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Introduction.

If K is a finite field, the "upper half plane" $\Omega/K = \mathbb{P}^1/K - \mathbb{P}^1(K)$ shares some features with the classical complex upper half-plane H or the Drinfeld upper half-plane Ω_{Drin}. Therefore, a meaningful theory of finite modular forms may be developed, i.e., of rational functions on Ω/K with the familiar transformation behavior under $\mathrm{GL}(2,K)$ and certain boundary conditions. It is some "kindergarten" variant of modular forms theory, deprived, of course, of any kind of analysis, but with some remaining algebraic and combinatorial subtleties.

The principal motivation comes from Drinfeld modular forms. In the most simple case (see [1]), Drinfeld modular forms are rigid analytic functions defined on Ω_{Drin}, a one-dimensional symmetric space over the infinite completion of the rational function field $K(T)$. In contrast with classical elliptic modular forms over $\mathbb{Q}$, they may be reduced not only at finite primes $\mathfrak{p}$ of $K(T)$, i.e., at primes $\mathfrak{p}$ of the polynomial ring $K[T]$, but also at the infinite prime "∞". Roughly speaking, our finite modular forms are the reductions of Drinfeld modular forms at ∞, see [2], sect. 8. Hence they provide information about Drinfeld modular forms, notably, about the location of zeroes of Eisenstein series and of other distinguished forms.

In the present article, we study finite modular forms in their own right. Apart from some fundamental properties (relation with rank-two lattices, structure and dimensions of spaces of modular forms, j-invariant, "Serre derivative", ...), which are surprisingly similar to their elliptic or Drinfeld counterparts, we describe finite modular forms in terms of the (modular) representation theory of the group $\mathrm{GL}(2,K)$.

Perhaps the most pleasant result is Theorem 9.5, where we construct an "Eichler-Shimura" isomorphism of the space of cusp forms with a certain space of harmonic cochains. It may be seen as a "finite" analogue of Teitelbaum's isomorphism ([8] Thm. 16) in the case of Drinfeld modular forms. As a main application, we find in Corollary 9.7 the multiplicity of twisted Steinberg representations in the k-th symmetric power $\mathrm{Sym}^k(\overline{K}^2)$ of the tautological representation of $\mathrm{GL}(2,K)$.

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0. Notations.

K	$=$	finite field $\mathbb{F}_q$ with q elements, of characteristic p
$\overline{K}$	$=$	a fixed algebraic closure of K
K_m	$=$	$\mathbb{F}_{q^m}$ = subfield of $\overline{K}$ of degree m over K
$\mathbb{P}^1/K$	$=$	projective line over K with K'-rational points $\mathbb{P}^1(K')$ (K' any extension of K)
Ω	$=$	$\mathbb{P}^1(\overline{K}) - \mathbb{P}^1(K) = \overline{K} - K$ "the upper half-plane" over K
Γ	$=$	$\mathrm{GL}(2,K)$ the "modular group" that acts on $\mathbb{P}^1$ and Ω through $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(z) = \frac{az+b}{cz+d}$
B	$=$	the Borel subgroup of upper triangular matrices of Γ, with unipotent radical U of strictly triangular matrices
Z	$\cong$	K^* the group of scalar matrices in Γ

If the group G acts on the set X and $x \in X$, we denote by G_x, Gx, $G \setminus X$ the stabilizer of x in G, the orbit of x, the space of all orbits, respectively.

1. Lattices and Goss polynomials.

Most of the statements collected in this section are well-known. Missing proofs can be found e.g. in [1] or in [4] Ch. 1.

A *lattice* in $\overline{K}$ is a K-subspace Λ of finite dimension. With Λ, we associate the polynomial

$$e_\Lambda(X) = X \prod_{\lambda \in \Lambda}{}' (1 - X/\lambda), \tag{1.1}$$

where $\prod'$ denotes the product over the non-vanishing elements of Λ. We do not distinguish between $e_\Lambda(X)$ and the map on $\overline{K}$ it induces. It is additive and even $\mathbb{F}_q$-linear and thus has the form

$$e_\Lambda(X) = \sum_{0 \leq i \leq r} \alpha_i X^{q^i}, \tag{1.2}$$

where $r = \dim\ \Lambda$, $\alpha_0 = 1$, $\alpha_r \neq 0$, and the $\alpha_i = \alpha_i(\Lambda)$ depend on Λ. They are homogeneous of weight $q^i - 1$ in Λ:

$$\alpha_i(c\Lambda) = c^{1-q^i} \alpha_i(\Lambda), \quad c \in \overline{K}^*. \tag{1.3}$$

(1.4) Consider the K-algebra of all polynomials $\sum a_i X^{q^i}$ with coefficients $a_i \in \overline{K}$, where the product $(f \circ g)(X)$ of two such polynomials is defined by $f(g(X))$. Writing τ^i for X^{q^i}, it may and will be identified with the non-commutative polynomial ring $\overline{K}\{\tau\}$ with commutator rule $\tau \circ c = c^q \circ \tau$ for constants $c \in \overline{K}$. Note that its identity is $\tau^0 = X$. Write $\overline{K}\{\{\tau\}\}$ for the "formal power series" in τ, and let

$$\log_\Lambda(X) = \sum_{i \geq 0} \beta_i X^{q^i} \tag{1.5}$$

be the inverse of e_Λ with respect to "$\circ$", which exists in $\overline{K}\{\{\tau\}\}$. We have for $k \geq 1$: $\sum_{i+j=k} \alpha_i \beta_j^{q^i} = 0 = \sum_{i+j=k} \alpha_i^{q^j} \beta_j$, in particular,

$$\beta_k = -\sum_{j<k} \alpha_{k-j} \beta_j^{q^{k-j}}.$$

Moreover (see e.g. [1] 2.8, 2.9),

$$\beta_k = -E_{q^k-1}(\Lambda),$$

where

(1.6) $$E_l(\Lambda) = \sum_{\lambda \in \Lambda}{}' \lambda^{-l}$$

is the *Eisenstein series of weight* l for Λ. (We use the convention $E_0(\Lambda) = -1$, which fits with the preceding and following formulas.) For Λ as above, write $t = t_\Lambda = \frac{1}{e_\Lambda}$. Taking logarithmic derivatives in (1.1), we find

(1.7.) $$t_\Lambda(z) = \sum_{\lambda \in \Lambda} \frac{1}{z - \lambda}.$$

The following basic observation is due to David Goss ([3] Ch. VI).

