1. Initial, terminal and zero objects

Definition

An object A is said to be an <u>initial object</u> provided that for each object B there is exactly one morphism from A to B.

Examples

1. Empty Set is the unique initial object for <u>Set</u>
   ~~Empty partially ordered set or~~ empty topological space is the unique initial object for ~~Pos/[illegible]~~ <u>Top</u>.

2. Every one element group is an initial object for <u>Grp</u> and then of course also for <u>Vec</u>.

Proposition

Initial objects are essentially unique, i.e.

(1) if A and B are initial, then A is iso to B

(2) if A initial, then every iso object to A is initial.

Proof: ~~[illegible]~~

(1) By def. there are morphisms $A \xrightarrow{k} B$ and $B \xrightarrow{h} A$ ~~[illegible]~~ $h \circ k = id_A$ since id_A is the unique morphism $A \to A$. Analogue, $k \circ h = id_B$. Thus k is an iso.

(2) $K: A' \to A$ iso. ~~Then~~ For each B there is $f: A \to B$.

Then $f \circ K: A' \to B$ is a morphism. Unique: if $g: A' \to B$ is another morphism. Then $g \circ K^{-1}: A^{0} \to B \Rightarrow g \circ K^{-1} = f$
$\Rightarrow g = f \circ K$ ok

Definition

An object A is a terminal object if for every object B there is exactly one morphism from B to A

Examples

1. Every one element set is a terminal obj. for Set.
2. For ~~ex~~ Vec, Top and Grp the construct that corresponds to $\{0\}$ is a terminal object.

Propoposition

Terminal objects are essentially unique

Proof:

Same proof for initial.

Definition

An object A is called zero object if it is an initial and terminal object.

Remark

Zero object is self dual since initial and terminal objects are dual to one another

Definition

A morphism $A \xrightarrow{f} B$ is called a <u>section</u> provided that there exists some morphism $B \xrightarrow{g} A$ such that $g \circ f = id_A$ („left-inverse")

Examples

1. A morphism is a <u>Set</u> is a section if and only if it is an injective function and not the empty function
2. In <u>Vec</u> sections are injective linear transformations.
3. Let X and Y be sets (or top. spaces) and if $a \in Y$, then the function $f: X \to X \times Y$ defined by $f(x) = (x, a)$ is a section in <u>Set</u> (resp. <u>Top</u>).

Proposition

(1) If $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ are section, then so is $A \xrightarrow{f} B \xrightarrow{g} C$

(2) If $A \xrightarrow{f} B \xrightarrow{g} C$ is a section, then so is $A \xrightarrow{f} B$.

Remark

1. Functors preserve sections
2. The same concept is given by <u>retractions</u> which represent „right-inverses" and the same propositions hold.

2. Monomorphisms and Epimorphisms

Definition

A morphism $A \xrightarrow{f} B$ is said to be a monomorphism provided that for all pairs $C \overset{h}{\underset{k}{\rightrightarrows}} A$ of morphisms with the entity $f \circ h = f \circ k$, it follows $h = k$.

Then f is also said to be „left-cancellable" w.r.t. composition

Examples

1. A function is a monomorphism in Set if and only if it is injective.
2. In Vec the following are equivalent:
   (a) f is a monomorphism
   (b) f is a section
   (c) f is injective

~~Bu~~ Propositions

(1) If $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ are mon., so is $A \xrightarrow{g \circ f} C$.

(2) If $A \xrightarrow{f} B \xrightarrow{g} C$ is mon., so is $A \xrightarrow{f} B$

(3) Every section is a monomorphism

(4) The following are equivalent:

(i) f is an isomorphism

(ii) f is a retraction and a monomorphism

<u>Proof:</u> (1)-(3) clear

(4): <u>(i) $\Rightarrow$ (ii)</u>:

Since every iso is a section and a retraction, then it is also a monomorphism and a retraction, follows from (3)

<u>(ii) $\Rightarrow$ (i)</u>:

Let f be mon. and since it is a retraction there exist $f \circ g = id$. Then

$$f \circ (g \circ f) = (f \circ g) \circ f = id \circ f = f \circ id$$

So that $g \circ f = id$ by def of a monomorphism.

<u>Proposition</u>

(1) Every repesentable functor <u>preserves monomorphisms</u>, i.e. if $F: A \to Set$ is representable and if f is a monomorphism in A, then $F(f)$ is a monomorphism.

(2) Every faithful functor <u>reflects monomorphisms</u>, i.e. if $F: A \to B$ is faithful and $F(f)$ is a B-monomorphism, then f is an A-monomorphism.

Proof

(1) It holds that

(i) hom-functors $hom(A,-): A \to \underline{Set}$ preserve monomorphisms, since for $g, h: hom(A,B) \to hom(A,C)$ ~~[illegible]~~ for which holds

$$hom(A,f)(g) = hom(A,f)(h)$$

$$\Leftrightarrow \quad f \circ g = f \circ h$$

$\Rightarrow \quad g = h$, such that ~~[illegible]~~ $hom(A,f)$ is an monomorphism.

(ii) whenever functors F, G are naturally isomorphic, if F preserves monomorphisms so does G.

