{p+1},\dots,a_n]\\
&=\sum_{\pi\in NC(n)} k_\pi[a_1,\dots,a_p,a_{p+1},\dots,a_n]\\
&\qquad\qquad-\sum_{\substack{\pi\in NC(n)\\ \pi|_{p=p+1}\neq\mathbf{1}_{n-1}}} k_\pi[a_1,\dots,a_p,a_{p+1},\dots,a_n]\\
&=\sum_{\substack{\pi\in NC(n)\\ \pi|_{p=p+1}=\mathbf{1}_{n-1}}} k_\pi[a_1,\dots,a_p,a_{p+1},\dots,a_n]\\
&=k_n(a_1,\dots,a_p,a_{p+1},\dots,a_n)+\sum_{\substack{\pi\in NC(n)\\ \#\pi=2\\ p\not\sim_\pi p+1}} k_\pi[a_1,\dots,a_p,a_{p+1},\dots,a_n].
\end{aligned}
$$

□

By using this lemma we can now prove our theorem in full generality: Consider $n\geq 2$ and $a_i\in\mathcal{A}_{j(i)}$ $(i=1,\dots,n)$. Assume that there exist k,l with $j(k)\neq j(l)$. We have to show

$$k_n(a_1,\dots,a_n)\overset{!}{=}0.$$

This follows so: If $j(1)\neq j(2)\neq\dots\neq j(n)$, then the assertion is already proved. Thus we can assume that there exists a p with $j(p)=j(p+1)$, implying $a_pa_{p+1}\in\mathcal{A}_{j(p)}$. In that case we can use the above lemma to obtain

$$
\begin{aligned}
k_n(a_1,\dots,a_p,a_{p+1},\dots,a_n)&=k_{n-1}(a_1,\dots,a_pa_{p+1},\dots,a_n)\\
&\quad-\sum_{\substack{\pi\in NC(n)\\ \#\pi=2\\ p\not\sim_\pi p+1}} k_\pi[a_1,\dots,a_p,a_{p+1},\dots,a_n].
\end{aligned}
$$

To show that this vanishes we will again use induction on n: The first term $k_{n-1}(a_1,\dots,a_pa_{p+1},\dots,a_n)$ vanishes by induction hypothesis, since two of its arguments are lying in the different algebras $\mathcal{A}_{j(k)}$ and $\mathcal{A}_{j(l)}$. Consider now the summand $k_\pi[a_1,\dots,a_p,a_{p+1},\dots,a_n]$ for $\pi\in NC(n)$ with

$$\pi=\{V_1,V_2\},\qquad \text{where } p\in V_1 \text{ and } p+1\in V_2.$$

Then we have $k_\pi = k_{V_1}k_{V_2}$, and by induction hypothesis this can be different from zero only in the case where all arguments in each of the two factors are coming from the same algebra; but this would impy that in the first factor all arguments are in $\mathcal{A}_{j(p)}$ and in the second factor all arguments are in $\mathcal{A}_{j(p+1)}$. Because of $j(p)=j(p+1)$ this would imply

$$j(1)=j(2)=\dots=j(n)$$

yielding a contradiction with $j(l)\neq j(k)$. Thus all terms of the right hand side have to vanish and we obtain

$$k_n(a_1,\dots,a_p,a_{p+1},\dots,a_n)=0.$$

(ii) $\Longrightarrow$ (i): (ii) gives an inductive way to calculate uniquely all mixed moments; according to what we have proved above this mixed moments must calculate in the same way as for free subalgebras; but this means of course that these subalgebras are free. □

2.8. **Notation.** For a random variable $a\in\mathcal{A}$ we put

$$k_n^a := k_n(a,\dots,a)$$

and call $(k_n^a)_{n\geq 1}$ the **(free) cumulants of** a.

Our main theorem on the vanishing of mixed cumulants in free variables specifies in this one-dimensional case to the linearity of the cumulants.

2.9. **Proposition.** Let a and b be free. Then we have

$$k_n^{a+b}=k_n^a+k_n^b\qquad \text{for all } n\geq 1.$$

Proof. We have

$$\begin{aligned} k_n^{a+b} &= k_n(a+b,\dots,a+b)\\ &= k_n(a,\dots,a)+k_n(b,\dots,b)\\ &= k_n^a+k_n^b, \end{aligned}$$

because cumulants which have both a and b as arguments vanish by Theorem 2.6. □

Thus, free convolution is easy to describe on the level of cumulants; the cumulants are additive under free convolution. It remains to make the connection between moments and cumulants as explicit as possible. On a combinatorial level, our definition specializes in the one-dimensional case to the following relation.

2.10. **Proposition.** Let $(m_n)_{n\geq 1}$ and $(k_n)_{n\geq 1}$ be the moments and free cumulants, respectively, of some random variable. The connection between these two sequences of numbers is given by

$$m_n = \sum_{\pi \in NC(n)} k_\pi,$$

where

$$k_\pi := k_{\#V_1} \cdots k_{\#V_r} \qquad \text{for} \qquad \pi = \{V_1, \dots, V_r\}.$$

Example: For $n = 3$ we have

$$\begin{aligned} m_3 &= k_{\sqcup\!\sqcup} + k_{|\,\sqcup} + k_{\sqcup\,|} + k_{\sqcup\!\!\!\!\,|\,} + k_{|\,|\,|} \\ &= k_3 + 3k_1k_2 + k_1^3. \end{aligned}$$

For concrete calculations, however, one would prefer to have a more analytical description of the relation between moments and cumulants. This can be achieved by translating the above relation to corresponding formal power series.