1.8 Proposition. *There exists a series $G_k = G_{k,\Lambda}(X)$ of polynomials with coefficients in the field $K(\Lambda)$ generated by Λ such that*

$$\sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^k} = G_k(t_\Lambda(z)).$$

We have

(i) *G_k is monic of degree k;*

(ii) *$G_k(0) = 0$;*

(iii) *$G_k(X) = X^k$ if $k \leq q$;*

(iv) *$G_{p^k} = (G_k)^p$ ($p = \mathrm{char}(K)$), and, putting $G_k = 0$ for $k \leq 0$,*

(v) *$G_k(X) = X(G_{k-1} + \alpha_1 G_{k-q} + \cdots + \alpha_i G_{k-q^i} + \cdots)$ ($k \geq 2$).*

Further, if $k = q^j - 1$ then

(vi) *$G_k(X) = \sum_{0 \leq i < j} \beta_i X^{q^j - q^i}$.*

(For a proof, see [1] sect. 3.) $\square$

The next two examples are crucial for modular forms.

1.9 Example. Consider $\Lambda = K$ itself, and omit subscripts Λ. Then $e(X) = X - X^q$, $\log(X) = \sum_{i\geq 0} X^{q^i}$, hence

$$G_{q^j-1}(X) = X^{q^j-1} + X^{q^j-q} + \cdots + X^{q^j-q^{j-1}}.$$

A quick calculation yields more generally

$$G_k(X) = \sum_{0\leq i\leq (k-1)/q} (-1)^i \binom{k-1-i(q-i)}{i} X^{k-i(q-1)}.$$

(1.10) Let us determine, for arbitrary Λ, the expansion of $\frac{z}{e_\Lambda(z)}$ as a formal power series in z. By (1.7),

$$\begin{aligned}\frac{z}{e_\Lambda(z)} &= \sum_{\lambda\in\Lambda} \frac{z}{z-\lambda} = 1 - \sum_{\lambda}{}' \frac{z/\lambda}{1-z/\lambda} = 1 - \sum_{\lambda}{}' (\sum_{k\geq 1} \lambda^{-k}) z^k \\ &= 1 - \sum_{k\geq 1} E_k(\Lambda) z^k.\end{aligned}$$

1.11 Example. Let Λ have dimension two. Up to scaling, we may assume that Λ is the K-span $\langle \omega, 1\rangle$ of ω and 1, where $\omega \in \overline{K} - K = \Omega$. The function e_Λ can be written

$$e_\Lambda(z) = z - g(\omega) z^q - \Delta(\omega) z^{q^2},$$

where $\Delta(\omega) \neq 0$ for each ω. Using (1.5) and (1.6), we find

$$\begin{aligned} g(\omega) &= -E_{q-1}(\omega),\ \ \Delta(\omega) = -E_{q^2-1}(\omega) - E_{q-1}^{q+1}(\omega) \quad \text{with} \\ E_k(\omega) &= E_k(\langle\omega, 1\rangle) = \sum_{a,b\in K}{}' \frac{1}{(a\omega+b)^k},\end{aligned}$$

the k-th Eisenstein series, considered as a function in ω.

Conversely, (1.10) yields the recursion

$$E_{k-1} = gE_{k-q} + \Delta E_{k-q^2},$$

valid for $k > 1$, which allows to express the E_k as polynomials in g and Δ. The functions E_k, g, Δ are prototypes of finite modular forms.

(1.12) Next, suppose that Λ is a lattice of dimension r, contained in K_l, $r \leq l$. Let Λ^* be the lattice $e_\Lambda(K_l)$, of dimension $l - r$. The two polynomials e_{K_l} and $e_{\Lambda^*} \circ e_\Lambda$ have the same degrees, kernels, and linear terms, and therefore agree, i.e., $1 - \tau^l = e_{\Lambda^*} \circ e_\Lambda$. In $\overline{K}\{\{\tau\}\}$ we have

$$e_{\Lambda^*} = (1-\tau^l) \circ \log_\Lambda \equiv \log_\Lambda \equiv -\sum_{0\leq i<l} E_{q^i-1}(\Lambda)\tau^i$$

modulo the ideal (τ^l) generated by τ^l. In view of $\deg_\tau(e_{\Lambda^*}) = \dim\ \Lambda^* = l - r$, we find

$$E_{q^i-1}(\Lambda) \text{ vanishes for } l - r < i < l. \tag{1.13}$$

Applying this to the situation of example 1.11, i.e., $r = 2$ and the functions $E_k(\omega) = E_k(\langle\omega, 1\rangle)$, yields the following result.

1.4 Proposition. *$E_{q^i-1}(\omega)$ vanishes on the subset $K_{i+1} - K$ of Ω.* $\square$

2. Modular forms.

(2.1) The orbits of Γ on $\mathbb{P}^1(\overline{K}) = \mathbb{P}^1(K) \cup \Omega$ are described as follows:

- $\mathbb{P}^1(K)$ forms one orbit, of length $q+1$; its elements are called the *cusps* of Ω;
- $K_2 - K$ forms one orbit, of length $q^2 - q$; its elements are the *elliptic points*;
- $\overline{K} - K_2$ splits into orbits of equal length $(q^2 - 1)q$.

The stabilizer of $x \in \mathbb{P}^1(\overline{K})$ is isomorphic with B, with K_2^*, with $Z \cong K^*$, respectively, according to whether x is a cusp, an elliptic point, neither a cusp nor elliptic.

2.2 Definition. *Let k be a non-negative integer and m a class* (mod $q-1$). *A modular form of weight k and type m is a function $f : \Omega \longrightarrow \overline{K}$ such that*

(i) *f is a rational function in the coordinate z of $\mathbb{P}^1$ (without poles on Ω);*

(ii) *f satisfies the functional equation*

$$f\left(\tfrac{az+b}{cz+d}\right) = \tfrac{(cz+d)^k}{(\det\ \gamma)^m} f(z) \quad \textit{for } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma;$$

(iii) *f is regular at the cusp ∞.*

We denote by $M_{k,m}$ the $\overline{K}$-vector space of forms of weight k and type m.

2.3 Remarks. (i) Condition (i) is the substitute for the usual holomorphy condition. Condition (iii) distinguishes the cusp ∞ from the finite cusps. Modular forms according to the present definition arise naturally from reducing Drinfeld modular forms over a rational function field $K(T)$ at the infinite place $T = \infty$ [2].
(ii) Applying the functional equation to scalar matrices, we see that $M_{k,m} \neq 0$ implies $k \equiv 2m(\text{mod } q-1)$.
(iii) Contrary to intuitive expectation, the different spaces $M_{k,m}$ fail to be linearly independent (see 2.4 (ii)). We therefore cannot form a graded algebra of modular forms as a subring of the ring of functions on Ω.
Some examples of modular forms are easily written down.

