Hom and Ext Functors

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The Hom Functors

- **$\mathrm{Hom}_R(-, X)$:** $(X$: fixed R-module$)$

$$M \in \mathcal{M}od_R \longmapsto \mathrm{Hom}_R(M, X) \in \mathcal{A}b$$

$$f : M \longrightarrow N \longmapsto \mathrm{Hom}_R(f, X) =: f^* : \mathrm{Hom}_R(N, X) \longrightarrow \mathrm{Hom}_R(M, X)$$

$$h \longmapsto h \circ f$$

Check : – f^* : group homomorphism

– $(\mathrm{Id}_M)^* = \mathrm{Id}$

– $(f \circ g)^* = g^* \circ f^*$

$\Longrightarrow$ $\mathrm{Hom}_R(-, X) : \mathcal{M}od_R \longrightarrow \mathcal{A}b$: contravariant functor

■ $\mathrm{Hom}_R(X,-)$: (X : fixed R-module)

$$M \in \mathcal{M}od_R \longmapsto \mathrm{Hom}_R(X,M) \in \mathcal{A}b$$

$$f : M \longrightarrow N \quad \longmapsto \mathrm{Hom}_R(X,f) =: f_* : \mathrm{Hom}_R(X,M) \longrightarrow \mathrm{Hom}_R(X,N)$$

$$h \longmapsto f \circ h$$

Check :
- f_* : group homomorphism
- $(\mathrm{Id}_M)_* = \mathrm{Id}$
- $(f \circ g)_* = f_* \circ g_*$

$$\Longrightarrow \quad \mathrm{Hom}_R(X,-) : \mathcal{M}od_R \longrightarrow \mathcal{A}b$$

covariant functor

- $\mathrm{Hom}_R(X,-)$ and $\mathrm{Hom}_R(-,X)$ are additive functors:

e.g. $\mathrm{Hom}_R(f+g, X)(h) = (f+g)^*(h) = h \circ (f+g) = h \circ f + h \circ g$

$$= f^*(h) + g^*(h)$$

$$= \mathrm{Hom}_R(f, X)(h) + \mathrm{Hom}_R(g, X)(h), \ \forall h$$

- $\mathrm{Hom}_R(X,-)$ and $\mathrm{Hom}_R(-,X)$ are left exact functors:

e.g. given an exact sequence $M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \longrightarrow 0$, we have to show that

$$0 \to \mathrm{Hom}_R(M_2, X) \xrightarrow{g^*} \mathrm{Hom}_R(M, X) \xrightarrow{f^*} \mathrm{Hom}_R(M_1, X)$$

is an exact sequence.

- g^*: injective:

$$g^*h = 0 \Rightarrow h \circ g = 0 \xrightarrow{g:\text{surj}} M_2 = \mathrm{Im}(g) \subseteq \mathrm{Ker}(h) \subseteq M_2 \Rightarrow h = 0.$$

- "$\mathrm{Ker}(f^*) \supseteq \mathrm{Im}(g^*)$": $h \in \mathrm{Im}(g^*) \Rightarrow \exists \varphi : g^*\varphi = h \Rightarrow \varphi \circ g = h$

$$\Rightarrow \varphi \circ (g \circ f) = h \circ f \xrightarrow{g \circ f = 0} h \circ f = 0 \Rightarrow f^*h = 0 \Rightarrow h \in \mathrm{Ker}(f^*).$$

- "$\mathrm{Ker}(f^*) \subseteq \mathrm{Im}(g^*)$": Take $h \in \mathrm{Ker}(f^*)$, namely $h \in \mathrm{Hom}_R(M, X)$ with $f^*h = 0$. We need to find $\theta \in \mathrm{Hom}_R(M_2, X)$ with $h = g^*\theta = \theta \circ g$. Note that

$$f^*h = 0 \Rightarrow h \circ f = 0 \Rightarrow \mathrm{Ker}(g) = \mathrm{Im}(f) \subseteq \mathrm{Ker}(h).$$

As g is surjective, for every $y \in M_2$ there is $x \in M$ such that $g(x) = y$. We now define $\theta : M_2 \longrightarrow X$, $y \longmapsto h(x)$ and note that θ is well-defined: If $x_1, x_2 \in M$ satisfy $g(x_1) = y = g(x_2)$, then $g(x_1 - x_2) = 0$, so

$x_1 - x_2 \in Ker(g) \subseteq Ker(h) \Rightarrow h(x_1 - x_2) = 0 \Rightarrow h(x_1) = h(x_2)$. Clearly, θ is an R-module homomorphism such that $h = \theta \circ g = g^* \theta$, and we are done.

