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Abstract

A mathematical model for the heat transfer in a system of exhaust pipes of an automobile is described. The forced convection of the exhaust gas, the heat conduction and the heat transfer due to radiation are coupled. An effective numerical solution of the boundary integral equation for the radiation heat transfer is presented. The analytical form of the matrix entries for the collocation method is derived. Adaptive cross approximation is used for the compression of the system matrix. Some numerical examples for the matrix compression are presented.

Contents

1 Introduction

Catalytic converters play an important role in the reduction of the pollution gases in the exhaust of an automobile engine. Modern catalytic converters are very efficient, the reduction of the CO, NO_x and of the various hydrocarbons contained in the exhaust being more than 90%. However, during the warm-up cycle the temperature of the catalytic converter is too low and the efficiency of the converter can be further improved by shortening the warm-up cycle. There are different technical ideas for this purpose. One of the most popular is to use the insulated exhaust pipes first introduced in 1991 [1]. The insulating split between two pipes should prevent the energy produced by the engine from getting lost on the way to the catalytic converter. The three-dimensional geometry of an insulated exhaust pipe is presented in **Figure 1.1**. Because of current and future government regulations for emission control there is permanent urgency not only to improve

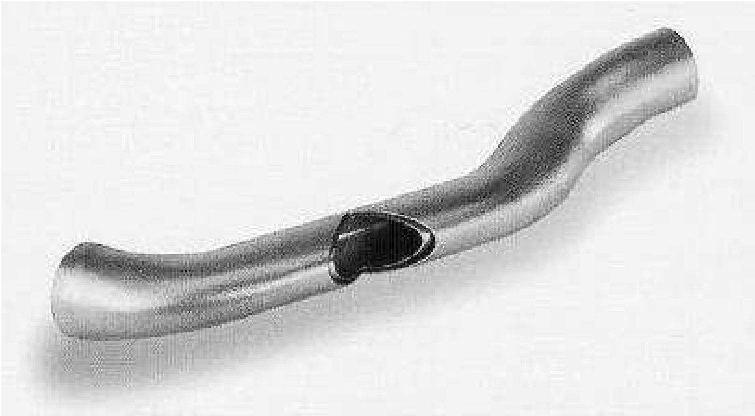

Figure 1: The three-dimensional geometry of an insulated exhaust pipe

the efficiency of exhaust systems but also to predict output during the development of new automobile models. Thus the mathematical modelling and numerical simulation of exhaust systems become more and more important. In the previous paper [6] we discussed the simplified situation of the straight insulated exhaust pipe. The special structure of the problem leads to special properties of the discrete system, i.e. the matrix is of Toeplitz structure. Using these properties we have designed a fast numerical algorithm for the numerical simulation of the warm-up cycle. In this paper we consider the most general situation of pipes having the geometry which is really used in industry.
The paper is organised as follows. In Section 2 we give a short description of the mathematical model for the heat transfer in an insulated exhaust pipe. In Section 3 we give a short review of the different approximation techniques for full dense matrices. In Section 4 we present the main ideas of our new approximation technique, called Adaptive Cross Approximation, and finally, in Section 5, we present the results of computations and draw some conclusions.

2 Mathematical model

We consider an insulated exhaust pipe of length L having the cross-section depicted in **Figure 2.1**. We consider the following physical processes in the pipe:

1. Heat transfer due to the forced convection in the inner pipe;

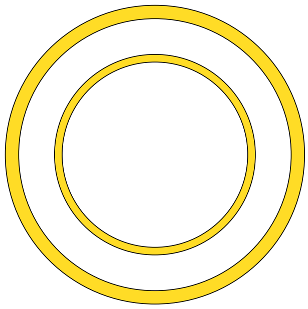

Figure 2: Cross-section geometry of an insulated exhaust pipe

2. Heat transfer due to conduction within and between both walls of the exhaust pipe;

3. Heat transfer due to radiation in the insulating split.

The exact description of the first two equations, the corresponding boundary and initial conditions as well as the standard numerical procedures can be found in our previous paper [6]. In this paper we concentrate on the numerical solution of the radiation heat transfer equation which can be written in the following form:

$$((\mathcal{I}-(1-\epsilon)\mathcal{B})\,R(x))\,(y)=(1-\epsilon)\left(\mathcal{B}^2U(x)\right)(y). \tag{1}$$