2.4 Examples. (i) The Eisenstein series

$$E_k(z) = \sum_{a,b\in K}{}' \frac{1}{(az+b)^k}$$

defines an element $E_k \in M_{k,0}$, non-trivial iff $k \equiv (0 \mod q-1)$ (see (3.1) for $E_k \neq 0$).
(ii) As in (1.11), let $\Lambda = \langle z, 1\rangle$ be the lattice spanned by 1 and $z \in \Omega$, and put $\alpha_i(z) := \alpha_i(\langle z,1\rangle)$, $\beta_i(z) := \beta_i(\langle z,1\rangle)$. Then $\alpha_1 = -g$, $\alpha_2 = -\Delta$, $\alpha_i = 0$ for $i > 2$, $\beta_i = -E_{q^i-1}$, and $\alpha_i, \beta_i \in M_{q^i-1,0}$. Here condition (2.2) (ii) is the translation of (1.3), condition (iii) follows from its validity for Eisenstein series. Note that $e_{\langle z,1\rangle}$ vanishes on 1, which yields the relation

$$1 - g(z) - \Delta(z) = 0$$

identically in z, i.e., a relation between modular forms of weights $1, q-1, q^2-1$.
(iii) $t(z) = \frac{1}{e_K(z)} = \frac{1}{z-z^q}$ defines an element t of $M_{q+1,1}$. Here we only have to verify condition (ii). Since its validity is stable under multiplications and Γ is generated by B and $w = \binom{0\,1}{1\,0}$, it suffices to check it for matrices in B (for which it is obvious) and for w (which is a one-line calculation).

We next introduce expansions around the cusp ∞. From now on, $e(z) = z - z^q$ and $t(z) = \frac{1}{e(z)} = \frac{1}{z-z^q} = \sum_{a\in K} \frac{1}{z-a}$ denote the functions associated with the one-dimensional lattice $\Lambda = K$.
Note first: The subfield $\overline{K}(z)^U$ of U-invariants of the rational function field $\overline{K}(z)$ equals $\overline{K}(e) = \overline{K}(t)$. Since any modular form f is U-invariant, i.e., $f(z) = f(z+b)$ for $b \in K$, it may be written $f(z) = \tilde{f}(t)$ with a rational function $\tilde{f}$.

2.5 Lemma. *In the above situation, $\tilde{f}$ is a polynomial in t.*

Proof. Since f is regular at $z = \infty$, $\tilde{f}$ is regular at $t = 0$, i.e., an element of $\overline{K}(t) \cap \overline{K}[[t]]$. Any non-constant factor in the denominator of $\tilde{f}$ would give rise to a pole of f on Ω. $\square$

2.6 Definition. *For any modular form f, we call the polynomial $\tilde{f}$ the t-expansion of f. (For obvious reasons, we cannot use "q-expansion".) For $0 \neq f \in M_{k,m}$, we let $\nu_z(f)$ be its order at $z \in \Omega$, $\nu_\infty(f)$ the vanishing order of $\tilde{f}(t)$ at $t = 0$, and $d(f)$ the degree of $\tilde{f}$. Clearly, $\nu_z(t)$ depends only on the Γ-orbit of z.*

The relations between the different quantities are described in the next result.

2.7 Proposition. *Let $f \neq 0$ be a modular form of weight k and type m. Then*

$$\text{(i)} \quad \sum_{z\in\Gamma\backslash\Omega}{}^{*} \nu_z(f) + \frac{\nu_e(f)}{q+1} + \frac{\nu_\infty(f)}{q^2-1} = \frac{d(f)}{q^2-1}$$

$$\text{(ii)} \quad \sum_{z\in\Gamma\backslash\Omega}{}^{*} \nu_z(f) + \frac{\nu_e(f)}{q+1} + \frac{\nu_\infty(f)}{q-1} = \frac{k}{q^2-1},$$

where $\epsilon \in K_2 - K$ is a fixed elliptic point and the sums $\sum^*$ are over the non-elliptic orbits.

Proof. Equation (i) multiplied by the degree $q^3 - q$ of the ramified covering $\mathbb{P}^1 \longrightarrow \Gamma \setminus \mathbb{P}^1$ yields

$$\text{(i')} \quad \sum_{z \in \Omega}{}^{*} \nu_z(f) + (q^2 - q)\nu_\epsilon(f) + q\nu_\infty(f) = q^d.$$

Since the divisor of t as a function on $\mathbb{P}^1$ is $q(\infty) - \sum_{x \text{ finite cusp}} (x)$, the left hand side of (i′) is the degree of the zero divisor of f, whereas the right hand side is the degree of its pole divisor, which agree.
For the proof of (ii), we may assume that $k \equiv 0 \pmod{q^2 - 1}$, $m = 0$, replacing f by a power if necessary. Since $t \in M_{q+1,1}$, the function $F(z) = f(z)t^{-k/(q+1)}$ on $\mathbb{P}^1$ is Γ-invariant. Moreover, the Ω-parts of the divisors of f and F agree. Therefore,

$$\sum_{z \in \Omega}{}^{*} \nu_z(f) + (q^2 - q)\nu_\epsilon(f) = \deg(\operatorname{div}(f)|_\Omega) = \deg(\operatorname{div}(F)|_\Omega).$$

Since F is invariant and holomorphic on Ω, the right hand side equals the degree of the pole divisor of F on $\mathbb{P}^1$, which is

$$-(q+1)\operatorname{ord}_{z=\infty} F(z) = -(q+1)q(\nu_\infty(f) - \frac{k}{q+1}) = -(q+1)q\nu_\infty(f) + qk.$$

Dividing by $q^3 - q$ yields the result. $\square$

2.8 Corollary. *For $0 \neq f \in M_{k,m}$ we have $k - d(f) = q \cdot \nu_\infty(f)$. Furthermore,* dim $M_{k,m} \leq [\frac{k}{q+1}] + 1$.

Proof. The equality follows from comparing (i) and (ii) in the proposition, the inequality from considering elements of $M_{k,m}$ as polynomials in t. $\square$

We call an Eisenstein series E_k *special* if the weight k has the form $q^i - 1$ for some i.

2.9 Corollary. *Let E_k ($k = q^i - 1$) be a special Eisenstein series. Then $E_k(z) = 0$ if and only if $z \in K_{i+1} - K$, and all these zeroes are simple.*

Proof. The vanishing of E_k on $K_{i+1} - K$ is (1.14). The fact that these are the only zeroes, and their simplicity, follows from (2.7) (ii). $\square$

3. Some t-expansions.

Let us first calculate the t-expansions of the Eisenstein series. Assume that $0 < k \equiv 0 \pmod{q-1}$. Then $\sum_{a \in K}{}' a^k = \sum{}' 1 = -1$ and

(3.1)

$$E_k(z) = \sum_{a,b \in K}{}' \frac{1}{(az+b)^k} = \sum_b{}' b^{-k} + \sum_{a \neq 0} a^{-k} \sum_b \frac{1}{(z + b/a)^k} = -1 - G_k(t(z)),$$

where G_k is the k-th Goss polynomial of the lattice K. We see from $G_k(0) = 0$ that $\nu_\infty(E_k) = 0$. Suppose that $k = q^j - 1$. Then (1.9) yields

$$E_k(z) = -(1 + \sum_{0 \le i < j} t^{q^j - q^i}) = - \sum_{0 \le i \le j} t^{q^j - q^i}. \tag{3.2}$$

Using this, we can give a direct proof of Corollary 2.9. Viz., for $z \in \Omega$, $t(z) = e(z)^{-1} \neq 0$, and $E_k(z) = 0 \Leftrightarrow \sum_{0 \le i \le j} t^{q^j - q^i} = 0 \Leftrightarrow \sum_{0 \le i \le j} e^{q^i}(z) = 0 \Leftrightarrow z - z^{q^{j+1}} = 0$ $\Leftrightarrow z \in K_{j+1}$.