2.11. **Theorem.** Let $(m_n)_{n\geq 1}$ and $(k_n)_{n\geq 1}$ be two sequences of complex numbers and consider the corresponding formal power series

$$M(z) := 1 + \sum_{n=1}^{\infty} m_n z^n,$$

$$C(z) := 1 + \sum_{n=1}^{\infty} k_n z^n.$$

Then the following three statements are equivalent:

(i) We have for all $n \in \mathbb{N}$

$$m_n = \sum_{\pi \in NC(n)} k_\pi = \sum_{\pi = \{V_1, \dots, V_r\} \in NC(n)} k_{\#V_1} \dots k_{\#V_r}.$$

(ii) We have for all $n \in \mathbb{N}$ (where we put $m_0 := 1$)

$$m_n = \sum_{s=1}^{n} \sum_{\substack{i_1, \dots, i_s \in \{0,1,\dots,n-s\} \\ i_1 + \dots + i_s = n-s}} k_s m_{i_1} \dots m_{i_s}.$$

(iii) We have

$$C[zM(z)] = M(z).$$

Proof. We rewrite the sum

$$m_n = \sum_{\pi \in NC(n)} k_\pi$$

in the way that we fix the first block V_1 of π (i.e. that block which contains the element 1) and sum over all possibilities for the other blocks; in the end we sum over V_1:

$$m_n = \sum_{s=1}^{n} \sum_{V_1 \text{ with } \#V_1 = s} \sum_{\substack{\pi \in NC(n) \\ \text{where } \pi = \{V_1, \dots\}}} k_\pi.$$

If

$$V_1 = (v_1 = 1, v_2, \dots, v_s),$$

then $\pi = \{V_1, \dots\} \in NC(n)$ can only connect elements lying between some v_k and v_{k+1}, i.e. $\pi = \{V_1, V_2, \dots, V_r\}$ such that we have for all $j = 2, \dots, r$: there exists a k with $v_k < V_j < v_{k+1}$. There we put

$$v_{s+1} := n + 1.$$

Hence such a π decomposes as

$$\pi = V_1 \cup \tilde{\pi}_1 \cup \cdots \cup \tilde{\pi}_s,$$

where

$$\tilde{\pi}_j \text{ is a non-crossing partition of } \{v_j + 1, v_j + 2, \dots, v_{j+1} - 1\}.$$

For such π we have

$$\begin{aligned} k_\pi &= k_{\#V_1} k_{\tilde{\pi}_1} \dots k_{\tilde{\pi}_s} \\ &= k_s k_{\tilde{\pi}_1} \dots k_{\tilde{\pi}_s}, \end{aligned}$$

and thus we obtain

$$
\begin{aligned}
m_n &= \sum_{s=1}^{n} \sum_{1=v_1<v_2<\dots<v_s\le n} \sum_{\substack{\pi=V_1\cup\tilde{\pi}_1\cup\dots\cup\tilde{\pi}_s\\ \tilde{\pi}_j\in NC(v_j+1,\dots,v_{j+1}-1)}} k_s k_{\tilde{\pi}_1}\dots k_{\tilde{\pi}_s}\\
&= \sum_{s=1}^{n} k_s \sum_{1=v_1<v_2<\dots<v_s\le n} \Big(\sum_{\tilde{\pi}_1\in NC(v_1+1,\dots,v_2-1)} k_{\tilde{\pi}_1}\Big)\dots\Big(\sum_{\tilde{\pi}_s\in NC(v_s+1,\dots,n)} k_{\tilde{\pi}_s}\Big)\\
&= \sum_{s=1}^{n} k_s \sum_{1=v_1<v_2<\dots<v_s\le n} m_{v_2-v_1-1}\dots m_{n-v_s}\\
&= \sum_{s=1}^{n} \sum_{\substack{i_1,\dots,i_s\in\{0,1,\dots,n-s\}\\ i_1+\dots+i_s+s=n}} k_s m_{i_1}\dots m_{i_s} \qquad (i_k := v_{k+1}-v_k-1).
\end{aligned}
$$

This yields the implication (i) $\Longrightarrow$ (ii).
We can now rewrite (ii) in terms of the corresponding formal power series in the following way (where we put $m_0 := k_0 := 1$):

$$
\begin{aligned}
M(z) &= 1+\sum_{n=1}^{\infty} z^n m_n\\
&= 1+\sum_{n=1}^{\infty}\sum_{s=1}^{n} \sum_{\substack{i_1,\dots,i_s\in\{0,1,\dots,n-s\}\\ i_1+\dots+i_s=n-s}} k_s z^s m_{i_1} z^{i_1}\dots m_{i_s} z^{i_s}\\
&= 1+\sum_{s=1}^{\infty} k_s z^s \Big(\sum_{i=0}^{\infty} m_i z^i\Big)^s\\
&= C[zM(z)].
\end{aligned}
$$

This yields (iii).
Since (iii) describes uniquely a fixed relation between the numbers $(k_n)_{n\ge1}$ and the numbers $(m_n)_{n\ge1}$, this has to be the relation (i). □

If we rewrite the above relation between the formal power series in terms of the Cauchy-transform

$$G(z) := \sum_{n=0}^{\infty} \frac{m_n}{z^{n+1}}$$

and the R-transform

$$R(z) := \sum_{n=0}^{\infty} k_{n+1} z^n,$$

then we obtain Voiculescu's formula.

2.12. **Corollary.** The relation between the Cauchy-transform $G(z)$ and the R-transform $R(z)$ of a random variable is given by

$$G[R(z) + \frac{1}{z}] = z.$$

Proof. We just have to note that the formal power series $M(z)$ and $C(z)$ from Theorem 2.11 and $G(z)$, $R(z)$, and $K(z) = R(z) + \frac{1}{z}$ are related by:

$$G(z) = \frac{1}{z} M(\frac{1}{z})$$

and

$$C(z) = 1 + zR(z) = zK(z), \qquad \text{thus} \qquad K(z) = \frac{C(z)}{z}.$$

This gives

$$\begin{aligned} K[G(z)] &= \frac{1}{G(z)} C[G(z)] \\ &= \frac{1}{G(z)} C[\frac{1}{z} M(\frac{1}{z})] \\ &= \frac{1}{G(z)} M(\frac{1}{z}) \\ &= z, \end{aligned}$$

thus

$$K[G(z)] = z$$

and hence also

$$G[R(z) + \frac{1}{z}] = G[K(z)] = z.$$

□

2.13. **Remark.** It is quite easy to check that the cumulants k_n^a of a random variable a are indeed the coefficients of the R-transform of a as introduced by Voiculescu: Remember that the distribution of a was modelled by the canonical variable (special formal power series in an isometry l^*, see [18])

$$b = l^* + \sum_{i=0}^{\infty} k_{i+1} l^i \in (\Theta(l), \tau).$$

Then we have

$$\begin{aligned} m_n &= \tau\big((l^* + \sum_{i=0}^{\infty} k_{i+1} l^i)^n\big) \\ &= \sum_{i(1),\dots,i(n)\in\{-1,0,1,\dots,n-1\}} \tau(l^{i(n)} \dots l^{i(1)}) k_{i(1)+1} \dots k_{i(n)+1}, \end{aligned}$$

where l^{-1} is identified with l^*,

$$l^{-1} \quad \hat{=} \quad l^*$$

and

$$k_0 := 1.$$

The sum is running over tuples $(i(1), \dots, i(n))$, which can be identified with paths in the lattice $\mathbb{Z}^2$:

$$\begin{aligned} i = -1 \quad &\hat{=} \quad \text{diagonal step upwards: } \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ i = 0 \quad &\hat{=} \quad \text{horizontal step to the right: } \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ i = k \quad (1 \le k \le n-1) \quad &\hat{=} \quad \text{diagonal step downwards: } \begin{pmatrix} 1 \\ -k \end{pmatrix} \end{aligned}$$