- An R-module P is projective iff $Hom_R(P,-)$ is exact.
- An R-module I is injective iff $Hom_R(-,I)$ is exact.

(Later: further characterizations via derived functors)

- Both $Hom_R(X,-)$ and $Hom_R(-,X)$ preserve split short exact sequences.

The Ext Functors

DEF.: Let X be a fixed R-module.

(a) For each $n \in \mathbb{N}$ the functor

$$\mathrm{Ext}_R^n(-,X) := \mathcal{R}^n \mathrm{Hom}_R(-,X) : \mathcal{M}od_R \longrightarrow \mathcal{A}b$$

is called the n-th extension functor of $\mathrm{Hom}_R(-,X) : \mathcal{M}od_R \longrightarrow \mathcal{A}b$.

(b) For each $n \in \mathbb{N}$ the functor

$$\overline{\mathrm{Ext}}_R^n(X,-) := \mathcal{R}^n \mathrm{Hom}_R(X,-) : \mathcal{M}od_R \longrightarrow \mathcal{A}b$$

is called the n-th extension functor of $\mathrm{Hom}_R(X,-) : \mathcal{M}od_R \longrightarrow \mathcal{A}b$.

- $\mathrm{Ext}_R^n(-,X)$ is an additive contravariant functor
- $\mathrm{Ext}_R^n(X,-)$ is an additive covariant functor

<u>Goal</u>: For every s.e.s. $0 \to L \to M \to N \to 0$ of R-module and R-module homomorphisms there is a long exact sequence in cohomology corresponding to $\mathrm{Ext}_R^n(-, X)$.

- There is an analogous statement for $\mathrm{Ext}_R^n(X, -)$, which can be obtained essentially by dual arguments.

- These results can be used to characterize injective and projective modules by Ext.

  e.g. T.f.a.e.:

  (i) I : injective.

  (ii) $\mathrm{Hom}(-, I)$: exact.

  (iii) $\mathrm{Ext}^n(M, I) = 0$, $\forall M \in \mathcal{M}od_R$, $\forall n \geq 1$.

  (iv) $\mathrm{Ext}^1(M, I) = 0$, $\forall M \in \mathcal{M}od_R$.

LEM. (Horse Shoe Lemma for Projectives): Consider the diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & P_1 \arrow[d, "\alpha_1"'] & & R_1 \arrow[d, "\gamma_1"'] & \\
 & P_0 \arrow[d, "\alpha_0"'] & & R_0 \arrow[d, "\gamma_0"'] & \\
0 \arrow[r] & L \arrow[r, "f"] \arrow[d] & M \arrow[r, "g"] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 &
\end{tikzcd}$$

where the bottom row is exact and $\mathbf{P}$ and $\mathbf{R}$ are projective resolutions of L and N, respectively. Then there is a projective resolution $\mathbf{Q}$ of M and chain maps $\mathbf{f}: \mathbf{P}_L \longrightarrow \mathbf{Q}_M$ and $\mathbf{g}: \mathbf{Q}_M \longrightarrow \mathbf{R}_N$ such that

$Q_n = P_n \oplus R_n, \forall n \in \mathbb{N}$ and

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"'] & Q_1 \arrow[r, "g_1"] \arrow[d, "\beta_1"'] & R_1 \arrow[r] \arrow[d, "\gamma_1"'] & 0 \\
0 \arrow[r] & P_0 \arrow[r, "f_0"] \arrow[d, "\alpha_0"'] & Q_0 \arrow[r, "g_0"] \arrow[d, "\beta_0"'] & R_0 \arrow[r] \arrow[d, "\gamma_0"'] & 0 \\
0 \arrow[r] & L \arrow[r, "f"] \arrow[d] & M \arrow[r, "g"] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

is a commutative row exact diagram.