Next, we consider the two modular forms $g(z) = -\alpha_1(\langle z, 1\rangle)$ and $\Delta(z) = -\alpha_2(\langle z, 1\rangle)$. By (1.5), (1.11), and (3.2),

$$\begin{array}{rcl} g & = & -E_{q-1} = 1 + t^{q-1} = \frac{z - z^{q^2}}{z^q - z^{q^2}} \\ \Delta & = & -E_{q^2-1} - E_{q-1}^{q+1} = -t^{q-1} = -\frac{1}{(z - z^q)^{q-1}}. \end{array} \tag{3.3}$$

Two lattices Λ, Λ' of dimension two are *similar* if $\Lambda' = c \cdot \Lambda$ for some $c \in \overline{K}^*$. As usual, the mapping $z \longmapsto \langle z, 1\rangle$ induces a canonical bijection of $\Gamma \setminus \Omega$ with the set $\mathcal{L}$ of similarity classes $[\Lambda]$ of such lattices.

3.4 Lemma. *The map $j : \mathcal{L} \longrightarrow \overline{K}$; $[\Lambda] \longmapsto g^{q+1}(\Lambda)/\Delta(\Lambda)$ is well-defined and bijective.*

Proof. j is well-defined in view of the homogenity properties of g and Δ, therefore only its injectivity needs to be checked. Suppose that $g^{q+1}(\Lambda)/\Delta(\Lambda) = g^{q+1}(\Lambda')/\Delta(\Lambda')$. Upon scaling Λ and Λ', we may assume that $\Delta(\Lambda) = \Delta(\Lambda') = 1$. Then $g(\Lambda)$ and $g(\Lambda')$ differ by a $(q+1)$-th root of unity. Scaling Λ further with $(q^2 - 1)$-th roots of unity ϵ, we get $\Delta(\epsilon\Lambda) = 1$, $g(\epsilon\Lambda) = \epsilon^{q-1} g(\Lambda)$, and with an appropriate choice of ϵ, $g(\epsilon\Lambda) = g(\Lambda')$. But g and Δ determine Λ = kernel of $e_\Lambda(z) = z - g(\Lambda) z^q - \Delta(\Lambda) z^{q^2}$, hence $\Lambda' = \epsilon\Lambda$, and we are done. $\square$

Combining the above, we get a Γ-invariant map

$$\begin{array}{rccl} j : & \Omega & \longrightarrow & \overline{K}, \\ & z & \longmapsto & g(z)^{q+1}/\Delta(z) \end{array}$$

the *modular invariant*, which identifies $\Gamma \setminus \Omega = \mathcal{L}$ with $\overline{K}$. Its t-expansion is

$$j(z) = -\frac{(1 + t^{q-1})^{q+1}}{t^{q-1}}. \tag{3.5}$$

Inserting $t = \frac{1}{z - z^q}$ and simplifying yields the following well-known description for the subfield of Γ-invariants of $\overline{K}(z)$.

3.6 Proposition. *The subfield of Γ-invariants of the rational function field $\overline{K}(z)$ is $\overline{K}(z)^\Gamma = \overline{K}(j)$, where $j = -\frac{(z - z^{q^2})^{q+1}}{(z - z^q)^{q^2+1}}$.* $\square$

Note that this property descends from $\overline{K}$ to K, i.e., for each subextension $K \subset K' \subset \overline{K}$, $K'(z)^\Gamma = K'(j)$ holds.

3.7 Proposition. *The modular invariant satisfies*

(i) $j(z) = 0 \Leftrightarrow z$ *elliptic*

(ii) $j(z) = \infty \Leftrightarrow z$ *a cusp*

(iii) $j(K_3 - K) = \{-1\}$

(iv) $K(j(K_4 - K)) = K_2$

(v) $K(j(K_n - K)) = K_n$ *if* $n \geq 5$.

Proof. (i) and (ii) are obvious, (iii) follows from $\Delta = -E_{q^2-1} - E_{q-1}^{q+1}$ since $E_{q^2-1}(K_3 - K) = \{0\}$. If $z \in K_4 - K$ then $j(z)^{q^2} = j(z)$. But $K_4 - K$ contains $q+1$ Γ-orbits, thus (iv). Similarly, if $n \geq 5$, $K_n - K$ contains strictly more than q^{n-3} orbits, which implies (v). $\square$

4. The vector spaces $M_{k,m}$.

In order to describe the spaces $M_{k,m}$, we observe:

(4.1) Let $f \in M_{k,m}$ have t-expansion $\sum a_i t^i$. If $a_i \neq 0$ then $i \equiv m \pmod{q-1}$. In particular, $d(f) \equiv v_\infty(f) \equiv m \pmod{q-1}$.

4.2 Theorem. *Let k be a non-negative integer with representative k^* $\pmod{q^2-1}$ (i.e., $0 \leq k^* < q^2-1$, $k \equiv k^* \pmod{q^2-1}$), and $0 \leq m < q-1$ such that $k \equiv 2m \pmod{q-1}$. Then*

(i) $M_{k+q+1,m+1} = \begin{cases} tM_{k,m} & m \not\equiv -1 \pmod{q-1} \\ tM_{k,m} \oplus \overline{K}E_{k+q+1} & m \equiv -1 \pmod{q-1} \end{cases}$

(ii) $\dim\ M_{k^*,m} = \begin{cases} 0 & k^* < m(q+1) \\ 1 & \textit{otherwise, in which case } g^n t^m \textit{ with } n = \frac{k^*-m(q+1)}{q-1} \textit{ is a basis vector.} \end{cases}$

(iii) $\dim\ M_{k,m} = [\frac{k}{q^2-1}] + \dim\ M_{k^*,m} = [\frac{k-m(q+1)}{q^2-1}] + 1$.