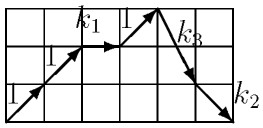
We have now

$$\tau(l^{i(n)} \dots l^{i(1)}) = \begin{cases} 1, & \text{if } i(1) + \dots + i(m) \le 0 \;\forall\, m = 1, \dots, n \text{ and} \\ & \qquad i(1) + \dots + i(n) = 0 \\ 0, & \text{otherwise} \end{cases}$$

and thus

$$m_n = \sum_{\substack{i(1),\dots,i(n)\in\{-1,0,1,\dots,n-1\} \\ i(1)+\dots+i(m)\le 0 \quad \forall m=1,\dots,n \\ i(1)+\dots+i(n)=0}} k_{i(1)+1} \dots k_{i(n)+1}.$$

Hence only such paths from $(0,0)$ to $(n,0)$ contribute which stay always above the x-axis. Each such path is weighted in a multiplicative way (using the cumulants) with the length of its steps.
Example:



$$\hat{=} \qquad \tau(l^1 l^2 l^* l^0 l^* l^*) k_1 k_3 k_2$$

The above summation can now be rewritten in terms of a summation over non-crossing partitions leading to the relation from Proposition 2.10. We will leave the proof of this lemma to the reader

2.13.1. *Lemma.* There exists a canonical bijection

$$NC(n) \longleftrightarrow \{(i(1),\dots,i(n))\,|\, i(m)\in\{-1,0,1,\dots,n-1\},\quad i(1)+\dots+i(m)\le 0 \quad \forall\, m=1,\dots,n;\quad i(1)+\dots+i(n)=0 \}.$$

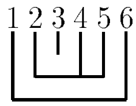
It is given by

$$\pi \mapsto \Pi = (i(1),\dots,i(n))$$

where

$$i(m)=\begin{cases}\#V_i-1, & \text{if } m \text{ is the last element in a block } V_i\\ -1, & \text{otherwise}\end{cases}$$

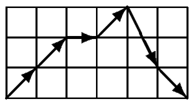
Note that a block consisting of one element gives a horizontal step; a block consisting of k $(k\ge 2)$ elements gives $k-1$ upward steps each of length 1 and one downward step of length $k-1$.
An example for this bijection is



$$\hat{=} \quad \pi=\{(1,6),(2,4,5),(3)\}$$

is mapped to

$$\Pi=(-1,-1,0,-1,2,1) \quad \hat{=}$$



Now note that with this identification of paths and non-crossing partitions the factor

$$k_\Pi = k_{i(1)+1}\dots k_{i(n)+1}$$

for

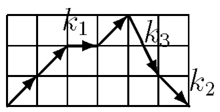
$$\Pi=(i(1),\dots,i(n)) \qquad \hat{=} \qquad \pi=\{V_1,\dots,V_r\}$$

goes over to

$$k_\pi := k_{\#V_1}\dots k_{\#V_r}.$$

Consider the above example:

$$\pi=\{(1,6),(2,4,5),(3)\}\mapsto \Pi \quad \hat{=}$$



thus

$$k_\pi = k_\Pi = k_1k_3k_2 = k_{\#(3)}k_{\#(1,6)}k_{\#(2,4,5)}.$$

This correspondence leads of course to the relation as stated in Proposition 2.10 ; thus the coefficients of the R-transform of Voiculescu coincide indeed with the free cumulants as defined in 2.8. Note that in this way we obtained easy combinatorial proofs of the main facts on the R-transform – namely, its additivity under free convolution and the formula relating it to the Cauchy-transform.

Finally, to show that our description of freeness in terms of cumulants has also a significance apart from dealing with additive free convolution, we will apply it to the problem of the product of free random variables: Consider $a_1, \dots, a_n, b_1, \dots, b_n$ such that $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ are free. We want to express the distribution of the random variables $a_1b_1, \dots, a_nb_n$ in terms of the distribution of the a's and of the b's.

2.14. **Notation.** 1) Analogously to k_π we define for

$$\pi = \{V_1, \dots, V_r\} \in NC(n)$$

the expression

$$\varphi_\pi[a_1 \dots, a_n] := \varphi_{V_1}[a_1, \dots, a_n] \dots \varphi_{V_r}[a_1, \dots, a_n],$$

where

$$\varphi_V[a_1, \dots, a_n] := \varphi(a_{v_1} \dots a_{v_l}) \qquad \text{for} \qquad V = (v_1, \dots, v_l).$$

Examples:

$$\varphi_{\sqcup\!\sqcup}[a_1, a_2, a_3] = \varphi(a_1a_2a_3)$$

$$\varphi_{|\,\sqcup}[a_1, a_2, a_3] = \varphi(a_1)\varphi(a_2a_3)$$

$$\varphi_{\sqcup\,|}[a_1, a_2, a_3] = \varphi(a_1a_2)\varphi(a_3)$$

$$\varphi_{\sqcup\!\!|\!\!\sqcup}[a_1, a_2, a_3] = \varphi(a_1a_3)\varphi(a_2)$$

$$\varphi_{|\,|\,|}[a_1, a_2, a_3] = \varphi(a_1)\varphi(a_2)\varphi(a_3)$$

2) Let $\sigma, \pi \in NC(n)$. Then we write

$$\sigma \leq \pi$$

if each block of σ is contained as a whole in some block of π, i.e. σ can be obtained out of π by refinement of the block structure.
Example:

$$\{(1), (2, 4), (3), (5, 6)\} \leq \{(1, 5, 6), (2, 3, 4)\}$$

With these notations we can generalize the relation

$$\varphi(a_1 \dots a_n) = \sum_{\pi \in NC(n)} k_\pi[a_1, \dots, a_n]$$

in the following way.