(2) Suppose that

$$f \circ h = f \circ k.$$

Then $Ff \circ Fh = Ff \circ Fk$ implies that $Fh = Fk$ and due to the faithfulness of F, it follows $h = k$.

Remark

In all constructs all morphisms with injective underlying functions are monomorphism

~~Thus~~ As always we introduce a dual concept to monomorphisms which are called epimorphisms.

Definition

A morphism $A \xrightarrow{f} B$ is said to be an epimorphism provided that for all pairs $B \overset{h}{\underset{k}{\rightrightarrows}} C$ of morphisms such that $h \circ f = k \circ f$, then it follows $h = k$ (i.e., f is „right-cancellable" w.r.t to composition)

Examples

(1) In Set and Vec the following are equivalent

(i) f is an epimorphism

(ii) f is a retraction

(iii) f is surjective

(2) A number of constructs have precisely surjective functions as epimorphisms. For instance in Top, Rel, Pos, Grp.

Propositions

(1) If $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ epimorphisms, then $A \xrightarrow{f} B \xrightarrow{g} C$ epimorphism

(2) If $A \xrightarrow{f} B \xrightarrow{g} C$ epimorphism, then g is epimorphism

(3) Every retraction is an epimorphism.

(4) Following are equivalent:

(i) f is an isomorphism

(ii) f is a section and an epimorphism.

(5) Every faithful functor reflects epimorphisms.

Remark

In any construct with surjective functions are epimorphisms.

Proposition

Every equivalence functor preserves and reflects each of the following: monomorphisms, epimorphisms, sections, retractions and isomorphisms.

Proof

By duality and previous propositions, we only need to show preservation of monomorphisms.

Let $F: A \to B$ be an equivalence, let $A' \xrightarrow{f} A$ be an A monomorphism and let $B \overset{r}{\underset{s}{\rightrightarrows}} FA'$ be morphisms such that $Ff \circ r = Ff \circ s$. Since F is isomorphism-dense, there exists an A-object A'' and a B-isomorphism $FA'' \xrightarrow{k} B$. By fullness there are A-morphisms $A'' \overset{r'}{\underset{s'}{\rightrightarrows}} A'$ with $Fr' = r \circ k$ and $Fs' = s \circ k$.

Thus

$$F(f \circ r') = Ff \circ Fr' = Ff \circ r \circ k = Ff \circ s \circ k = Ff \circ Fs'$$

$$= F(f \circ s')$$

Since F is faithful and f a monomorphism $r' = s'$ and hence $r = s$

$\square$

Remark

The above proposition is typical for equivalences. They reflect and preserve all categorical ~~exact~~ properties.

Definition

(1) A morphism is called a bimorphism if it is a mono- and epimorphism.

(2) A category is called balanced if all bimorphisms are isomorphisms.

Examples

(1) Set, Vec, Grp are balanced categories

(2) The inclusion $\mathbb{Z} \to \mathbb{Q}$ is a non-isomorphic bimorphism in Rng

(3) The fuction $\ell^\infty \xrightarrow{f} c_0$ defined by $f(x_n) = (\frac{x_n}{n})$ is a non-isomorphic bimorphism in Ban.

3. Subobjects and Quotients

Definition

Let M be a class of monomorphisms. An <u>M-subobject</u> of an object B is a pair (A, m) where $A \xrightarrow{m} B$ belongs to M. In case M consists of all monomorphisms, M-subobjects are called <u>subobjects</u>.

~~Remark~~

~~[illegible]~~

Definition

Let (A, m) and (A', m') be subobjects of B.

(1) (A, m) and (A', m') are called <u>isomorphic</u> provided that there exists an isomorphism $h: A \to A'$ with $m = m' \circ h$.

(2) (A, m) is said to be <u>smaller than</u> (A', m') - denoted by $(A, m) \leq (A', m')$ - provided that there exists some (unique) morphism $h: A \to A'$ with

$$m = m' \circ h$$

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[dr, "m"'] & A' \arrow[d, "m'"] \\ & B \end{tikzcd}$$

Definition

Let M be a class of monomorphisms of a category $\underline{A}$.

(1) $\underline{A}$ is called <u>M-wellpowered</u> provided that no $\underline{A}$-object has a proper class of pairwise non-isomorphic M-subobjects.

(2) If M consists of all monomorphisms, then it is called <u>wellpowered</u>.

Definition

Let E be a class of epimorphisms. An <u>E-quotient object</u> of an object A is a pair (e, B) where $A \xrightarrow{e} B$. In case consists of all epimorphisms, it is just a <u>quotient object</u>.

Definition

Let (e, B) and (e', B') be quotient objects of A.

(1) (e, B) and (e', B') are <u>isomorphic</u> if there is an isomorphism $h : B \to B'$ with $e' = h \circ e$

(2) (e, B) is said to be <u>larger than</u> (e', B') - denoted by $(e, B) \geq (e', B')$ - provided that there exists some morphism $h : B \to B'$ with $e' = h \circ e$

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[dr, "e'"'] & B \arrow[d, "h"] \\ & B' \end{tikzcd}$$

4. Embeddings and Quotient morphisms

Definition

Let $\mathbf{A}$ be a concrete category over $\mathbf{X}$.