(iv) *The monomials $g^a t^b$ such that $a, b \geq 0$, $a(q-1) + b(q+1) = k$, $b \equiv m \pmod{q-1}$ form a basis. Each $f \in M_{k,m}$ may uniquely be written as an isobaric polynomial $F(g,t)$ of weight k in g and t.*

Proof. If $m \not\equiv -1 \pmod{q-1}$, each $f \in M_{k+q+1,m+1}$ has a t-expansion divisible by t. If $m \equiv -1$, E_{k+q+1} has a non-vanishing constant term in t, thus (i) in both cases. Let $0 \neq f \in M_{k^*,m}$. Then from (2.7) (ii), $(q-1)\nu_\epsilon(f) + (q+1)\nu_\infty(f) = k^*$, and from (4.1), $v_\infty(f) = m$. Therefore, $k^* - m(q+1) = (q-1)\nu_\epsilon(f) \geq 0$ and, again from (2.7) (ii), $f/t^m = \text{const.}\ g^{\nu_\epsilon(f)}$. Conversely, if $k^* - m(q+1) > 0$, then $g^n t^m \in M_{k^*,m}$. As to (iii), it follows from $(q-1)$ times applying (i). Assertion (iv) results e.g. from induction on k. $\square$

4.3 Corollary. *The following three filtrations F^i on $M_{k,m}$ agree:*

(a) $F^i M_{k,m} = \{f \in M_{k,m} \mid v_\infty(f) \geq i(q-1)\}$

(b) $F^i M_{k,m} = \{f \in M_{k,m} \mid d(f) \leq k - i(q^2-q)\}$

(c) $F^i M_{k,m} = \langle g^a t^b \mid b \geq i(q-1)\rangle$

Proof. This is a restatement of (iv) above, together with the equality of (2.8). $\square$

4.4 Corollary. *Let* $P_m(u) = \sum_{k\geq 0} \dim(M_{k,m})u^k$ *be the Poincaré function that encodes* $\dim M_{k,m}$*, where* $0 \leq m < q-1$*. Then*

$$P_m(u) = \frac{u^{m(q+1)}}{(1-u^{q-1})(1-u^{q^2-1})}.$$

Proof. Immediate from (4.2) (iii). $\square$

4.5 Corollary. *With hypotheses as in* (4.2)*,*

$$\dim F^1 M_{k+2,m+1} = \begin{cases} \dim M_{k,m} & k \not\equiv m(q+1) \pmod{q^2-1} \\ \dim M_{k,m} - 1 & k \equiv m(q+1) \pmod{q^2-1}, \end{cases}$$

and thus

$$\sum_{k\geq 0} \dim(F^1 M_{k+2,m+1})u^k = u^{q-1}P_m(u).$$

Proof. In the first case, each of the basis vectors $g^a t^b$ of $M_{k,m}$ contains at least one factor g. Replacing it with t yields the basis $\{g^{a-1}t^{b+1}\}$ of $F^1 M_{k+2,m+1}$. In the second case, the argument is similar, but we have to omit the basis vector $t^{k/q+1}$. The last equation follows from $P_m(u) - \frac{u^{m(q+1)}}{1-u^{q^2-1}} = u^{q-1}P_m(u)$. $\square$

5. The derivative ∂.

Similar to the classical case, we construct new modular forms from given ones through derivation. First note that

$$\frac{dt}{dz} = \frac{d}{dz}\frac{1}{e(z)} = -\frac{1}{e(z)^2} = -t^2, \tag{5.1}$$

since $e'(z) = 1$. Therefore, $\frac{d}{dz} = -t^2\frac{d}{dt}$ and

$$\frac{\Delta'}{\Delta} = t = -\frac{t'}{t}, \tag{5.2}$$

where $(\ \ )'$ denotes $\frac{d}{dz}(\ \ )$. As in [1] or [2], we get through a straightforward calculation:

5.3 Proposition.

(i) *For* $f \in M_{k,m}$ *put* $\partial f = \partial_k f = f' + k\frac{\Delta'}{\Delta}f$*. Then* $\partial f \in M_{k+2,m+1}$*.*

(ii) *For* $i = 1, 2$ *let* $f_i \in M_{k_i,m_i}$. *Then* $\partial_{k_1+k_2}(f_1 f_2) = \partial_{k_1}(f_1)f_2 + f_1\partial_{k_2}(f_2)$. $\square$

For example, $\partial g = -t$, $\partial t = 0$, and therefore $\partial(g^a t^b) = -ag^{a-1}t^{b+1}$. More generally, if $f \in M_{k,m}$ is written as $F(g,t)$ with an isobaric polynomial $F(X,Y)$ of weight k then

$$\partial f = -\frac{\partial F}{\partial X}(g,t)t. \tag{5.4}$$

If thus $g_k = -E_{q^k-1}$ denotes the normalized ($g_k(\infty) = 1$) special Eisenstein series and $h_k = \partial g_k$ its "derivative", (3.2) and (5.1) yield

$$h_k = -t \sum_{1 \le i \le k} t^{q^k - q^i} = -t g_{k-1}^q. \tag{5.5}$$

Hence the zeroes of h_k are precisely the points of $K_k - K$, and they all have order q.

6. Generating polynomials for Eisenstein series.

Since $E_k \in M_{k,0}$ and $\Delta = -t^{q-1}$, there exists an isobaric polynomial $r_k(X,Y)$ of weight k (where $wt(X) = q-1$, $wt(Y) = q^2 - 1$) such that $-E_k = r_k(g, \Delta)$. We calculate r_k.

Let Λ be the two-dimensional lattice $\langle z, 1 \rangle$. The formal identity

$$X = \frac{X}{e_\Lambda(X)} \cdot e_\Lambda(X) = (-\sum_{i \ge 0} E_i(\Lambda)X^i)(X - g(\Lambda)X^q - \Delta(\Lambda)X^{q^2})$$

gives the recursion for $k > 1$:

$$E_{k-1} = gE_{k-q} + \Delta E_{k-q^2}, \tag{6.1}$$

where all the terms depend on Λ, i.e., on z, $E_0 = -1$, and $E_k = 0$ if $k < 0$. In terms of the r_k,

$$r_k = Xr_{k-(q-1)} + Yr_{k-(q^2-1)}.$$

As $E_k = 0$ for $k \not\equiv 0 \pmod{q-1}$, we need only consider r_k with k divisible by $q-1$. Put

$$\rho_k(Z) = r_{k(q-1)}(X,Y)/X^k, \tag{6.2}$$

a polynomial in $Z := Y/X^{q+1}$ that satisfies

$$\rho_k = \rho_{k-1} + Z\rho_{k-(q+1)} \quad (k \ge 1,\ \rho_0 = 1,\ \rho_k = 0 \text{ if } k < 0).$$