2.15. **Proposition.** Consider $n \in \mathbb{N}$, $\sigma \in NC(n)$ and $a_1, \dots, a_n \in \mathcal{A}$. Then we have

$$\varphi_\sigma[a_1, \dots, a_n] = \sum_{\substack{\pi \in NC(n) \\ \pi \leq \sigma}} k_\pi[a_1, \dots, a_n].$$

Proof. Each

$$\pi \leq \sigma = \{V_1, \dots, V_r\}$$

can be decomposed as

$$\pi = \pi_1 \cup \dots \cup \pi_r \qquad \text{where} \qquad \pi_i \in NC(V_i) \quad (i = 1, \dots, r).$$

In such a case we have of course

$$k_\pi = k_{\pi_1} \dots k_{\pi_r}.$$

Thus we obtain (omitting the arguments)

$$\begin{aligned}
\varphi_\sigma &= \varphi_{V_1} \dots \varphi_{V_r} \\
&= \Big(\sum_{\pi_1 \in NC(V_1)} k_{\pi_1} \Big) \dots \Big(\sum_{\pi_r \in NC(V_r)} k_{\pi_r} \Big) \\
&= \sum_{\pi = \pi_1 \cup \dots \cup \pi_r \leq \sigma} k_{\pi_1} \dots k_{\pi_r} \\
&= \sum_{\pi \leq \sigma} k_\pi.
\end{aligned}$$

□

Consider now

$$\{a_1, \dots, a_n\}, \{b_1, \dots, b_n\} \qquad \text{free.}$$

We want to express alternating moments in a and b in terms of moments of a and moments of b. We have

$$\varphi(a_1 b_1 a_2 b_2 \dots a_n b_n) = \sum_{\pi \in NC(2n)} k_\pi[a_1, b_1, a_2, b_2, \dots, a_n, b_n].$$

Since the a's are free from the b's, Theorem 2.6 tells us that only such π contribute to the sum whose blocks do not connect a's with b's. But this means that such a π has to decompose as

$$\pi = \pi_1 \cup \pi_2 \qquad \text{where} \quad \begin{aligned} &\pi_1 \in NC(1, 3, 5, \dots, 2n-1) \\ &\pi_2 \in NC(2, 4, 6, \dots, 2n). \end{aligned}$$

Thus we have

$$\begin{aligned}
&\varphi(a_1b_1a_2b_2\dots a_nb_n)\\
&= \sum_{\substack{\pi_1\in NC(odd)\\ \pi_2\in NC(even)\\ \pi_1\cup\pi_2\in NC(2n)}} k_{\pi_1}[a_1,a_2,\dots,a_n]\cdot k_{\pi_2}[b_1,b_2,\dots,b_n]\\
&= \sum_{\pi_1\in NC(odd)} \Big(k_{\pi_1}[a_1,a_2,\dots,a_n]\cdot \sum_{\substack{\pi_2\in NC(even)\\ \pi_1\cup\pi_2\in NC(2n)}} k_{\pi_2}[b_1,b_2,\dots,b_n]\Big).
\end{aligned}$$

Note now that for a fixed π_1 there exists a maximal element σ with the property $\pi_1\cup\sigma\in NC(2n)$ and that the second sum is running over all $\pi_2\le\sigma$.

2.16. **Definition.** Let $\pi\in NC(n)$ be a non-crossing partition of the numbers $1,\dots,n$. Introduce additional numbers $\bar 1,\dots,\bar n$, with alternating order between the old and the new ones, i.e. we order them in the way

$$1\bar 1 2\bar 2\dots n\bar n.$$

We define the **complement** $K(\pi)$ of π as the maximal $\sigma\in NC(\bar 1,\dots,\bar n)$ with the property

$$\pi\cup\sigma\in NC(1,\bar 1,\dots,n,\bar n).$$

If we present the partition π graphically by connecting the blocks in $1,\dots,n$, then σ is given by connecting as much as possible the numbers $\bar 1,\dots,\bar n$ without getting crossings among themselves and with π.

(This natural notation of the complement of a non-crossing partition is also due to Kreweras [3]. Note that there is no analogue of this for the case of all partitions.)

With this definition we can continue our above calculation as follows:

$$\begin{aligned}
&\varphi(a_1b_1a_2b_2\dots a_nb_n)\\
&\qquad= \sum_{\pi_1\in NC(n)} \Big(k_{\pi_1}[a_1,a_2,\dots,a_n]\cdot \sum_{\substack{\pi_2\in NC(n)\\ \pi_2\le K(\pi_1)}} k_{\pi_2}[b_1,b_2,\dots,b_n]\Big)\\
&\qquad= \sum_{\pi_1\in NC(n)} k_{\pi_1}[a_1,a_2,\dots,a_n]\cdot \varphi_{K(\pi_1)}[b_1,b_2,\dots,b_n].
\end{aligned}$$

Thus we have proved the following result.

2.17. **Theorem.** Consider

$$\{a_1, \dots, a_n\}, \{b_1, \dots, b_n\} \qquad \text{free.}$$

Then we have

$$\varphi(a_1 b_1 a_2 b_2 \dots a_n b_n) = \sum_{\pi \in NC(n)} k_\pi[a_1, a_2, \dots, a_n] \cdot \varphi_{K(\pi)}[b_1, b_2, \dots, b_n].$$

Examples: For $n = 1$ we get

$$\begin{aligned} \varphi(ab) &= k_1(a)\varphi(b) \\ &= \varphi(a)\varphi(b); \end{aligned}$$

$n = 2$ yields

$$\begin{aligned} \varphi(a_1 b_1 a_2 b_2) &= k_1(a_1)k_1(a_2)\varphi(b_1 b_2) + k_2(a_1, a_2)\varphi(b_1)\varphi(b_2) \\ &= \varphi(a_1)\varphi(a_2)\varphi(b_1 b_2) + \big(\varphi(a_1 a_2) - \varphi(a_1)\varphi(a_2)\big)\varphi(b_1)\varphi(b_2) \\ &= \varphi(a_1)\varphi(a_2)\varphi(b_1 b_2) + \varphi(a_1 a_2)\varphi(b_1)\varphi(b_2) - \varphi(a_1)\varphi(a_2)\varphi(b_1)\varphi(b_2). \end{aligned}$$

3. Free stochastic calculus

In this lecture, we will develop the analogue of a stochastic calculus for free Brownian motion. Free Brownian motion is characterized by the same requirements as classical Brownian motion, one only has to replace 'independent increments' by 'free increments' and the normal distribution by the semi-circle. In the same way as classical Brownian motion can be written as $a_t + a_t^*$ for a_t and a_t^* being annihilation and creation operators, respectively, on the Bosonic Fock space, the free Brownian motion has a canonical realization as $l_t + l_t^*$ for l_t and l_t^* being (left) annihilation and creation operators on the full Fock space. Thus, instead of developping a stochastic calculus for free Brownian motion $S_t = l_t + l_t^*$, one could also split S_t into its two summands and develop a free stochastic calculus for l_t and l_t^*, in analogy to the Hudson-Parthasarathy calculus for a_t and a_t^*. This was done by Kümmerer and Speicher [4]. The free stochastic calculus with respect to S_t, which is due to Biane and Speicher [2], however, has some advantages and we will here restrict to that theory.