(1) An $\underline{A}$-morphism $A \xrightarrow{f} B$ is called <u>initial</u> if for any $\underline{A}$-object C an X-morphism $|C| \xrightarrow{g} |A|$ is an $\underline{A}$-morphism whenever $|C| \xrightarrow{f \circ g} |B|$ is an $\underline{A}$-morphism.

(2) An initial morphism $A \xrightarrow{f} B$ that has a monomorphic underlying X-morphism $|A| \xrightarrow{f} |B|$ is called an <u>embedding</u>

(3) If $A \xrightarrow{f} B$ is an embedding, then (f, B) is called an <u>extension</u> of A and (A, f) is called an <u>initial subobject</u> of B.

Proposition

For any concrete category the following hold:

(1) Each embedding is a monomorphism

(2) Each section (and in particular each isomorphism) is an embedding.

(3) If the forgetful functor preserves regular monomorphisms, then each regular monomorphism is an embedding.

Proof

(1) Since faithful functors ~~embeddings~~ reflect monomorphisms, this is clear.

(2) Suppose that $A \xrightarrow{s} B$ and $B \xrightarrow{r} A$ are $\underline{A}$-morphisms with $r \circ s = id_A$. Let $|C| \xrightarrow{g} |A|$ be an $\underline{X}$-morphism ~~and~~ such that $|C| \xrightarrow{s \circ g} |A|$ is an $\underline{A}$-morphism.

Then
$$g = r \circ (s \circ g)$$
is an $\underline{A}$-morphism and hence $A \xrightarrow{s} B$ is an embedding since each section is a monomorphism.

Examples

(1) If an abstract category $\underline{A}$ is considered to be concrete over itself via the identity functor, then every morphism is initial. Hence,
$$Emb(\underline{A}) = Mono(\underline{A})$$

(2) If $\underline{A} = \underline{Top}$ then a cts. map $f:(X,\tau) \to (Y,\sigma)$ is initial if and only if τ is the induced topology of f, i.e. $\tau = \{ f^{-1}[S] \mid S \in \sigma \}$.

(3) In the constructs $\underline{Grp}$ or $\underline{Vec}$ the initial morphisms coincide with the monomorphisms, i.e.
$$Init(\underline{A}) = Emb(\underline{A}) = Mono(\underline{A})$$

Proposition

(1) If $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ are initial morphisms (resp. embeddings), then $A \xrightarrow{g\circ f} C$ is an initial morphism (resp. embedding

(2) If $A \xrightarrow{g\circ f} C$ is an initial morphism (resp. embedding), then f is initial (resp. embedding)

Definition

Let $\underline{A}$ be a concrete category over $\underline{X}$.

(1) An $\underline{A}$-morphism $A \xrightarrow{f} B$ is called <u>final</u> provided that for any $\underline{A}$-object C, an $\underline{X}$-morphism $|B| \xrightarrow{g} |C|$ is an $\underline{A}$-morphism if $|A| \xrightarrow{g\circ f} |C|$ is an $\underline{A}$-morphism.

(2) A final morphism $A \xrightarrow{f} B$ withe epimorphic underlying $\underline{X}$-morphism $|A| \xrightarrow{f} |B|$ is called a <u>quotient morphism</u>.

(3) If $A \xrightarrow{f} B$ is a quotient morphism, then (f, B) is called a <u>final quotient object</u> of A.

Examples

(1) In $\underline{\text{Top}}$ a cts. function $f: (X, \tau) \to (Y, \sigma)$ is final if and only if $\sigma = \{A \subseteq Y \mid f^{-1}[A] \in \tau\}$.

(2) In $\underline{\text{Grp}}$, $\underline{\text{Vec}}$ one has that $A \xrightarrow{f} B$ is a final morphism if and only if it is a quotient morphism.

Proposition

For any concrete category the following hold:

(1) Each quotient morphism is an epimorphism.

(2) Each retraction (and in particular each isomorphism) is a quotient morphism.

(3) If the forgetful functor preserves regular epimorphisms, then each regular epimorphism is a quotient morphism.

Proposition

The following are equivalent for each $\underline{A}$-morphism f:

(1) f is an $\underline{A}$-iso morphism

(2) f is an initial morphism and an $\underline{X}$-iso morphism.

(3) f is a final morphism and an $\underline{X}$-iso morphism.

Proof

(1) $\Rightarrow$ (2):

Follows from the fact that functors preserve isomorphisms and that each isomorphism is an embedding.

(2) $\Rightarrow$ (1):

Let $A \xrightarrow{f} B$ be an initial $\underline{X}$-isomorphism, then $|B| \xrightarrow{f^{-1}} |A| \xrightarrow{f} |B| = |B| \xrightarrow{id_B} |B|$ implies by initality that f^{-1} is an $\underline{A}$-morphism. Hence f is an $\underline{A}$-isomorphism.

(1) $\Leftrightarrow$ (3):

Follows from the fact that (3) is the dual concept to (2).

$\square$