Here $R(x), x\in\Gamma$ denotes the reflected part of the radiation energy on the inner surfaces of the insulating split Γ the integral operator $\mathcal{B}$ is defined as

$$(\mathcal{B}\,R(x))\,(y)=\frac{1}{\pi}\int_\Gamma \kappa(x,y)\frac{(n_x,y-x)(n_y,x-y)}{|x-y|^4}\,R(x)\,ds_x\,,y\in\Gamma \tag{2}$$

whereby the function $U(x)$ is an abbreviation of

$$U(x)=\epsilon\,\sigma\,T^4(x)\ \ x\in\Gamma.$$

The constant ϵ is the emissivity of the steel and σ denotes the Boltzmann constant

$$\sigma=5.669\cdot 10^{-8}.$$

The numerical solution of equation (1) begins with the discretisation of the surface Γ using a system of plane, triangle panels

$$\Gamma\approx\Gamma_h=\bigcup_{j=1}^{N}\Gamma_j$$

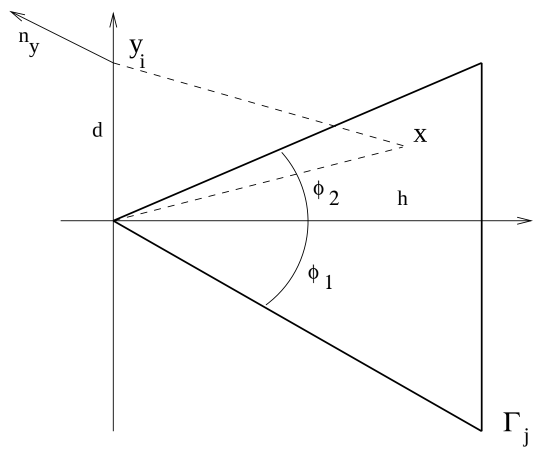

Figure 3: Computation of the element of the matrix

The centres of the mass of the panels Γ_i build the system of collocation points y_i

$$y_i = \frac{1}{3}\left(x_i^{(1)} + x_i^{(2)} + x_i^{(3)}\right)\,,\ i = 1, \dots N\,.$$

The piecewise constant ansatz r for the unknown function $R(x)$ and u for the function $U(x)$ leads to the following system of collocation equations

$$A\,r = f\,,\ A \in \mathbb{R}^{N\times N}\,,\ r, f \in \mathbb{R}^N\,.$$

The matrix A and the right hand side of this system are of the following form:

$$A = I - (1-\epsilon)B\,,\ f = (1-\epsilon)B^2 u\,.$$

The elements of the matrix B are defined as

$$b_{i,j} = \frac{1}{\pi}\int_{\Gamma_j} \kappa(x, y_i)\frac{(n_x, y_i - x)(n_{y_i}, x - y_i)}{|x - y_i|^4}\,ds_x.$$

If the triangle Γ_j is completely visible from the collocation point y_i then the corresponding element of the matrix B can be computed analytically in the following way. We first consider the situation where the projection of the point y_i into the plane defined by the triangle Γ_j coincides with one of its edges. First we rotate and translate the system of coordinates in such a way that the origin coincides with a projection of the point y_i, the x_1-axis is directed along the height of the triangle and the x_3-axis is directed along the vector n_x. Let us denote the corresponding rotation matrix by V. In these new coordinates, the unit normal vectors n_x and n_y take the following form:

$$Vn_x = (0,0,1)^T \,,\; Vn_{y_i} = (\alpha_1, \alpha_2, \alpha_3)^T \,.$$

This situation is depicted in **Figure 2.2**. The integral will be evaluated in polar coordinates in the plane of the triangle Γ_j. Using the notation from **Figure 2.2** we obtain

$$b_{i,j} = \frac{d}{\pi} \int\limits_{\phi_1}^{\phi_2} \int\limits_{0}^{\frac{h}{\cos(\phi)}} \frac{\alpha_1 \rho \cos(\phi) + \alpha_2 \rho \sin(\phi) - \alpha_3 d}{(\rho^2 + d^2)^2} \, \rho \, d\rho d\phi.$$

After simple calculations we obtain the following expression for $b_{i,j}$.