In our presentation we will put the emphasis on two main points:

- **appropriate norms:** on a linear level all stochastic calculi have formally the same structure, the main point lies in establishing the integrals with respect to appropriate norms; in contrast to all other known examples, the free calculus has the very strong feature that one has estimates with respect to the uniform operator norm; i.e. the free stochastic integrals can be defined in L^p with $p = \infty$

- **Ito formula:** on a formal level the difference between stochastic calculi lies in their multiplicative structure; at least formally, a stochastic calculus is characterized by its Ito formula; for free stochastic calculus this is very similar to the Ito-formula for classical Brownian motion, however, due to non-commutativity there is a small, but decisive difference

3.1. **Definition.** A **free Brownian motion** consists of

- a von Neumann algebra $\mathcal{A}$
- a faithful normal tracial state τ on $\mathcal{A}$
- a filtration $(\mathcal{A}_t)_{t\geq 0}$ – i.e. $\mathcal{A}_t$ are von Neumann subalgebras of $\mathcal{A}$ with
$$\mathcal{A}_s \subset \mathcal{A}_t \qquad \text{for } s \leq t.$$
- a family of operators $(S_t)_{t\geq 0}$ with
  - $S_t = S_t^* \in \mathcal{A}_t$
  - for each $t \geq 0$, the distribution of S_t is a semi-circle with mean 0 and variance t
  - for all $0 \leq s < t$, the increment $S_t - S_s$ is free from $\mathcal{A}_s$
  - for all $0 \leq s < t$, the distribution of the increment $S_t - S_s$ is a semi-circle with mean 0 and variance $t - s$

Usually, we will call $(S_t)_{t\geq 0}$ the free Brownian motion.

In the same way as the classical Brownian motion can be realized on Bosonic Fock space, the free Brownian motion has a concrete realization on the full Fock space – as follows by Proposition 1.3. Note, however, that for the development of our free stochastic calculus we will not need this concrete realization but just the abstract properties of $(S_t)_{t\geq 0}$.

3.2. **Theorem.** Let
$$l_t := l(\chi_{(0.t)}), \qquad l_t^* = l^*(\chi_{(0.t)})$$
be the left annihilation and creation operators for the characteristic functions of the interval $(0, t)$ on the full Fock space $\mathcal{F}(\mathcal{H})$ for $\mathcal{H} = L^2(0, \infty)$. Put
$$\tau[A] := \langle \Omega, A\Omega \rangle,$$
and
$$S_t := l_t + l_t^*.$$

Then $(S_t)_{t\geq 0}$ is a free Brownian motion with respect to the filtration

$$\mathcal{A}_t := vN(S_s \mid s \leq t).$$

3.3. **Remark.** According to the connection between freeness and random matrices there is also a random matrix realization of free Brownian motion:

Consider random matrices

$$B_t^{(N)} := \frac{1}{\sqrt{N}}\big(B_{ij}(t)\big)_{i,j=1}^N,$$

where

- $B_{ij}(t)$ are classical real-valued Brownian motions for all i, j
- the matrices $B_t^{(N)}$ are symmetric, i.e. $B_{ij}(t) = B_{ji}(t)$ for all i, j
- apart from the symmetry condition, all entries are independent, i.e. $\{B_{ij}(\cdot) \mid 1 \leq i \leq j < \infty\}$ are independent Brownian motions.

Consider now the state

$$\varphi := \mathrm{E} \circ (\frac{1}{N}\mathrm{tr}),$$

where E denotes the expectation with respect to the above specified probability space and $\frac{1}{N}\mathrm{tr}$ is the normalized trace on $N \times N$ matrices. Then we have

$$S_t \qquad \hat{=} \qquad \lim_{N\to\infty} B_t^{(N)},$$

i.e. for all $n \in \mathbb{N}$ and $t_1, \dots, t_n \geq 0$ we have:

$$\tau[S_{t_1} \dots S_{t_n}] = \lim_{N\to\infty} \mathrm{E}[\frac{1}{N}\mathrm{tr}(B_{t_1}^{(N)} \dots B_{t_n}^{(N)})].$$

We will not use this realization, but it shows that our free stochastic calculus can also be viewed as the large N limit of stochastic calculus with respect to $N \times N$ hermitian matrix valued Brownian motion.

3.4. **Remarks.** 1) Note that we have the non-commutative L^p-spaces associated with our free Brownian motion. Namely, $L^p(\mathcal{A})$, for $1 \leq p < \infty$, is the completion of $\mathcal{A}$ with respect to the norm

$$\|A\|_{L^p} := \tau[|A|^p]^{1/p}.$$

For $p = \infty$, we put

$$\|A\|_{L^\infty} := \|A\|, \qquad \text{i.e.} \qquad L^\infty(\mathcal{A}) = \mathcal{A}.$$

In the concrete realisation of S_t on the full Fock space, we can identify $L^2(\mathcal{A})$ with the full Fock space $\mathcal{F}(\mathcal{H})$ and we can embed $\mathcal{A}$ into the full

Fock space by the injective mapping

$$\begin{aligned} \mathcal{A} &\subset \mathcal{F}(\mathcal{H}) \\ A &\mapsto A\Omega. \end{aligned}$$

2) For our latter norm estimates it will be important that we can obtain the operator norm as the limit $p \to \infty$ of the L^p-norms: For $A \in \mathcal{A}$ one has

$$\begin{aligned} \|A\| &= \lim_{p\to\infty} \|A\|_{L^p} \\ &= \lim_{m\to\infty} \tau\left[(A^*A)^m\right]^{1/2m}. \end{aligned}$$

3.5. **Remarks.** 1) Let A_t, B_t be adapted processes. Then we want to define the stochastic integral

$$\int A_t dS_t B_t.$$

In contrast to the stochastic theories considered in the other courses, we have to face now the new phenomenon of two-sided integrals. In the usual cases, adaptedness of the process implies that the differentials commute with the process (or anti-commute in the fermionic case), thus a two-sided integral can always be reduced to a one-sided one and there is no need to consider two-sided integrals. But in our case there is no such reduction. Adaptedness implies that the differential and the process are free, but this does not result in any commutation relation. Thus we should consider as the most general integral the two-sided one. Note that one could of course restrict to one-sided integrals in the beginning, but then a meaningful form of Ito formulas would result automatically in two-sided integrals.