PROP.: If $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ is a s.e.s. of R-modules and R-module homomorphisms, then for any R-module X, there exists a long exact cohomology sequence

$$0 \longrightarrow \mathrm{Hom}_R(N,X) \xrightarrow{g^*} \mathrm{Hom}_R(M,X) \xrightarrow{f^*} \mathrm{Hom}_R(L,X) \xrightarrow{\Phi^0}$$
$$\xrightarrow{\Phi^0} \mathrm{Ext}_R^1(N,X) \longrightarrow \mathrm{Ext}_R^1(M,X) \longrightarrow \mathrm{Ext}_R^1(L,X) \xrightarrow{\Phi^1}$$
$$\xrightarrow{\Phi^1} \mathrm{Ext}_R^2(N,X) \longrightarrow \mathrm{Ext}_R^2(M,X) \longrightarrow \mathrm{Ext}_R^2(L,X \xrightarrow{\Phi^2} \cdots$$

where Φ^n is a connecting homomorphism for each $n \in \mathbb{N}$.

The above sequence is called the long exact Ext-sequence in the first variable.

- There is also a long exact Ext-sequence in the second variable.

PROOF: If $\mathbf{P}$ and $\mathbf{R}$ are projective resolutions of L and N, respectively, then by the Horse Shoe Lemma we can find a projective resolution $\mathbf{Q}$ of M and chain maps $\mathbf{f}: \mathbf{P} \to \mathbf{Q}$ and $\mathbf{g}: \mathbf{Q} \to \mathbf{R}$ lifting f and g, respectively, such that

$$0 \longrightarrow \mathbf{P} \xrightarrow{\mathbf{f}} \mathbf{Q} \xrightarrow{\mathbf{g}} \mathbf{R} \longrightarrow 0$$

is a short exact sequence of chain complexes

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P_n \arrow[r, "f_n"] \arrow[d, "\alpha_n"'] & Q_n = P_n \oplus R_n \arrow[r, "g_n"] \arrow[d, "\beta_n"'] & R_n \arrow[r] \arrow[d, "\gamma_n"'] & 0 \\
0 \arrow[r] & P_{n-1} \arrow[r, "f_{n-1}"] \arrow[d] & Q_{n-1} = P_{n-1} \oplus R_{n-1} \arrow[r, "g_{n-1}"] \arrow[d] & R_{n-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

and in fact every row is a split short exact sequence by construction. Since $\mathrm{Hom}_R(-,X)$ preserves split short exact sequences we get a diagram

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \mathrm{Hom}_R(R_2,X) \arrow[r] \arrow[u] & \mathrm{Hom}_R(Q_2,X) \arrow[r] \arrow[u] & \mathrm{Hom}_R(P_2,X) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(R_1,X) \arrow[r] \arrow[u] & \mathrm{Hom}_R(Q_1,X) \arrow[r] \arrow[u] & \mathrm{Hom}_R(P_1,X) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(R_0,X) \arrow[r] \arrow[u] & \mathrm{Hom}_R(Q_0,X) \arrow[r] \arrow[u] & \mathrm{Hom}(P_0,X) \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

whose rows are exact and whose columns compute $\mathrm{Ext}_R^n(N,X)$, $\mathrm{Ext}_R^n(M,X)$ and $\mathrm{Ext}^n(L,X)$ for $n\in\mathbb{N}$. In particular,

$$0 \longrightarrow \mathrm{Hom}_R(R_N, X) \xrightarrow{g^*} \mathrm{Hom}_R(Q_M, X) \xrightarrow{f^*} \mathrm{Hom}_R(P_L, X) \longrightarrow 0$$

is a s.e.s. of (positive) cochain complexes, and therefore there exists a long exact sequence in cohomology. Since $\mathrm{Hom}_R(-,X)$ is a left exact additive contravariant functors, by a previous PROP/FACT we know that

$$\mathrm{Ext}_R^0(L,X) \cong \mathrm{Hom}_R(L,X)$$
$$\mathrm{Ext}_R^0(M,X) \cong \mathrm{Hom}_R(M,X)$$
$$\mathrm{Ext}_R^0(N,X) \cong \mathrm{Hom}_R(N,X)$$

This concludes the proof. $\blacksquare$