$$\begin{aligned} b_{i,j} &= -\frac{1}{2\pi} \frac{\alpha_1 d + \alpha_3 h}{\sqrt{d^2+h^2}} \left[\arctan\left(h^{-1} \left(\sqrt{d^2+h^2} \tan(\frac{\phi}{2}) + d \right) \right) \right]_{\phi_1}^{\phi_2} \\ &- \frac{1}{2\pi} \frac{\alpha_1 d + \alpha_3 h}{\sqrt{d^2+h^2}} \left[\arctan\left(h^{-1} \left(\sqrt{d^2+h^2} \tan(\frac{\phi}{2}) - d \right) \right) \right]_{\phi_1}^{\phi_2} \\ &+ \frac{1}{2\pi} \left[\arctan\left(\frac{h}{d\cos(\phi)} \right) (\alpha_1 \sin(\phi) - \alpha_2 \cos(\phi)) \right]_{\phi_1}^{\phi_2} . \end{aligned} \tag{3}$$

If the projection of the point y_i does not coincide with one of the edges of the triangle Γ_j then we can always reduce the calculation of the integral to the calculation of the three integrals of the previous type. Thus we obtain the analytical result in any case. If the triangle Γ_j is not completely visible from y_i then the visible part has a polygonal form and therefore the integral can be computed using some decomposition of the visible part into a set of triangles. This can be a very complicated and rather technical task.

3 Approximation of the full dense matrices

The system matrix A of the usual boundary element method has the following properties. The matrix is dense, i.e. the memory requirements are $Mem(A) = O(N^2)$. N denotes the size of the matrix here. The matrix can be asymptotically badly conditioned, i.e. in many applications the condition number is $\kappa(A) = O(N)$. The numerical solution of the resulting system

of linear equations $Ay = b$ using classical direct solvers is extremely expensive, i.e. the numerical work is $Op(y = A^{-1}b) = O(N^3)$. The only possible increase in the efficiency of the boundary element methods is due to using iterative methods for the resulting system of linear equations. Two main ingredients are needed for the iterative methods. The first is an effective preconditioner B with $\kappa(B^{-1}A) = O(1)$. The second is an effective procedure for the fast matrix-vector multiplication with $Op(As) = O(N)$. These two problems have been of continuous interest in research over the last 15 years. There are two main preconditioning ideas. The first is based on the spectral equivalence of the boundary element matrices constructed using the same discretisation of the same problem on different but topologically equivalent surfaces. Thus one can choose some structured surface (i.e. surface of revolution) to obtain a specially structured system matrix (i.e. multi-level Toeplitz-like matrix) (see [5],[7],[8]). Since such matrices can be handled very efficiently, the optimal preconditioner is constructed. There are clear disadvantages of this preconditioning. It is not easy to construct a well-structured surface if the given surface is complicated (i.e. multiple connected, etc.). The preconditioning will also require uniform discretisation on the structured domain and it can come into conflict with the necessity of the adaptive discretisation of the given domain. The second preconditioning strategy is based on using the operator of the opposite order on the same surface and using the same discretisation (see [12]). It is much more convenient for the majority of problems but the application of the preconditioning matrix is now as expensive as the matrix-vector multiplication itself. Thus research on fast matrix-vector multiplication becomes even more important.
The first fast matrix-vector multiplication tool, **the Fast Multipole Method**, was presented in 1985 by V. Rokhlin [13],[14]. The problem was to compute the force applied to each of the large number of charged particles in each time step. The main ideas here were to divide the whole space into a near and a far field and to cluster particles in the far field. The Fast Multipole Method is widely used for many applications (see also [15]). **The Panel Clustering** algorithms for boundary integral equations, first published by Hackbusch and Novak [4], are based on very similar ideas.

4 Adaptive cross approximation

4.1 Matrix partitioning

Let $P = \{\Gamma_1, \ldots, \Gamma_N\}$ be the set of panels. If we use a quadrature formula to approximate the integrals, the approximant's properties with respect to availability of low-rank approximants come from the kernel. Thus we concentrate on the investigation of the matrices

$$A \in \mathbb{R}^{N\times N}, \quad a_{ij} = \kappa(x_i, x_j).$$

We first subdivide the index set $I \times I$, where $I = \{1, \ldots, N\}$, into subsets $t_1^i \times t_2^i$, $i = 1, \ldots, n$ so that with $S_t = \{x_j,\, j \in t\}$ either the admissibility condition

$$\operatorname{diam} S_{t_2^i} \leq \eta \operatorname{dist}(S_{t_1^i}, S_{t_2^i})$$