2) Since $\int A_t dS_t B_t$ is bilinear in A_t, B_t it is natural to consider more general linear combinations

$$\sum_i \int A_t^i dS_t B_t^i$$

for adapted processes A_t^i and B_t^i. We will also write this as

$$\int U_t \sharp dS_t, \qquad \text{with} \qquad U_t := \sum_i A_t^i \otimes B_t^i \in \mathcal{A} \otimes \mathcal{A}^{op}$$

and call $U = (U_t)_{t\geq 0}$ a **biprocess**. ($\mathcal{A}^{op}$ is the opposite algebra of $\mathcal{A}$, i.e. with the same linear structure and the order of multiplication reversed; it is quite natural to consider U as an element in this space, since A_t multiplies from the left, wheras B_t multiplies from the right.)

3) The definition of the integral proceeds now as usual: First define the

integral for simple biprocesses, prove some adequate norm estimates for such cases and then extend the definition with respect to the involved norms.

3.6. **Definition.** Let $U_t = A_t \otimes B_t$ be a simple adapted biprocess, i.e. there exist $0 = t_0 < t_1 < \cdots < t_n < \infty$ such that

$$U_t = \begin{cases} A_i \otimes B_i & t_i \le t < t_{i+1} \\ 0 & t_n \le t. \end{cases}$$

(Adaptedness means here of course: $A_i, B_i \in \mathcal{A}_{t_i}$.) For such a simple biprocess we define the integral

$$\int U_t \sharp dS_t = \int A_t dS_t B_t := \sum_{i=0}^{n-1} A_i (S_{t_{i+1}} - S_{t_i}) B_i.$$

For simple adapted biprocesses of the general form $U_t = \sum_i A_t^i \otimes B_t^i$ we extend the definition by linearity.

As usual, it is quite simple to obtain the isometry of the integral in L^2-norm.

3.7. **Proposition (Ito isometry).** For all adapted simple biprocesses U and V, one has

$$\tau\big[\int U_t \sharp dS_t \cdot (\int V_t \sharp dS_t)^*\big] = \langle U, V\rangle := \int \langle U_t, V_t\rangle_{L^2 \otimes L^2} dt.$$

Proof. By bilinearity, it is enough to prove the assertion for processes

$$U_t = A \otimes B \cdot 1_{[t_0, t_1[}(t)$$

and

$$V_t = C \otimes D \cdot 1_{[t_2, t_3[}(t).$$

Then the left hand side is

$$\tau\big[A(S_{t_1} - S_{t_0}) B D^* (S_{t_3} - S_{t_2}) C^*\big].$$

Note that by linearity it suffices to consider the cases where the two time intervals are either the same or disjoint. In the first case we have

$$\begin{aligned} \tau\big[A(S_{t_1} - S_{t_0}) B D^* (S_{t_1} - S_{t_0}) C^*\big] &= \tau[AC^*]\tau[BD^*]\tau\big[(S_{t_1} - S_{t_0})^2\big] \\ &= \tau[AC^*]\tau[BD^*](t_1 - t_0) \end{aligned}$$

(because $\{A, B, D^*, C^*\}$ is free from the increment $S_{t_1} - S_{t_0}$), whereas in the second case one of the increments is free from the rest and thus, because of the vanishing mean of the increment, the expression vanishes. But his gives exactly the assertion. □

3.8. **Notation.** We endow the vector space of simple biprocesses with the norms ($1 \leq p \leq \infty$)

$$\|U\|_{\mathcal{B}_p} := \Big(\int \|U_t\|^2_{L^p(\tau\otimes\tau^{op})} dt \Big)^{1/2}.$$

(For $p = \infty$, $L^\infty(\tau \otimes \tau^{op})$ is the von Neumann algebra tensor product of $\mathcal{A}$ and $\mathcal{A}^{op}$.)
The completion of the space of simple biprocesses for these norms will be denoted by $\mathcal{B}_p$. The closed subspaces of $\mathcal{B}_p$ generated by adapted simple processes will be denoted by $\mathcal{B}^a_p$.

Thus we have shown that the map $U \mapsto \int U_t \sharp dS_t$ can be extended isometrically to a mapping

$$\mathcal{B}^a_2 \to L^2(\mathcal{A}).$$

But it is now an essential feature of the free calculus, which distinguishes it from all other ones, that one can even show a norm estimate for $p = \infty$, i.e. the stochastic integral is for a quite big class of (bi)processes a bounded operator. Thus we do not have to think about possible domains of our operators and the multiplication of such integrals (as considered for the Ito formula) presents no problems.

3.9. **Theorem (Burkholder-Gundy inequality).** For any simple adapted biprocess U one has

$$\| \int U_t \sharp dS_t \| \leq 2\sqrt{2} \|U\|_{\mathcal{B}_\infty}.$$

3.10. **Corollary.** The stochastic integral map $U \mapsto \int U_t \sharp dS_t$ can be extended continuously to a mapping

$$\mathcal{B}^a_\infty \to \mathcal{A}.$$

In particular, the stochastic integral of an adapted bounded biprocess from $\mathcal{B}^a_\infty$ is a bounded operator.