By construction,

$$-E_{k(q-1)} = g^k \rho_k(\frac{\Delta}{g^{q+1}}) = g^k \rho_k(j^{-1})$$

and

$$\rho_k(x) = 0 \Leftrightarrow x = j(z)^{-1}$$

with a non-elliptic zero z of $E_{k(q-1)}$. An easy calculation shows that more precisely

$$\nu_z(E_{k(q-1)}) = \mathrm{ord}_x \rho_k \tag{6.3}$$

for z non-elliptic, $x = j(z)^{-1}$. About the vanishing of E_k at elliptic points, we have:

6.4 Lemma. *E_k vanishes at an elliptic point ϵ if and only if $k \not\equiv 0 \pmod{q^2-1}$.*

Proof. Let k be divisible by q^2-1. Then $E_k(\epsilon) = \sum\limits_{a \in K_2}{}' a^{-k} = -1$. Conversely, if $E_k(\epsilon) \neq 0$, the functional equation of E_k, applied to some non-scalar $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\epsilon$, yields $(c\epsilon + d)^k = 1$. Therefore, $\epsilon'^k = 1$ for each element ϵ' of K_2, and so $k \equiv 0 \pmod{q^2-1}$. $\square$

Some properties of the ρ_k are summarized in the next result.

6.5 Proposition.

(i) $\rho_k(Z) = \sum\limits_{0 \le i \le [k/q+1]} \binom{k-qi}{i} Z^i$

(ii) $\rho_{pk} = (\rho_k)^p$

(iii) $\deg \rho_k = \frac{k - \nu_\epsilon(E_{k(q-1)})}{q+1}$

Proof. (i) may be shown e.g. by induction on k, (ii) reflects $E_{pk} = (E_k)^p$, and (iii) comes from (6.3), counting zeroes of $E_{k(q-1)}$. $\square$

The polynomials ρ_k also enjoy the following mysterious property, which seems to be difficult to prove without their relationship to Eisenstein series.

6.6 Proposition. *Let $k = \frac{q^i-1}{q-1}$. Then $\rho_k(Z)$ is separable with splitting field K if $i = 1$ or 2, K_2 if $i = 3$, and K_{i+1} if $i \ge 4$.*

Proof. (2.9) + (3.7), combined with (6.3). $\square$

6.7 Remark. Knowledge of the zeroes of our present Eisenstein series allows, through reduction, to locate the zeroes of certain distinguished Drinfeld modular forms, see [2]. In contrast with the case of special Eisenstein series described by the proposition, the zeroes of non-special Eisenstein series behave in a rather unpredictable way. Neither are they simple (e.g., $E_{2(q^2-1)} = -(E_{q^2-1})^2$ independently of q), nor can we prescribe the field they generate. If for example $q = 2$, the splitting field of ρ_k ist K_8, K_{10}, K_{14} for $k = 25,\ 35,\ 45$, respectively. Also, $\nu_\epsilon(E_{k(q-1)})$ can be strictly larger than $k - (q+1)[\frac{k}{q+1}]$, e.g. for $q = 2$, $k = 11$, $\nu_\epsilon(E_{11}) = 5$.

7. Homogeneous description of $M_{k,m}$.

In this section, we identify $M_{k,m}$ in terms of homogeneous polynomials.

(7.1) We consider finite-dimensional $\overline{K}[\Gamma]$-modules C, for which $C^\wedge = \mathrm{Hom}(C, \overline{K})$ denotes the contragredient module, $C^{(m)} = \{x \in C \mid \gamma x = (\det\ \gamma)^m x\}$, and $C(m) = C \otimes (\det)^m$, the module C with the action twisted by the m-th power of the determinant character. Note that $C^\wedge$ is a right Γ-module under $f \circ \gamma$ or a left Γ-module under $\gamma f = f \circ \gamma^{-1}$ if Γ acts from the left on C ($\gamma \in \Gamma$, $f \in G^\wedge$).

(7.2) Let $V = \overline{K}^2$ be the two-dimensional vector space over $\overline{K}$ with standard basis $\{e_1, e_2\}$, $\mathrm{Sym}(V) = \bigoplus_{k \geq 0} \mathrm{Sym}^k(V)$ its symmetric algebra, provided with its natural Γ-action ($\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) e_1 = ae_1 + be_2$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) e_2 = ce_1 + de_2$), and put for brevity $S_k = \mathrm{Sym}^k(V)^\wedge$, $S_{-k} = S_k^\wedge = \mathrm{Sym}^k(V)$ ($k \geq 0$). We regard S_k as the space of forms of degree k in the coordinates x and y of V.

7.3 Lemma. *Let $\eta \in S_{q+1}$ be the form $\eta(x,y) = xy^q - x^q y$. Then $\eta \in S_{q+1}^{(1)}$, and as a form on $\mathbb{P}^1$, it has its zeroes at $\mathbb{P}^1(K)$, all simple.*

Proof. The second assertion is obvious. As for the first, it suffices (see (2.4) (iii)) to verify $\eta \circ \gamma = (\det\ \gamma)\eta$ for $\gamma \in B$ or $\gamma = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. $\square$

Let $f \in M_{k,m}$ be a modular form for Γ, and consider

$$f^*(x,y) = y^{-k} f(x/y).$$

Note that we may insert x/y into the rational function f. The condition $f \in M_{k,m}$ translates to $f^* \circ \gamma = (\det\ \gamma)^{-m} f^*$ for $\gamma \in \Gamma$. Now since f has no poles on Ω, we must have

$$f^* = F/\eta^n \tag{7.4}$$

for some n, where $F \in S_{n(q+1)-k}^{(n-m)}$. Using de l'Hôpital's rule, the condition $f(\infty) \neq \infty$ gives $\deg_x F(x,y) \leq qn$.

7.5 Lemma. *Suppose that n is minimal subject to (7.4). Then $n = d(f)$. In particular, $n \leq k$.*

Proof. Let f have t-expansion $\sum a_i t^i$ ($0 \leq i \leq d$, $a_d \neq 0$). Calculation yields

$$f^*(x,y) = \eta^{-d} y^{-k} \sum_{0 \leq i \leq d} a_i y^{(q+1)i} \eta^{d-i},$$

thus $n = d$. $\square$

Hence, suppressing the minimality condition, each f^* can be uniquely written as F/η^k, where $F(x,y) \in S_{kq}^{(k-m)} = S_{kq}^{(m)}$ (since $k \equiv 2m \pmod{q-1}$ if $0 \neq f \in M_{k,m}$).