is fulfilled or one of the index sets t_1^i and t_2^i has just one element.
In [4] a set of clusters $\mathcal{T}$ that possesses a tree structure is used to suitably subdivide the set P with respect to a fixed point. We will use a set of cluster pairs having a tree structure for the partitioning of $P \times P$ (the Cartesian product of the set of panels with itself). This set $\mathcal{T}'$ is constructed from the set $\mathcal{T}$ by applying the following recursion to (Γ_h, Γ_h). Take a cluster pair (t_1, t_2), $t_1, t_2 \in \mathcal{T}$. If t_1 and t_2 both have children t_{11}, t_{12} and t_{21}, t_{22} in $\mathcal{T}$ respectively, then assign the pairs (t_{11}, t_{21}), (t_{11}, t_{22}), (t_{12}, t_{21}) and (t_{12}, t_{22}) as children to the cluster pair (t_1, t_2) and add them to $\mathcal{T}'$. Now repeat these steps with each child. It is obvious that the covering of $\Gamma_h \times \Gamma_h$ with the smallest number of cluster pairs can be computed by the following algorithm.
Set $D = \emptyset$ and call $Partition((I, I), D)$, where *Partition* is the following recursive procedure.

procedure $Partition((t_1, t_2), D)$
  if (t_1, t_2) is admissible
  then $D := D \cup \{(t_1, t_2)\}$
  else apply the procedure to each of the children of (t_1, t_2) in $\mathcal{T}'$.

Each generated index pair $(t_1, t_2) \in D$ is admissible. Recently Hackbusch and Khoromskij [2], [3] have proven that the storage requirement for such matrices is $O(N \log N)$. However, they did not investigate the dependency on the parameter η which is essential for our algorithm since the approximation error will be controlled by this parameter.
In [9] it is shown that under the assumption of a quasi-uniform panel set P, i.e. there is a constant c_u so that

$$c_u \operatorname{diam} \Gamma_j > h, \text{ for all } \Gamma_j \in P,$$

where $h = \max_{\Gamma_j \in P} \operatorname{diam} \Gamma_j$, for the number of generated blocks $\mathcal{N}$ it holds that

$$\mathcal{N} = O(N^{1+\alpha}\eta^{-2(d-1)}), \ \forall \alpha > 0.$$

The partitioning of the matrix can be done so that the number of operations for this purpose is of the same order as $\mathcal{N}$.

4.2 Low-rank approximation

If the generating function $\kappa : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is asymptotically smooth (see [10]), i.e. there are constants c_1, $c_2 > 0$ and $g \in \mathbb{R}$ so that for any multi-index $\alpha \in \mathbb{N}_0^d$

$$|\partial_y^\alpha \kappa(x,y)| \le c_1\, p!\, c_2^p\, |x-y|^{g-p}, \quad p = |\alpha|,$$

it can be shown using a Taylor expansion that an admissible block can be well approximated by low-rank matrices. It is known that the best low-rank approximant is obtained by using singular value decomposition. But for SVD it is necessary to generate the whole block, which finally leads to a complexity of order N^2.

In [11] a result on the existence of low-rank approximants using only a small part of the original matrix can be found. In this result the so-called maximum volume submatrices are used. But to find them within a block in general all entries have to be investigated. In [9] we proposed an algorithm that computes the same kind of approximant but uses only a small part of the original matrix.

Assume that we have an admissible block of dimension $m \times n$. For $k = 1, 2, \ldots$ calculate

$$\begin{aligned}
(\widetilde{u}_k)_i &= \kappa(x_i, y_{j_k}) - \sum_{l=1}^{k-1} (v_l)_{j_k} (u_l)_i, \quad i = 1, \ldots, m \\
u_k &= (\widetilde{u}_k)_{i_k}^{-1}\, \widetilde{u}_k \\
(v_k)_j &= \kappa(x_{i_k}, y_j) - \sum_{l=1}^{k-1} (u_l)_{i_k} (v_l)_j, \quad j = 1, \ldots, n,
\end{aligned}$$

where in each step j_k is chosen so that $\widetilde{u}_k \neq 0$ and i_k so that $|(\widetilde{u}_k)_{i_k}| \ge |(\widetilde{u}_k)_i|$. For the approximant S_k it holds that

$$S_k = \sum_{l=1}^{k} u_l v_l^T.$$

Thus for the whole approximation we need only the evaluation of $\kappa(x_i, y_{j_k})$, $i = 1, \ldots, m$, and $\kappa(x_{i_k}, y_j)$, $j = 1, \ldots, n$. The rest are algebraic transformations, which are easy to implement, whereby it should be remembered that the entries of u_k at the positions i_l and the entries of v_k at the positions j_l, $l < k$ are zero.