Proof. Let us just give a sketch of the proof. We restrict here to biprocesses of the form $U_t = A_t \otimes B_t$. The extension to sums of such biprocesses follows the same ideas.
Put

$$M_s := \int_0^s U_t \sharp dS_t = \int_0^s A_t dS_t B_t.$$

We want to obtain an operator norm estimate for M_s by using

$$\|M_s\| = \lim_{m\to\infty} \tau\big[(M_s^* M_s)^m\big]^{1/2m}.$$

This means we must estimate the p-th moment of our integral for $p \to \infty$. This is much harder than the case $p = 2$, but nevertheless it can be done by using again the crucial property

$$\tau\big[A(S_{t_1} - S_{t_0})B(S_{t_1} - S_{t_0})C\big] = \tau[AC]\tau[B](t_1 - t_0),$$

if $\{A, B, C\}$ is free from the increment $S_{t_1} - S_{t_0}$. By using also Hölder inequality for non-commutative L^p-spaces one can finally derive an inequality of the form

$$\tau[|M_s|^{2m}] \leq 2m \sum_{k=0}^{m-1} \int_0^s \big(\tau[|M_t|^{2k}] \cdot \tau[|M_t|^{2m-2-2k}] \cdot \|A_t\|^2 \cdot \|B_t\|^2\big) dt.$$

Note that the structure of this inequality resembles the recursion formula for the Catalan numbers c_n,

$$c_n = \sum_{k=0}^{n-1} c_k c_{n-k-1}.$$

By induction, one derives now from the above implicit inequality the explicit one

$$\tau[|M_s|^{2m}] \leq c_m \big(2 \int_0^s \|A_t\|^2 \|B_t\|^2 dt\big)^m.$$

We take now the $2m$-th root and note that

$$\lim_{m\to\infty} c_m^{1/2m} = 2.$$

Thus we obtain

$$\begin{aligned}\|M_s\| &\leq 2\sqrt{2}\big(\int_0^s \|A_t\|^2 \|B_t\|^2 dt\big)^{1/2}\\ &= 2\sqrt{2}\big(\int_0^s \|U_t\|^2_{L^\infty(\tau\otimes\tau^{op})} dt\big)^{1/2}.\end{aligned}$$

$s = \infty$ gives the assertion. □

Let us now present the Ito formula for the free calculus. First, we will do this on a formal differential level. As stated above, if we work with biprocesses from $\mathcal{B}^a_\infty$, then our integrals are bounded operators and multiplication presents no problem. We will show that the Ito formula holds even with respect to operator norm.
On a formal level, the Ito formula makes the difference between different stochastic calculi. On a first look, free Brownian motion $(S_t)_{t\geq 0}$ has the same Ito formula as classical Brownian motion $(B_t)_{t\geq 0}$, namely

$$dB_t dB_t = dt \qquad \text{and} \qquad dS_t dS_t = dt.$$

In our non-commutative context, however, this does not contain all necessary information, since we must now also specify

$$dS_t A dS_t \qquad \text{for } A \in \mathcal{A}_t.$$

In the classical case, A commutes with the increment dB_t and we have there

$$dB_t A dB_t = A dB_t dB_t = A dt.$$

But for free Brownian motion we have a different result.

3.11. **Theorem (Ito formula – product form).** For a free Brownian motion $(S_t)_{t\geq 0}$ we have the following Ito formula

$$dS_t A dS_t = \tau[A] dt \qquad \text{for } A \in \mathcal{A}_t.$$

Proof. Let $I \subset [0, \infty)$ be an interval and consider decompositions into disjoint subintervals I_i, $I = \cup I_i$. For an interval I we denote by $S(I)$ the corresponding increment of the free Brownian motion, i.e.

$$S(I) := S_t - S_s \qquad \text{for} \qquad I = [s, t[.$$

The main point is now to show that (with λ denoting Lebesgue measure)

$$\sum_i S(I_i) A S(I_i) \to \tau[A]\lambda(I),$$

where we take the usual limit with width $\max_i \lambda(I_i)$ of our decomposition going to zero. As said above we want to see that this convergence even holds in operator norm.

We will sketch two proofs of this fact, one using the abstract properties of freeness, whereas the other works in a concrete representation on full Fock space.

1) The assertion follows from the following two facts about freeness:

- let $\{s_1, \dots, s_n\}$ be a semicircular family, i.e. each s_i is semicircular and $s_1, \dots, s_n$ are free; then, for a random variable a which is free from $\{s_1, \dots, s_n\}$ we have [6]: $s_1 a s_1, \dots, s_n a s_n$ are free
- let $x_1, \dots, x_n$ be free random variables with $\tau[x_i] = 0$ for all $i = 1, \dots, n$; then we have [14]

$$\|x_1 + \cdots + x_n\| \leq \max_{1\leq i\leq n} \|x_i\| + 2\Big(\sum_{i=1}^n \tau[|x_i|^2]\Big)^{1/2}$$

2) We realize $S(I)$ on the full Fock space as $S(I) = l(I) + l^*(I)$; then we have to estimate the four terms $\sum l(I_i) A l(I_i)$, $\sum l(I_i) A l^*(I_i)$,

$\sum l^*(I_i)Al(I_i)$, and $\sum l^*(I_i)Al^*(I_i)$. Three of these terms tend to zero by simple norm estimates, the only problematic case is

$$\sum_i l(I_i)Al^*(I_i) \to \tau[A]\lambda(I).$$

(This corresponds of course to the Ito formulas for l_t and l_t^*, namely the only non-zero term is $dl_tAdl_t^* = \tau[A]dt$, see [4].) To prove this later statement, one can model A by the sum of creation and annihilation operators on the full Fock space as $A = \sum \alpha_n(l + l^*)^n$ for $l = l(f)$ with f being orthogonal to $L^2(I)$, hence l, l^* free from all $l(I_i), l^*(I_i)$. Then one has to expand this representation of A and bring it, by using the Cuntz relations $l(f)l^*(g) = \langle f, g\rangle 1$, into a normal ordered form $A = \sum \beta_{n,m}l^{*n}l^m$. Finally, note that, again by the Cuntz relations, only the term for $n = m = 0$ contributes to the sum of our statement. □

3.12. **Example.** The Ito formula contains also the germ for the combinatorial difference between independence and freeness – all partitions versus non-crossing partitions. For a classical Brownian motion $(B_t)_{t\geq 0}$ we can calculate the fourth moment $\tau[B_t^4]$ with the help of the Ito formula as follows:

$$d(B_t^4) = 3B_t^3dB_t + 6B_t^2dt,$$

thus

$$\frac{d\tau[B_t^4]}{dt} = 6\tau[B_t^2] = 6t$$

yielding

$$\tau[B_t^4] = 3t^2.$$

In the case of the free Brownian motion we obtain in the same way

$$\begin{aligned} d(S_t^4) = dS_tS_t^3 &+ S_tdS_tS_t^2 + S_t^2dS_tS_t + S_t^3dS_t \\ &+ dS_tdS_tS_tS_t + dS_tS_tdS_tS_t + dS_tS_tS_tdS_t \\ &+ S_tdS_tdS_tS_t + S_tdS_tS_tdS_t + S_tS_tdS_tdS_t \\ = dS_tS_t^3 &+ S_tdS_tS_t^2 + S_t^2dS_tS_t + S_t^3dS_t + 3S_t^2dt + \tau[S_t^2]dt, \end{aligned}$$

thus

$$\frac{d\tau[S_t^4]}{dt} = 4\tau[S_t^2] = 4t$$

yielding

$$\tau[S_t^4] = 2t^2.$$

This difference between the fourth moments in the classical and free case reflects the fact that there are 3 pairings of 4 elements, but only 2 of them are non-crossing.