Next, we let L_k be the space of rational functions on $\mathbb{P}^1(\overline{K})$ that have their poles at $K \hookrightarrow \mathbb{P}^1(\overline{K})$, all of order less or equal to k, and L_k' the subspace of functions that vanish at ∞ and have poles of order strictly less than k. Obviously,

$$\dim L_k = kq + 1, \ \ \dim L_k' = (k-1)q. \tag{7.6}$$

7.7 Proposition.

(i) *L_k is a right Γ-module under*

$$f_{[\gamma]} := f_{[\gamma]_k}(z) = (cz+d)^{-k} f(\gamma z) \quad (\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma);$$

(ii) *L_k' is a Γ-submodule of L_k;*

(iii) *The left Γ-module attached to L_k ($\gamma f := f_{[\gamma^{-1}]}$) is isomorphic with S_{kq} under $i_k : \ f \longmapsto F, \ F(x,y) = \eta(x,y)^k y^{-k} f(x/y)$.*

Proof. (i) and (ii) are obvious or result from a straightforward calculation. The map i_k is a well-defined bijection from L_k to S_{kq}, and $i_k(f_{[\gamma]}) = F \circ \gamma$, thus (iii). $\square$

7.8 Corollary. $M_{k,m} = L_k^{(m)}$.

Proof. $M_{k,m}$ is contained in L_k since by (2.8), each $f \in M_{k,m}$ has degree $d(f) \leq k$ in $t(z) = \frac{1}{z - z^q}$. It is now clear that as a subspace of L_k, it agrees with $L_k^{(m)}$. $\square$

7.9 Theorem.

(i) *The restriction of i_k to $M_{k,m}$ identifies $M_{k,m}$ with $S_{kq}^{(m)}$.*

(ii) *Under i_k, the filtration F^i of* (4.3) *on $M_{k,m}$ is mapped to the filtration on $S_{kq}^{(m)}$ defined through divisibility by powers of η.*

(iii) *The diagram*

$$\begin{array}{ccccc} M_{k_1,m_1} & \times & M_{k_2,m_2} & \longrightarrow & M_{k_1+k_2,m_1+m_2} \\ & \downarrow & & & \downarrow \\ S_{k_1 q}^{(m_1)} & \times & S_{k_2 q}^{(m_2)} & \longrightarrow & S_{(k_1+k_2)q}^{(m_1+m_2)} \end{array}$$

is commutative, where horizontal maps are multiplications and vertical maps are the different maps i_*.

(iv) *For $f \in M_{k,m}$, the identity $i_{k+2}(\partial f) = (\frac{\eta^2}{y})\frac{\partial}{\partial x} i_k(f)$ holds. Hence ∂ corresponds to the differential operator $(\frac{\eta^2}{y})\frac{\partial}{\partial x}$ on $\bigoplus_{k \geq 0} S_{kq}$.*

Proof. For $k \not\equiv 2m \pmod{q-1}$, both $M_{k,m}$ and $S_{kq}^{(m)}$ are zero. For $k \equiv 2m$, items (i) and (ii) follow from (7.5), (7.7) and (7.8), as is easily seen. (iii) is immediate from the definition of i_k, and (iv) results from a straightforward calculation. $\square$

In view of the theorem, we are entitled to identify $M_{k,m}$ with $S_{kq}^{(m)}$ through i_k. In contrast with the various $M_{k,m}$, the $S_{kq}^{(m)}$ are linearly independent, and we can form $S = \bigoplus_{k,m} S_{kq}^{(m)}$, which we regard as an analogue for the classical algebra of modular forms. Note that S is the algebra of semi-invariants (i.e., of invariants under $\mathrm{SL}(2,K)$) of $\bigoplus_{\substack{k\geq 0 \\ k\equiv 0 \pmod q}} S_k \hookrightarrow \mathrm{Sym}(V^\wedge)$. Such rings of invariants are described through Dickson invariants (see [6] Ch.VIII, [10]), to which our functions are closely related.

7.10 Example. We list the elements $F \in S_{kq}^{(m)}$ to which some standard modular forms f correspond.

f	t	Δ	g	g_k	$h_k = \partial g_k$
F	η^q	$-\eta^{q(q-1)}$	$\frac{xy^{q^2}-x^{q^2}y}{\eta}$	$\frac{xy^{q^{k+1}}-x^{q^{k+1}}y}{\eta}$	$x^{q^{k+1}}y^q - x^q y^{q^{k+1}}$

Here the first two are obvious, the assertion about g and g_k follows from (2.9) and $g_k(\infty) = 1$, the last one from calculation.

8. The representations $St(m)$.

In order to describe the relations with modular representation theory of Γ, we recall some known facts, proofs of which may be found in [7], see also [9]. Our notation will not distinguish between representations and the spaces on which Γ acts. We are especially interested in the so-called *Steinberg representation* $St = S_{q-1}$, which is characterized as follows.

8.1 Proposition. *St is equivalent with the canonical representation of Γ on the space $C_0(\mathbb{P}^1(K), \overline{K})$ of $\overline{K}$-valued functions φ on $\mathbb{P}^1(K)$ that satisfy $\sum_{x\in\mathbb{P}^1(K)} \varphi(x) = 0$. It is projective, simple and self-dual. If 1_B (resp. 1_Γ) denotes the respective trivial one-dimensional representation, the induced representation $\mathrm{Ind}_B^\Gamma(1_B)$ splits as $St \oplus 1_\Gamma$.* $\square$

Here the first assertion results from the fact that each function on $K = \mathbb{F}_q$ is induced by a unique polynomial of degree $\leq q-1$. The occurrence of St on $\mathrm{Ind}_B^\Gamma(1_B)$ now comes from the isomorphy of the two Γ-spaces $\mathbb{P}^1(K)$ and Γ/B. This also shows the projectivity and self-duality of St.

We want to determine the number

$$(8.2) \qquad a_k(m) = \text{multiplicity of } St(m) := St \otimes (\det)^m$$

in a composition series of the Γ-module $\mathrm{Sym}^k(V)$.

That number has been calculated in [5], see also [8], provided that $m = 0$. (In fact, Kuhn and Mitchell found the multiplicity of the Steinberg representation of $\mathrm{GL}(r, K)$ in $\mathrm{Sym}^k(\overline{K}^r)$ for any $r \geq 2$.) We will use an idea similar to Teitelbaum's to also cover the twisted representations $St(m)$.

We have $St^\wedge = St$ and, more generally, $St(m)^\wedge = St(-m)$. Now, as $St(m)$ is still simple and projective,

$$
\begin{aligned}
a_k(m) &= \dim \mathrm{Hom}_\Gamma(St(m), S_{-k}) = \dim \mathrm{Hom}(St(m), S_{-k})^\Gamma \\
&\quad \text{(dimension of invariants under the canonical left action} \\
&\quad \gamma\varphi := \gamma(\varphi \circ \gamma^{-1}) \text{ of } \Gamma \text{ on } \mathrm{Hom}(St(m), S_{-k})) \\
&= \dim \mathrm{Hom}(St, S_{-k})^{(-m)} = \dim C_k^{(-m)},
\end{aligned}
\tag{8.3}
$$

where $C_k := C_0(\mathbb{P}^1(K), S_{-k})$ is the $\overline{K}$-space of functions

$$\varphi : \mathbb{P}^1(K) \longrightarrow S_{-k} \text{ that satisfy } \sum_{x \in \mathbb{P}^1(K)} \varphi(x) = 0.$$

Here the left Γ-action on C_k is as follows:

$$(\gamma\varphi)(s)(F) = \varphi(\gamma^{-1}s)(F \circ \gamma) \tag{8.4}$$

for $\gamma \in \Gamma$, $s \in \mathbb{P}^1(K)$, $F \in S_k$.