To obtain S_k we need $(m+n)k$ units of storage and $O((m+n)k^2)$ operations to generate the approximant S_k.

In [9] it has been shown that using this method to generate an approximant of accuracy ε, i.e. $\|A-\widetilde{A}\|_F \leq \varepsilon$, we need

$$\mathcal{N}_{st} = O(N^{1+\alpha}\varepsilon^{-\alpha})$$

storage and the number of operations for a matrix-vector multiplication and for the generation of the approximant must not exceed $O(N^{1+\alpha}\varepsilon^{-\alpha})$.
Although we will use a bounded number of iteration steps it is useful to stop the algorithm if a prescribed accuracy is reached. Since we do not want to calculate the whole original matrix we cannot compute the error exactly. Thus a good approximation for the error is the only way to control the algorithm.
Since

$$R_{n_{p+1}} = R_{n_p} - \sum_{l=n_p+1}^{n_{p+1}} u_l v_l^T$$

and $\|R_{n_{p+1}}\|_F$ is of one order smaller than $\|R_{n_p}\|_F$ the value

$$\|\sum_{l=n_p+1}^{n_{p+1}} u_l v_l^T\|_F$$

may be used as a good approximation to $\|R_{n_p}\|_F$.
When computing $\|\sum_{l=n_p+1}^{n_{p+1}} u_l v_l^T\|_F$ it should be noted that

$$\|\sum_{l=n_p+1}^{n_{p+1}} u_l v_l^T\|_F^2 = \sum_{l,k=n_p+1}^{n_{p+1}} \left(\sum_{i=1}^{m}(u_l)_i(u_k)_i\right)\left(\sum_{j=1}^{n}(v_l)_j(v_k)_j\right).$$

The evaluation of the last expression can be made in $(m+n)(r_p^d)^2$ operations.

5 Numerical examples

We begin this subsection with the following trivial situation. Let Γ be the inner surface of the unit sphere

$$\Gamma = \left\{x \in \mathbb{R}^3,\ |x| = 1\right\}.$$

Then the kernel

$$\frac{1}{\pi}\frac{(n_x, y-x)(n_y, x-y)}{|x-y|^4}$$

of the radiation heat transfer equation can be simplified as follows:

$$n_x = -x\,,\ n_y = -y$$

Table 1: The results of the compression

Dimension	MBytes	MBytes	Percentage
512	2	2	100%
1024	8	5.0	63%
2048	32	12.6	39.5%
4096	128	27.4	21.4%
8192	512	63.6	12.4%
16384	2048	163.8	8.0%
32768	8192	380.9	4.6%

and therefore

$$\frac{1}{\pi}\frac{(n_x, y-x)(n_y, x-y)}{|x-y|^4} = \frac{1}{\pi}\frac{(1-(x,y))(1-(y,x))}{(2-2(x,y))^2} = \frac{1}{4\pi}.$$

Thus, any appropriate discretisation of this problem will lead to the matrix B of the rank one.
For the numerical tests of the Adaptive Cross Approximation procedure we will consider a geometry of a simple pipe given by the following parametric representation

$$\Gamma = \left\{ x \in \mathbb{R}^3,\ x = \begin{pmatrix} \cos(2\pi\, t) \\ \sin(2\pi\, t) \\ 4\, z \end{pmatrix},\ (t,z) \in (0,1)^2 \right\}.$$

The panels are plane triangles appearing after the canonical discretisation of the parameter domain $(0,1)^2$ using n_t and n_z discretisation points. The whole number of panels is then $N = 2\, n_t\, n_z$. In the following table we present the results of the matrix compression with accuracy $\varepsilon = 10^{-6}$ in the Frobenius norm achieved by the Adaptive Cross Approximation procedure. The first column of this matrix contains the dimension of the matrix N, the second column gives the amount of memory in megaabytes which would be needed without the approximation, the third the corresponding value for the compressed matrix, and the last the percentage of memory saved.

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