The Ito formula can also be put into a functional form to calculate the differential of a function $f(S_t)$ for nice functions f – not just for polynomials. The main message of the classical Ito formula is that we have to make a Taylor expansion, but we should not stop after the first order in the differentials, but take also the second order into account using $dB_t dB_t = dt$, hence

$$\begin{aligned} df(B_t) &= f'(B_t)dB_t + \frac{1}{2}f''(B_t)dB_t dB_t \\ &= f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt \end{aligned}$$

There exists also a free analogue of this; whereas the first order term is essentially the same as in the classical case, the second order term is different; one of the two derivatives is replaced by a difference expression.

3.13. **Theorem (Ito formula - functional form).** Let f be a sufficiently nice function (for example, a function of the form $f(x) = \int e^{ixy}\mu(dy)$ for a complex measure μ with $\int |y|^2|\mu|(dy) < \infty$). Then we have

$$df(S_t) = \partial f(S_t)\sharp dS_t + \frac{1}{2}\Delta_t f(S_t)dt,$$

where $\partial f(X)$ is the extension of the derivation

$$\partial X^n = \sum_{k=0}^{n-1} X^k \otimes X^{n-k-1}$$

and $\Delta_t f$ denotes the function

$$\Delta_t f(x) = \frac{\partial}{\partial x}\int \frac{f(x)-f(y)}{x-y}\nu_t(dy),$$

where ν_t is the distribution of S_t, i.e. a semicircular distribution with variance t.

Proof. One has to check the statement for polynomials by using the product form of the Ito formula. All expressions in the statement make also sense for nice functions and the statement extends by continuity. For a polynomial $f(x) = x^n$ we have

$$dS_t^n = \sum_{k=0}^{n-1} S_t^k dS_t S_t^{n-k-1} + \sum_{\substack{k,l\geq 0 \\ k+l\leq n-2}} S_t^k dS_t S_t^l dS_t S_t^{n-k-l-2}.$$

The first term gives directly the first term in our assertion (this is just a non-commutative first derivative), whereas the second yields

$$\begin{aligned}\sum_{\substack{k,l\geq 0\\ k+l\leq n-2}} S_t^k dS_t S_t^l dS_t S_t^{n-k-l-2} &= \sum_{\substack{k,l\geq 0\\ k+l\leq n-2}} S_t^k \tau[S_t^l] dt S_t^{n-k-l-2}\\ &= \sum_{\substack{k,l\geq 0\\ k+l\leq n-2}} S_t^{n-l-2}\tau[S_t^l]dt\\ &= \sum_{m=1}^{n-1} mS_t^{m-1}\tau[S_t^{n-m-1}]dt,\end{aligned}$$

which can be identified with the second term in our statement. One should note: the fact that the trace and not the identity acts on the expression between two differentials results finally in the unusual form of the second order term in the functional form of the Ito formula; it is not a non-commutative version of the second derivative, but a mixture of derivative and difference expression. □

References

1. P. Biane, *Representations of symmetric groups and free probability*, Preprint (1998).
2. P. Biane and R. Speicher, *Stochastic calculus with respect to free Brownian motion and analysis on Wigner space*, Preprint (1997).
3. G. Kreweras, *Sur les partitions non-croisees d'un cycle*, Discrete Math. **1** (1972), 333–350.
4. B. Kümmerer and R. Speicher, *Stochastic integration on the Cuntz algebra O_∞*, J. Funct. Anal. **103** (1992), 372–408.
5. A. Nica, *R-transforms of free joint distributions, and non-crossing partitions*, J. Funct. Anal. **135** (1996), 271–296.
6. A. Nica and R. Speicher, *On the multiplication of free n-tuples of non-commutative random variables* (with an appendix by D. Voiculescu), Amer. J. Math. **118** (1996), 799–837.
7. A. Nica and R. Speicher, *R-diagonal pairs—a common approach to Haar unitaries and circular elements*, Free Probability Theory (D.-V. Voiculescu, ed.), AMS, 1997, pp. 149–188.
8. A. Nica and R. Speicher, *Commutators of free random variables*, to appear in Duke Math. J.; also available under `http://www.rzuser.uni-heidelberg.de/~L95`
9. R. Speicher, *A new example of 'independence' and 'white noise'*, Probab. Theory Rel. Fields **84** (1990), 141–159.

10. R. Speicher, *Free convolution and the random sum of matrices*, Publ. RIMS **29** (1993), 731–744.
11. R. Speicher, *Multiplicative functions on the lattice of non-crossing partitions and free convolution*, Math. Ann. **298** (1994), 611–628.
12. R. Speicher, *Combinatorial theory of the free product with amalgamation and operator-valued free probability theory* (Habilitationsschrift, 1994), Memoirs of the AMS **627** (1998).
13. R. Speicher, *Free probability theory and non-crossing partitions*, Seminaire Lotharingien de Combinatoire **B39c** (1997).
14. D. Voiculescu, *Addition of certain non-commuting random variables*, J. Funct. Anal. **66** (1986), 323–346.
15. D. Voiculescu, *Limit laws for random matrices and free products*, Invent. math. **104** (1991), 201–220.
16. D. Voiculescu, *Free probability theory: random matrices and von Neumann algebras*, Proceedings of the ICM 1994, Birkhäuser, 1995, pp. 227–241.
17. D. Voiculescu (ed.), *Free Probability Theory* (Fields Institute Communications, vol. 12), AMS, 1997.
18. D.V. Voiculescu, K.J. Dykema, and A. Nica, *Free Random Variables* (CRM Monograph Series, vol. 1), AMS, 1993.