9. The residue map.

We define a map $\mathrm{res} = \mathrm{res}_k$ from L_k to C_{k-2} ($k \geq 2$) as follows:

$$(\mathrm{res}\ f)(s)(x^i y^{k-2-i}) := \mathrm{res}_s z^i f(z) dz. \tag{9.1}$$

Here $\{x^i y^{k-2-i} \mid 0 \leq i \leq k-2\}$ is the canonical basis for S_{k-2}, z the coordinate in $\mathbb{P}^1(\overline{K})$, and the right hand side is the residue in $s \in \mathbb{P}^1(K)$ of the differential $z^i f(z) dz$.

9.2 Lemma. *With the left Γ-actions on L_k and C_{k-2} described by (7.7) and (8.4), we have*

$$\mathrm{res}(\gamma f) = (\det\ \gamma)\gamma(\mathrm{res}(f))$$

for $\gamma \in \Gamma$.

Proof. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. It suffices to verify that both sides evaluate equally on the basis elements of S_{k-2}. Now

$$
\begin{aligned}
\mathrm{res}(\gamma f) &= \mathrm{res}(f_{[\gamma^{-1}]}) = \mathrm{res}\left(\left(\tfrac{(-cz+a)}{\det\ \gamma}\right)^{-k} f(\gamma^{-1}z)\right), \quad \text{hence} \\
\mathrm{res}(\gamma f)(s)(x^i y^{k-2-i}) &= \mathrm{res}_s\left(\left(\tfrac{-cz+a}{\det\ \gamma}\right)^{-k} z^i f(\gamma^{-1}z) dz\right) \\
&= \mathrm{res}_{\gamma^{-1}s}\left(\left(\tfrac{-c\gamma z+a}{\det\ \gamma}\right)^{-k} (\gamma z)^i f(z) d\gamma z\right) \quad \text{(invariance of the residue)} \\
&= \mathrm{res}_{\gamma^{-1}s}\left((cz+d)^k \left(\tfrac{az+b}{cz+d}\right)^i f(z) \tfrac{\det\ \gamma}{(cz+d)^2} dz\right) \\
&= (\det\ \gamma)\mathrm{res}_{\gamma^{-1}s}((az+b)^i (cz+d)^{k-2-i} f(z) dz) \\
&= (\det\ \gamma)(\mathrm{res}\ f)(\gamma^{-1}s)(x^i y^{k-2-i} \circ \gamma) \\
&= (\det\ \gamma)(\gamma\ \mathrm{res}\ f)(s)(x^i y^{k-2-i}). \quad \square
\end{aligned}
$$

9.3 Corollary. $res_k : L_k \longrightarrow C_{k-2}(1)$ *is a* Γ*-morphism.* $\square$

The kernel R_k of res_k is easy to find. In view of the residue theorem:

$$\sum_{s\in\mathbb{P}^1(K)} \mathrm{res}_s f(z)dz = 0 \quad \text{for any } f \in L_k,$$

it is the set of those $f \in L_k$ such that $\mathrm{res}_s p(z)f(z)dz = 0$ for all $s \in K$ and all polynomials $p(z) \in \overline{K}[z]$ of degree $\leq k-2$, and is freely generated by the linearly independent functions $(z-s)^{-k}(s \in K)$ and the constants in L_k. Therefore, $\dim R_k = q+1$, and the sequence of Γ-modules

$$0 \longrightarrow R_k \longrightarrow L_k \xrightarrow{\mathrm{res}_k} C_{k-2}(1) \longrightarrow 0 \tag{9.4}$$

is exact, for $\dim L_k = kq+1$, $\dim C_{k-2} = (k-1)q$. Moreover, since obviously $L'_k \cap R_k = 0$, (9.4) splits and res_k restricted to L'_k yields a Γ-isomorphism of L'_k with $C_{k-2}(1)$. Now

$$\begin{aligned} L'^{(m)}_k &= \{f \in M_{k,m} \mid f \text{ has poles of order } < k \text{ at finite cusps}\} \\ &= F^1M_{k,m} = \{\text{cusp forms in } M_{k,m}\} \quad (\text{see } (4.3)), \end{aligned}$$

and we have proved the following main result.

9.5 Theorem. *Let* $k \geq 2$*. The residue map* res_k *identifies the space* $F^1M_{k,m}$ *of cusp forms of weight* k *and type* m *with* $C^{(m)}_{k-2}(1) = C^{(m-1)}_{k-2} = C_0(\mathbb{P}^1(K), S_{2-k})^{(m-1)}$*.* $\square$

Together with (8.3) we get:

9.6 Corollary. *The multiplicity* $a_k(-m)$ *of* $St(-m)$ *in* $\mathrm{Sym}^k(V)$*, where* $k \geq 0$*, equals the dimension* $\dim F^1M_{k+2,m+1}$ *of the space of cusp forms in* $M_{k+2,m+1}$*.* $\square$

Now the dimension in question has been determined in Corollary 4.5. We finally find the following formula for the Poincaré function.

9.7 Corollary. *Let* m *be the representative with* $0 \leq m < q-1$*. Then*

$$\sum_{k\geq 0} a_k(-m)u^k = \frac{u^{m(q+1)+q-1}}{(1-u^{q-1})(1-u^{q^2-1})}. \quad \square$$

Remark. Since R_k contains the constants, on which $B \subset \Gamma$ acts through the character $\chi_k : \left(\begin{smallmatrix} a & b \\ 0 & d\end{smallmatrix}\right) \longmapsto d^{-k}$, R_k is the induced representation $\mathrm{Ind}^\Gamma_B(\chi_k)$. We have $R^{(m)}_k \neq 0$ if and only if $k \equiv m \equiv 0 \pmod{q-1}$, in which case $R^{(m)}_k = R^\Gamma_k = \overline{K}E_k$ is the one-dimensional space spanned by the Eisenstein series E_k of weight k. The splitting $L_k = R_k \oplus L'_k$ of Γ-modules, along with its interpretation through modular forms, may be viewed as a simple but not trivial analogue of the splitting (familiar from classical modular forms) of the $\mathrm{SL}(2,\mathbb{R})$-module $L^2(\mathrm{SL}(2,\mathbb{Z}) \setminus \mathrm{SL}(2,\mathbb{R}))$ into (essentially) "Eisenstein series" and "cusp forms".

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